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EMPHASIS OF COEFFICIENTS ON THE CONVERGENCE RATE OF FIXED POINT ITERATIVE ALGORITHM IN BANACH SPACE

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ABSTRACT. This paper deals with the fixed point iterative schemesthat are being utilized to solve the systems of nonlinear equations in various fields and spaces. The convergence speed of iterative processes is highly focused, calculated and comparedwith various original methods to check the efficiency of these methods towards its optimal solution. The effect on the speed of convergence by the coefficients included in these iterative algorithms are investigated and analyzed in this manuscript. Moreover, the results on the convergence rate of various fixed point iterative schemesare supported by comparing the convergence rate of these iterative plans using the methodology of interchanging the coefficients of these algorithms. The convergence behaviour of these iterative processes is also shown graphically. In a nut shell, the comparison analysis shows that the coefficients involved in such type of schemes may vary the convergence rate of the schemes towards their fixed points.

1. INTRODUCTION

This paper deals with the fixed point iteration methods that utilized to solve linear and nonlinear systems. It is assumed that each problem of can be expressed using the fixed point algorithm. Fixed Point Algorithm (FPA) is a type of tools which is available everywhere and use tofind more accurate solutions

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to non-linear systemsused in Analysis, Algebra, Geometry, Logic etc.The convergence speed of these procedures playan effective role in the solution of non-linear systems. This paper discusses about the fixed point iterative schemes with it various applications in every field science and technology. A very common and fundamental application of fixed point in computer science is in recursion theory and mathematical model loops. The utilize of its algorithm is because of the semantics of recursion that are depicted by fixed points of functions. One of the first occurrence of fixed points is in the field of automata (robotics). The theorem about the existence $\&$ axioms of the FPs is popularlyregarded as the 'fixed point theorem' (FPT). Several remarks, results and consequences in the theory of automata are obtained from the basicresults of fixed points.

Several prominent applications related to the fixed point theory (FPT) are there in every field of Mathematical science and technology. In analysis, fixed point has aimperative job in software engineering and in data-flow analysis as programming languages compilers use fixed point calculation for program statement analysiswhich is often required for code optimization. In numerical analysis, one can calculate the FPs of the iterationfunctions using FP-iteration. The fixed point iteration for a point τ in the domain of function φ is defined as $\tau_{p+1} = \varphi(\tau_p)$ for $p \geq 0$. In Numerical analysis, it is a method of determining fixed point by doing a number of iteration steps to the function. The fixed point iteration process may be called as a chain point, because the current result of the immediate solution treated as an input to next solution. So it creates a chain point in more appropriate values of the solution.

The Fixed Point Algorithm (FPA) generates a recursive sequence that find the fixed point for some standard functions. The major advantages and benefits of these algorithmsare that these are not difficult to implement on Mathematical models and functions. The Fixed Point Algorithm (FPA) uses a value, ideally chosen very close to the fixed point thatwe want to find and a function $\varphi(\tau)$ that generates a recursively defined sequence τ_p for $p = 0$ and $\tau_{p+1} = \varphi(\tau_p)$ where $p \geq 0$.

The Carl Gustav Jacob's (CGJ) or Jacobi iterative procedure (JIP) is a prominent method for finding solution of diagonal system ofequations. Similarly, Gauss Seidel Iteration method (GSIM) utilizes the latest approximate values and scan the mesh points symmetrically from left rows to right rows along with successive rows.

2. Preliminaries

A fixed point of a function $\varphi(\tau)$ is a value τ_0 in the domain of function such that $\varphi(\tau) = \tau$. Geometrically, a fixed point occurs at which the graph of the function $\rho = \varphi(\tau)$ crosses the graph of $\rho = \tau$.

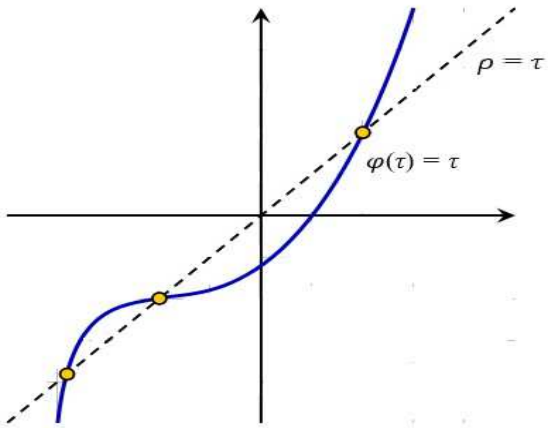

Fig. 1: Fixed Point

Following are some basic definitions and preliminaries that are used in this paper.

