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VAGUE FILTER AND FANTASTIC FILTER OF BL-ALGEBRAS

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ABSTRACT. In this paper, we investigate some properties of vague filter of a *BL*-algebra. Also, introduce the notion of a vague fantastic filter with illustration. Further, we discuss some of related properties. Finally, we obtain equivalent condition and extension property of a vague fantastic filter.

1. INTRODUCTION

Zadeh, [11] introduced the concept of fuzzy set theory in 1965. The notion of intuitionistic fuzzy sets was introduced by Atanassov, [1, 2] in 1986 as an extension of fuzzy set. Gau and Buehrer [3] proposed the concept of vague set in 1993, by replacing the value of an element in a set with a subinterval of $[0, 1]$. Namely, there are two membership functions: a truth membership function t_S and a false membership function f_S, where $t_S(x)$ is a lower bound of the grade of membership of x derived from the "evidence of *x*" and $f_S(x)$ is a lower bound on the negation of *x* derived from the "evidence against *x*" and $t_S(x)+f_S(x) \leq 1$. Thus, the grade of membership in vague set S is subinterval $[t_S(x), 1 - f_S(x)]$ of [0, 1]. Hajek, [4] introduced the notion of *BL*-algebras as the structures for Basic Logic. Recently, the authors [7–10] introduce the definitions of vague filter, vague prime, Boolean filters, and vague implicative and vague positive

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implicative filters of *BL*-algebra and discussed some of their related properties with illustrations.

2. Preliminaries

In this section, we recall some basic definitions and their properties which are helpful to develop the main results.

Definition 2.1. *[4] A BL-algebra is an algebra $(\mathcal{B}, \vee, \wedge, *, \rightarrow, 0, 1)$ of type $(2,2,2,2,0,0)$ such that*

(i) $(\mathcal{B}, \vee, \wedge, 0, 1)$ *is a bounded lattice*
(ii) $(\mathcal{B},\ *,\ 1)$*is a commutative monoid*
(iii) $'*'$ *and* $'\rightarrow'$ *form an adjoint pair, that is,* $z \leq x \rightarrow y$ *if and only if* $x * z \leq y$ *for all* $x, y,\ z \in \mathcal{B}$
(iv) $x \wedge y = x * (x \rightarrow y)$
(v) $(x \rightarrow y) \vee (y \rightarrow x) = 1.$

Proposition 2.1. *[4] In a BL- algebra $\mathcal{B}$, the following properties hold for all* $x, y,\ z \in \mathcal{B}$,

(i) $y \rightarrow (x \rightarrow z) = x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z,$
(ii) $1 \rightarrow x = x,$
(iii) $x \leq y$ *if and only if* $x \rightarrow y = 1$
(iv) $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x),$
(v) $x \leq y$ *implies* $y \rightarrow z \leq x \rightarrow z,$
(vi) $x \leq y$ *implies* $z \rightarrow x \leq z \rightarrow y,$
(vii) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y),$
(viii) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z),$
(ix) $x \leq (x \rightarrow y) \rightarrow y,$
(x) $x * (x \rightarrow y) = x \wedge y$
(xi) $x * y \leq x \wedge y$
(xii) $x \rightarrow y \leq (x * z) \rightarrow (y * z),$
(xiii) $x * (y \rightarrow z) \leq y \rightarrow (x * z),$
(xiv) $(x \rightarrow y) * (y \rightarrow z) \leq x \rightarrow z,$
(xv) $(x * x^{-}) = 0.$

Note.

(1) In the sequel, we shall use $\mathcal{B}$ to denote as *BL*- algebras and the operation $\vee, \wedge, *$ have priority towards the operations $'' \rightarrow''$.
(2) In a *BL*- algebra $\mathcal{B}$, we can define $x^{-} = x \rightarrow 0$ for all $x \in \mathcal{B}$.

Definition 2.2. *[6] A filter of a BL- algebra $\mathcal{B}$ is a non-empty subset F of $\mathcal{B}$ such that for all $x, y \in \mathcal{B}$,*

(i) *If $x,\ y \in F,$ then $x * y \in F$,*
(ii) *If $x \in F$ and $x \leq y$, then $y \in F$.*

Proposition 2.2. *[6] Let F be a non-empty subset of a BL- algebra $\mathcal{B}$. Then F is a filter of $\mathcal{B}$ if and only if the following conditions hold:*

(i) $1 \in F$,
(ii) $x,\ x \rightarrow y \in F$ *implies* $y \in F$.
(iii) *A filter F of a BL-algebra $\mathcal{B}$ is proper if $F \neq \mathcal{B}$.*

Definition 2.3. *[5] A non-empty subset F of BL-algebra $\mathcal{B}$ is called a fantastic filter, if it satisfied the following axioms for all $x,\ y,\ z \in \mathcal{B}$,*

(i) $1 \in F$
(ii) $z \rightarrow (y \rightarrow x) \in F$ *and* $z \in F$ *imply* $((x \rightarrow y) \rightarrow y) \rightarrow x \in F$.

Definition 2.4. *[3] A vague set S in the universe of discourse X is characterized by two membership functions given by*

(i) *A truth membership function $t_S : X \rightarrow [0,\ 1]$,*
(ii) *A false membership function $f_S : X \rightarrow [0,\ 1]$.*

Where $t_S(x)$ is lower bound of the grade of membership of x derived from the 'evidence for x', and $f_S(x)$ is a lower bound of the negation of x derived from the 'evidence against x' and $t_S(x)+f_S(x) \leq 1$. Thus the grade of membership of x in the vague set S is bounded by a subinterval $[t_S(x),\ 1 - f_S(x)]$ of [0, 1].

The vague set S is written as $S = \{(x, [t_S(x), f_S(x)]) : x \in X\}$, where the interval $[t_S(x), 1 - f_S(x)]$ is called the value of *x* in the vague set S and denoted by $\vartheta_S(x)$.

Definition 2.5. *[3] A vague set S of a set X is called:*

(i) *the zero vague set of X if $t_S(x) = 0$ and $f_S(x) = 1$ for all $x \in X$,*
(ii) *the unit vague set of X if $t_S(x) = 1$ and $f_S(x) = 0$ for all $x \in X$,*
(iii) *the ρ-vague set of X if $t_S(x) = \rho$ and $f_S(x) = 1 - \rho$ for all $x \in X$ where* $\alpha \in (0, 1)$.

Definition 2.6. *[3] Let S be a vague set of X with truth membership functiont_S and the false membership functionf_S. For any $\rho, \sigma \in [0,1]$, the (ρ, σ)-cut of the vague set X is crisp subset $S_{(\rho,\sigma)}$ of the set X by $S_{(\rho,\sigma)} = \{\vartheta_S(x) \geq [\rho, \sigma]\ x \in X\}$, where $\rho \leq \sigma$. Obviously, $S_{(0,0)} = X$. The (ρ, σ)-cut is called vague-cut of the vague set S.*

Definition 2.7. *[3] Let S be a vague set of X. Then a ρ-cut of S is a crisp subset $S_\rho = S_{(\rho,\rho)}$ is defined as $S_\rho = \{t_S(x) \geq \rho : x \in X\}$.*

Definition 2.8. *[3] Let $D[0,1]$ denote the family of all closed subintervals of $[0,1]$. Now we define refined maximum (rmax) and $'' \geq ''$ on elements $D_1 = [a_1, b_1]$ and $D_2[a_2, b_2]$ of $D[0,1]$ as $rmax(D_1, D_2) = [max\{a_1, a_2\}, max\{b_1, b_2\}]$. Similarly we can define $\leq, =$ and rmin*

Definition 2.9. *[7] Let S be vague set of a BL-algebra $\mathcal{B}$ is called a vague filter of $\mathcal{B}$, Then, if it satisfied the following axioms for all $x,\ y \in \mathcal{B}$,*

(i) $\vartheta_S(1) \geq \vartheta_S(x)$,
(ii) $\vartheta_S(y) \geq rmin\{\vartheta_S(x \to y),\ \vartheta_S(x)\}$.