Definition 2.1. *Let $\mathcal{B}$ be a non-void set and φ be a self-mapping on $\mathcal{B}$. A FP of φ is an object in $\mathcal{B}$ that is drawn to itself, i.e. $\tau \in \mathcal{B}$ implies $\varphi(\tau) = \tau$.*

Definition 2.2. *Assume that $C' \subseteq \mathcal{B}$ be a non-void closed and convex subset. A function $\varphi : C' \to C'$ is non-expansive if*

$$|| \varphi\tau - \varphi\rho|| \leq ||\tau - \rho \in C'. \tag{2.1}$$

Definition 2.3. *Let $(\mathcal{B}, \omega)$ and $\varphi : \mathcal{B} \to \mathcal{B}$ be a map on $\mathcal{B}$. Define $F\varphi = \{\tau \in \mathcal{B} | \varphi\tau = \tau\}$ as the collection of fixed points of φ, then*

$$\omega(\varphi\tau, \varphi\rho) \leq \mu\omega(\tau, \rho) \forall \tau, \rho \in \mathcal{B} \tag{2.2}$$

and $\mu \in [0, 1)$ is known as Banach contraction's condition.

Definition 2.4. *A mapping $\varphi : \mathcal{B} \to \mathcal{B}$ is supposed to be non expensive if $\omega(\varphi\tau, \varphi\rho) \leq \omega(\tau, \rho) \forall \tau, \rho \in \mathcal{B}$. It is called a contraction if for all $\tau, \rho \in \{\mathcal{B}\}, \exists \mu \in (\mathbf{0}, \mathbf{1})$ such that*

$$||\varphi\tau - \varphi\rho|| \leq \mu||\tau - \rho|| \tag{2.3}$$

Definition 2.5. *Let $\{\sigma_p\}$ and $\{\sigma_p^{'}\}$ be two sequences approaches to n and m, then $\{\sigma_p\}$ moves faster than $\{\sigma_{p'}^{'}\}$ if*

$$\lim_{p\to\infty} \frac{||\ \sigma_p\ -\ \ n\ ||}{||\ \sigma\prime_p\ -\ \ m\ ||} = 0. \tag{2.4}$$

In the following section, we discuss the comparison of various possible iterations obtained by exchanging the coefficients and their comparison analysis is recorded as improved version.

3. Numerical Convergence

Example 1. *Let $\mathcal{B} = [1, 50]$ with initial values $\sigma_0 = 20$ and $\tau_0 = 30$ where the coefficient $\theta_p = 0.70$ and $\delta_p = 0.80$ for all $p \geq 0$. Define two maps $\varphi_1 : \mathcal{B} \to \mathcal{B}$ and $\varphi_2 : \mathcal{B} \to \mathcal{B}$ by $\varphi_1(\sigma) = \sigma^{\frac{1}{2}}$ and $\varphi_2(\sigma) = \sigma^{\frac{1}{3}}$ where φ_1 and φ_2 are the contraction maps.*

Solution. Consider the following fixed point iteration scheme:

$$\varphi_2\ \sigma_p + 1 = (1-\theta_p)\varphi_2\sigma_p + \theta_p\varphi_1\tau_p, \tag{3.1}$$

$$\varphi_2\tau_p = (1-\delta_n)\varphi_2\sigma_p + \delta_p\varphi_1\sigma_p, \tag{3.2}$$

where $\sigma_0 = 20, \tau_0 = 30, \theta_n = 0.70$ and $\delta_0 = 0.80$ for all $p \geq 0$,

$$\varphi_2\ \sigma_p + 1 = \theta_p\varphi_2\sigma_p + (1-\theta_p)\varphi_1\tau_p$$

$$\varphi_2\tau_p = \delta_p\varphi_2\sigma_p + (1-\delta_p)\varphi_1\sigma_p, \tag{3.3}$$