Proposition 2.3. *[7] Let S be vague set of BL-algebra $\mathcal{B}$ is a vague filter of $\mathcal{B}$ if and only if the following hold for all $x,\ y \in \mathcal{B}$,*

(i) $t_S(1) \geq t_S(x)$ *and* $1 - f_S(1) \geq 1 - f_S(x)$,
(ii) $t_S(y) \geq \min\{t_S(x \to y),\ t_S(x)\}$, and $1 - f_S(y) \geq \min\{1 - f_S(x \to y),\ 1 - f_S(y)\}$.

Proposition 2.4. *[7] Every vague filter S of BL- algebra A is order preserving.*

Proposition 2.5. *[7] Let S be a vague set of BL- algebra $\mathcal{B}$. S is a vague filter of A if and only if:*

(i) If $x \leq y,$ then $\vartheta_S(x) \leq \vartheta_S(y)$,
(ii) $\vartheta_S(x * y) \geq rmin\{V_S(x),\ V_S(y)\}$ *for all* $x,\ y, \in \mathcal{B}$.

Proposition 2.6. *[7] Let S be a vague set of BL- algebra $\mathcal{B}$. Let S be a vague filter of $\mathcal{B}$. Then the following are hold for all $x,\ y,\ z \in \mathcal{B}$,*

(i) *If* $\vartheta_S(x \to y) = \vartheta_S(1)$, *then* $\vartheta_S(x) \leq \vartheta_S(y)$
(ii) $\vartheta_S(x \wedge y) = rmin\ \{\vartheta_S(x),\ \vartheta_S(y)\}$
(iii) $\vartheta_S(x * y) = rmin\ \{\vartheta_S(x),\ \vartheta_S(y)\}$
(iv) $\vartheta_S(0) = rmin\ \{\vartheta_S(x),\ \vartheta_S(x^-)\}$

(v) $\vartheta_S(x \to z) \geq rmin\ \{\vartheta_S(x \to y), \vartheta_S(y \to z)\}$
(vi) $\vartheta_S(x \to y) \leq\ \vartheta_S(x * z\ \ \to y * z)$
(vii) $\vartheta_S(x \to y) \leq\ \vartheta_S((y \to z)\ \to (x \to z))$
(viii) $\vartheta_S(x \to y) \leq\ \vartheta_S((z \to x)\ \to (z \to y))$.

3. Main results

3.1. Vague Filters. In this section, we discuss some properties of vague filter.

Proposition 3.1. *A ρ- vague set and unit vague set of a BL-algebra $\mathcal{B}$ are vague filter of $\mathcal{B}$.*

Proof. Let S be a ρ-vague set of *BL*-algebra $\mathcal{B}$. Then from (i) of the proposition 2.5, we have if $x \leq y,\ then \vartheta_S(x) \leq \vartheta_S(y)$ for all $x,\ y, \in \mathcal{B}$.
To prove: $\vartheta_S(x * y) \geq rmin\{\vartheta_S(x),\ \vartheta_S(y)\}$ for all $x,\ y, \in \mathcal{B}$.
Now,

$$t_S(x * y) = \rho \\ = \min\{\rho, \rho\} \\ = \min\{t_S(x), t_S(y)\} \tag{3.1}$$

and

$$1 - f_S(x * y) = \rho \\ = \min\{\rho, \rho\} \\ = \min\{1 - f_S(x), 1 - f_S(y)\}, \forall x, y \in \mathcal{B}. \tag{3.2}$$

From (3.1) and (3.2), we have $\vartheta_S(x * y) \geq rmin\{\vartheta_S(x), \vartheta_S(y)\}$.
Thus, ρ- vague set is a vague filter of $\mathcal{B}$.
Similarly, we prove unit set is a vague of $\mathcal{B}$.

□

Theorem 3.1. *Let S be a vague set of BL-algebra $\mathcal{B}$. Then S is a vague filter of $\mathcal{B}$ if and only if the set $S_{(\rho,\sigma)}$ is either empty or a filter of $\mathcal{B}$ for all $\rho, \sigma \in [0, 1]$, where $\rho \leq \sigma$.*

Proof. Let S be vague filter of *BL*-algebra $\mathcal{B}$ and $S_{(\rho,\sigma)} \neq \emptyset$ for all $\rho, \sigma \in [0, 1]$.
To prove: $S_{(\rho,\ \ \sigma)}$ is a filter of $\mathcal{B}$.
If $x \leq y$ and $x \in S_{(\rho,\sigma)}$.

From (i) of the proposition 3.1, we have $\vartheta_S(y) \geq \vartheta_S(x) \geq [\rho, \sigma]$ for all $x, y \in \mathcal{B}$.
Thus, $y \in S_{(\rho,\sigma)}$.
If $x, y \in S_{(\rho,\sigma)}$, then $\vartheta_S(x)$ and $\vartheta_S(y) \geq [\rho, \sigma]$.
From (ii) of the proposition, we have $\vartheta_S(x * y) \geq rmin\{\vartheta_S(x), \vartheta_S(y)\} \geq [\rho, \sigma]$.
Thus $x * y \in S_{(\rho,\sigma)}$. Hence $S_{(\rho,\sigma)}$ is a filter of $\mathcal{B}$.
Conversely, if for all $\rho, \sigma[0, 1]$, the set $S_{(\rho,\sigma)}$ is either empty or a filter of $\mathcal{B}$.
Let $t_S(x) = \rho_1$, $t_S(y) = \rho_2$, $1 - f_S(x) = \sigma_1$ and $1 - f_S(y) = \sigma_2$. Put $\rho = \min\{\rho_1, \rho_2\}$ and $\sigma = \min\{1 - \sigma_1, 1 - \sigma_2\}$.
Then $t_S(x), t_S(y) \geq \rho$ and $1 - f_S(x), 1 - f_S(y) \geq \sigma$.
Thus $\vartheta_S(x)$ and $\vartheta_S(y) \geq [\rho, \sigma]$, that is $x, y \in S_{(\rho, \sigma)}$.
Thus $S_{(\rho, \sigma)} \neq \emptyset$. Hence by the assumption $S_{(\rho, \sigma)}$ is a filter of $\mathcal{B}$.
To prove: S is a vague filter of $\mathcal{B}$.
If $x \leq y$, $t_S(x) = \rho$ and $1 - f_S(x) = \sigma$.
Then $x \in S_{(\rho, \sigma)}$. Since $S_{(\rho, \sigma)}$ is a filter, $y \in S_{(\rho, \sigma)}$, that is,

$$\vartheta_S(y) \geq [\rho, \sigma] \tag{3.3}$$

Since $S_{(\rho,\sigma)}$ is filter of $\mathcal{B}$, $x * y \in S_{(\rho,\sigma)}$.
That is, $\vartheta_S(x * y) \geq [\rho, \sigma]$ for all $x, y \in \mathcal{B}$

$$\begin{aligned} &= [\min\{\rho_1, \rho_2\}, \min\{1 - \sigma_1, 1 - \sigma_2\}] \\ &= rmin\{[t_S(x), 1 - f_S(x)], [t_S(y), 1 - f_S(y)]\} \\ &= rmin\{\vartheta_S(x), \vartheta_S(y)\} \end{aligned} \tag{3.4}$$

for all $x, y \in \mathcal{B}$.
From (3.3) and (3.4), S is a vague filter of $\mathcal{B}$. □

Note. The filter $S_{(\rho, \sigma)}$ is called a vague-cut filter of *BL*-algebra $\mathcal{B}$.

Proposition 3.2. *Let S be a vague filter of BL-algebra $\mathcal{B}$. Then S_ρ is either empty or a filter of $\mathcal{B}$ for all $\rho \in [0, 1]$.*

Proof. Let S be a vague filter of *BL*-algebra $\mathcal{B}$. Then from theorem 3.1, the proof is obvious. □

Proposition 3.3. *Any filter F of a BL-algebra $\mathcal{B}$ is a vague-cut filter of some vague filter of $\mathcal{B}$.*

Proof. Let F be filter of a *BL*-algebra $\mathcal{B}$. Define a set as,

$$t_S(x) = 1 - f_S(x) = \begin{cases} 1 & if \quad x \in F \\ \rho & otherwise \end{cases} . \ \forall x \in \mathcal{B}. \tag{3.5}$$

Then, we get $\vartheta_S(x) = [1,\ 1]$, if $x \in F$ and $\vartheta_S(x) = [\rho,\ \ \rho]$, otherwise, $\rho \in (0,\ 1)$.

Thus $F = S_{(1,1)}$.

Case (i): If $x,\ y \in F$, then $x * y \in F$, and

From (3.5), we have $\vartheta_S(x * y) = [1, 1] = rmin\{\vartheta_S(x), \vartheta_S(y)\}$ for all $x,\ y \in \mathcal{B}$.

Case (ii): If one of x or $y \notin F$, then one of $\vartheta_S(x)$ and $\vartheta_S(y)$ is equal to $[\rho, \rho]$.

Thus $\vartheta_S(x * y) \geq [\rho,\ \rho] = rmin\{\vartheta_S(x), \vartheta_S(y)\}$ for all $x,\ y \in \mathcal{B}$.

If $x \leq y$ and $x \in F$, then $y \in F$. Therefore $\vartheta_S(x) = \vartheta_S(y)$

If $x \notin F$, then $\vartheta_S(x) = [\rho,\ \ \rho]$, and we have $\vartheta_S(x) \leq \vartheta_S(y)$ for all $x,\ y \in \mathcal{B}$.

Hence from the both the cases (i) and (ii) S is vague filter of $\mathcal{B}$. □

3.2. **Vague fantastic filter.** In this section we introduce a notion of vague fantastic filter and investigate some related properties.

Definition 3.1. *Let S be a vague subset of BL- algebra $\mathcal{B}$. Then S is called a vague fantastic filter of $\mathcal{B}$, if it satisfies the following axioms for all $x,\ y,\ z \in \mathcal{B}$,*

(i) $\vartheta_S(1) \geq \vartheta_S(x)$,
(ii) $\vartheta_S((x \to y) \to y) \to x) \geq rmin\{\vartheta_S(z \to (y \to x)), \vartheta_S(z)\}$.

Example 1. *Let $\mathcal{B} = \{0, p, q, r, 1\}$. The binary operations "$*$" and "$\to$" given by the Table 1 and Table 2.*

$\to$	0	p	q	r	1
0	1	1	1	1	1
p	0	1	1	1	1
q	0	r	1	r	1
r	0	q	q	1	1
1	0	p	q	r	1

TABLE 1

$*$	0	p	q	r	1
0	0	0	0	0	0
p	0	p	p	p	p
q	0	p	q	p	q
r	0	p	p	r	r
1	0	p	q	r	1

TABLE 2

Then $(\mathcal{B}, \ \vee, \ \wedge, \ *, \ \rightarrow, \ 0, \ 1)$ *is a BL- algebra. Define a vague set* S of A *as follows:* $S = \{(0, [0.2, 0.5]), (p, [0.2, 0.5]), (q, [0.2, 0.5]), (r, [0.4, 0.7]), (1, [0.7, 0.7])\}$. *It is easily verify that* S *is a vague fantastic filter of* A.

Proposition 3.4. *Every vague fantastic filter of a* BL*-algebra* $\mathcal{B}$ *is a vague filter of* $\mathcal{B}$.

Proof. Let S be a vague fantastic filter of *BL*-algebra $\mathcal{B}$. Then from (i) of the definition 3.1, we have

$$\vartheta_S(1) \geq \vartheta_S(x) \ \forall \ x \in \mathcal{B} \tag{3.6}$$

From (ii) of the definition 3.1, $\vartheta_S((x \rightarrow y) \rightarrow y) \rightarrow x) \geq rmin\{\vartheta_S(z \rightarrow (y \rightarrow x)), \vartheta_S(z)\}$ for all $x, \ y, \ z \in \mathcal{B}$.

Put $y = 1$, we have

$$\begin{aligned} \vartheta_S((x \rightarrow 1) \rightarrow 1) \rightarrow x) &\geq rmin\{\vartheta_S(z \rightarrow (1 \rightarrow x)), \vartheta_S(z)\} \\ \vartheta_S((1 \rightarrow 1) \rightarrow x) &\geq rmin\{\vartheta_S(z \rightarrow x), \ \vartheta_S(z)\} \qquad (3.7) \\ \vartheta_S(x) &\geq rmin\{\vartheta_S(z \rightarrow x), \vartheta_S(z)\} forall x, \ z \in \mathcal{B}. \end{aligned}$$

Thus from (3.6) and (3.7), S is vague filter of $\mathcal{B}$. □

Proposition 3.5. *Let* S *be a vague filter of a BL-algebra* $\mathcal{B}$. *Then* S *is a vague fantastic filter of* $\mathcal{B}$ *if and only if* $\vartheta_S(((x \rightarrow y) \rightarrow y) \rightarrow x) \geq \vartheta_S(y \rightarrow x)$ *for all* $x, y \in \mathcal{B}$.

Proof. Let S be a vague fantastic filter of a *BL*-algebra $\mathcal{B}$.
Then (ii) of the definition 3.1, we have $\vartheta_S((x \rightarrow y) \rightarrow y) \rightarrow x) \geq rmin\{\vartheta_S(z \rightarrow (y \rightarrow x)), \ \vartheta_S(z)\}$ for all $x, \ y, \ z \in \mathcal{B}$.

Put $z = 1,$ we get,

$$\begin{aligned}\vartheta_S((x \to y) \to y) \to x) &\geq rmin\{\vartheta_S(1 \to (y \to x)),\ \ \vartheta_S(1)\}\\ &= rmin\{\vartheta_S(y \to x),\ \vartheta_S(1)\}\\ &= \vartheta_S(y \to x)\,.\end{aligned}$$

Thus, we have $\vartheta_S(((x \to y) \to y) \to x) \geq \vartheta_S(y \to x)$ for all $x, y \in \mathcal{B}$.
Conversely, let S be a vague filter *BL*-algebra $\mathcal{B}$ and satisfies $\vartheta_S(((x \to y) \to y) \to x) \geq \vartheta_S(y \to x)$ for all $x, y \in \mathcal{B}$.
Then from the (ii) of the Definition 2.9, we have

$$\begin{aligned}\vartheta_S(((x \to y) \to y) \to x) &\geq \vartheta_S(y \to x)\\ &\geq rmin\{\vartheta_S(z \to (y \to x)),\ \ \vartheta_S(z)\},\ \forall x, y, z \in \mathcal{B}.\end{aligned} \tag{3.8}$$

Since S is vague filter of $\mathcal{B}$, we have

$$\vartheta_S(1) \geq \vartheta_S(x),\ \forall x \in \mathcal{B}. \tag{3.9}$$

Thus from (3.8) and (3.9), S is a vague fantastic filter of $\mathcal{B}$. □

Theorem 3.2. *Let S be a vague set of BL-algebra $\mathcal{B}$. Then S is a vague fantastic filter of $\mathcal{B}$ if and only if the set $S_{(\rho,\sigma)}$ is either empty or a fantastic filter of $\mathcal{B}$ for all $\rho, \sigma \in [0, 1]$, where $\rho \leq \sigma$.*

Proof. Let S be a vague filter of *BL*-algebra $\mathcal{B}$ and $S_{(\rho,\sigma)} \neq \emptyset$ for all $\rho, \sigma \in [0,\ \ 1]$. Since S is a vague filter of $\mathcal{B}$, from the theorem 3.1, $S_{(\rho,\ \ \sigma)}$ is a filter of $\mathcal{B}$. To prove: $S_{(\rho,\ \ \sigma)}$ is a fantastic filter of $\mathcal{B}$. Let $((x \to y) \to y) \to x \in S_{(\rho,\ \ \sigma)}$ for all $x,\ y \in \mathcal{B}.$ Then $\vartheta_S(((x \to y) \to y) \to x) \geq [\rho,\ \sigma]$ Then from the proposition 3.5, we have $\vartheta_S(y \to x) \geq [\rho,\ \sigma]$. Therefore, $y \to x \in S_{(\rho,\ \ \sigma)}$. Hence $S_{(\rho,\ \ \sigma)}$ is a fantastic filter of $\mathcal{B}$.

Conversely, let $t_S(((x \to y) \to y) \to x) = \rho$ and $1 - f_S(((x \to y) \to y) \to x) = \sigma$ for all $x,\ y \in \mathcal{B}$. Then $\vartheta_S(((x \to y) \to y) \to x) \geq [\rho,\ \sigma]$. Thus, we have $((x \to y) \to y) \to x \in S_{(\rho,\ \ \sigma)}$ and $S_{(\rho,\ \ \sigma)} \neq \emptyset$. Therefore, $y \to x \in S_{(\rho,\ \ \sigma)}$,

That is, $\vartheta_S(y \to x) \geq [\rho,\ \sigma] = \vartheta_S(((x \to y) \to y) \to x)$. From the proposition 3.5, we have S is a vague fantastic filter of $\mathcal{B}$.. □

Corollary 3.1. *Let S be a vague set of BL-algebra $\mathcal{B}$. Then S is a vague fantastic filter of $\mathcal{B}$. Then the set S_ρ for all $\rho \in [0, 1]$ is either empty or fantastic filter of $\mathcal{B}$.*

Theorem 3.3. *Let $S_1$and S_2 be two vague filters of BL-algebra $\mathcal{B}$, $S_1 \subseteq S_2$ and $V_{S_1}(1) = V_{S_2}(1)$. If S_1 is a vague fantastic filter, then so is S_2.*

Proof. Let S_1 and S_2 be two vague filters of *BL*-algebra $\mathcal{B}$, $S_1 \subseteq S_2$ and $V_{S_1}(1) = V_{S_2}(1)$. Then, we have

$$\begin{aligned}
&\vartheta_{S_1}(((((y \to x) \to x) \to y) \to y) \to ((y \to x) \to x)) \\
&\qquad \geq \vartheta_{S_2}(((((y \to x) \to x) \to y) \to y) \to ((y \to x) \to x)) \forall\ x, y\ \in \mathcal{B} \\
&\qquad \geq \vartheta_{S_1}(y \to (y \to x) \to x)) \\
&\qquad = \vartheta_{S_1}((y \to x) \to (y \to x)) \\
&\qquad = \vartheta_{S_1}(1) \\
&\qquad = \vartheta_{S_2}(1)\,.
\end{aligned}$$

Since, $(((((y \to x) \to x) \to y) \to y) \to x) \to (((x \to y) \to y) \to x)$

$$\begin{aligned}
&\geq ((x \to y) \to y) \to ((((y \to x) \to x) \to y) \to y) \\
&\geq (((y \to x) \to x) \to y) \to (x \to y) \\
&\geq x \to ((y \to x) \to x) \\
&= (y \to x) \to (x \to x) \\
&= (y \to x) \to 1 = 1\,.
\end{aligned}$$

Now,

$$\begin{aligned}
& \vartheta_{S_2}(((x \to y) \to y) \to x) \\
\geq\ & rmin\{\vartheta_{S_2}(1), \vartheta_{S_2}(((((y \to x) \to x) \to y) \to y) \to x),\ \forall x, y \in \mathcal{B} \\
=\ & \vartheta_{S_2}(((((y \to x) \to x) \to y) \to y) \to x) \\
\geq\ & rmin\{\vartheta_{S_2}((y \to x) \to (((((y \to x) \to x) \to y) \to y) \to x), \vartheta_{S_2}(y \to x)\} \\
=\ & rmin\{\vartheta_{S_2}((((y \to x) \to x) \to y) \to y), \vartheta_{S_2}((y \to x))\} \\
\geq\ & rmin\{\vartheta_{S_2}(1), \vartheta_{S_2}(y \to x)\} \\
=\ & \vartheta_{S_2}(y \to x)\,.
\end{aligned}$$

Thus from proposition 3.5, S_2 is a fuzzy fantastic implicative filter. □

4. CONCLUSIONS

This paper mainly focused as investigation of some properties of the vague filter in *BL*-algebra. Also, we have introduced the notion of the vague fantastic filter and discussed some results. Finally, we have investigated equivalent condition and extension property of the vague fantastic filter of BL-algebra.

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