$$\varphi_2\ \sigma_p + 1 = \theta_p\varphi_2\sigma_p + (1-\theta_p)\varphi_1\tau_p,$$

$$\varphi_2\tau_p = (1-\delta_p)\varphi_2\sigma_p + \delta_p\varphi_1\sigma_p, \tag{3.4}$$

and

$$\varphi_2\ \sigma_p + 1 = (1-\theta_p)\varphi_2\sigma_p + \theta_p\varphi_1\tau_p,$$

$$\varphi_2\tau_p = \delta_p\varphi_2\sigma_p + (1-\delta_p)\varphi_1\sigma_p, \tag{3.5}$$

where $\sigma 0 = 20, \tau 0 = 30, \theta 0 = 0.70 and \delta 0 = 0.80$.

Here in the following table, we compare four different cases of this iterative method:

Table 3.1: Comparison of iteration ofφ_2 $\sigma_{\gamma-1}$

Steps	Iteration (5)	Iteration (6)	Iteration (7)	Iteration (8)
1	20	20	20	20
2	4.648383871	3.543260003	3.543260004	4.648383188
3	3.032758079	3.182386197	3.182386198	3.032758080
4	1.982147798	2.686616150	2.686616151	1.980700986
5	1.460445409	2.251688906	2.251688907	1.460311366
6	1.216632658	1.909824677	1.909824679	1.216592446
7	1.103196627	1.653251704	1.653251631	1.103184565
8	1.049808611	1.465354602	1.465354552	1.049804993
9	1.024304786	1.329760594	1.329760560	1.024303702
10	1.011956989	1.232793443	1.232831915	1.011956664
11	1.005915998	1.163953511	1.163980442	1.005915902
12	1.002938283	1.115266094	1.115284946	1.002938255
13	1.001462984	1.080935480	1.080948677	1.001462977
14	1.000729583	1.056779417	1.056788655	1.000729583
15	1.000364205	1.039807876	1.039814343	1.000364205
16	1.001892300	1.027896654	1.027932324	1.000181923
17	1.000090907	1.022366281	1.019583767	1.000090907
18	1.000045437	1.015664181	1.013724207	1.000045437
19	1.000033789	1.010968819	1.009614730	1.000022714
20	1.000014678	1.007680119	1.006734203	1.000011550
21	1.000006674	1.005377057	1.004715888	1.000005677
22	1.000003137	1.003764426	1.003302095	1.000002838

Table 3.2: Comparison of iteration of $\varphi_2\, \tau_n$

Steps	Iteration (5)	Iteration (6)	Iteration (7)	Iteration (8)
1	4.120592286	3.065961283	4.120592300	3.065961285
2	2.621470658	3.257556507	2.400446023	4.141655056
3	1.769923962	2.836752044	1.799849601	2.849154969
4	1.361156399	2.390474630	1.502050077	1.875403875
5	1.165478769	2.020978859	1.328848724	1.409430803
6	1.081676801	1.737443965	1.220301834	1.192901692
7	1.029590004	1.527337400	1.149594475	1.092131876
8	1.019381602	1.374637844	1.102487564	1.044580031
9	1.009557113	1.264982086	1.070646665	1.021797125
10	1.004736039	1.186820715	1.048910226	1.010738969
11	1.002354662	1.131515575	1.035322900	1.005318682
12	1.001173174	1.092359204	1.024319399	1.002643370
13	1.000585302	1.064821535	1.016820691	1.001316708
14	1.000292256	1.045460106	1.011673147	1.000656816
15	1.000146007	1.031704586	1.008120561	1.000327936
16	1.000072967	1.025552812	1.005665307	1.000163823
17	1.000036472	1.017897595	1.003956174	1.000081868
18	1.000021002	1.012533631	1.002764551	1.000040921
19	1.000011331	1.008776198	1.001932801	1.000020457
20	1.000005222	1.006144667	1.001351768	1.000010227
21	1.000002478	1.004301931	1.000945641	1.000005113

From the above tables, we analyze the iteration (5) converges more rapid than all iterations (6)-(8). The following pie graph represents the convergence behaviour of this iterative method:

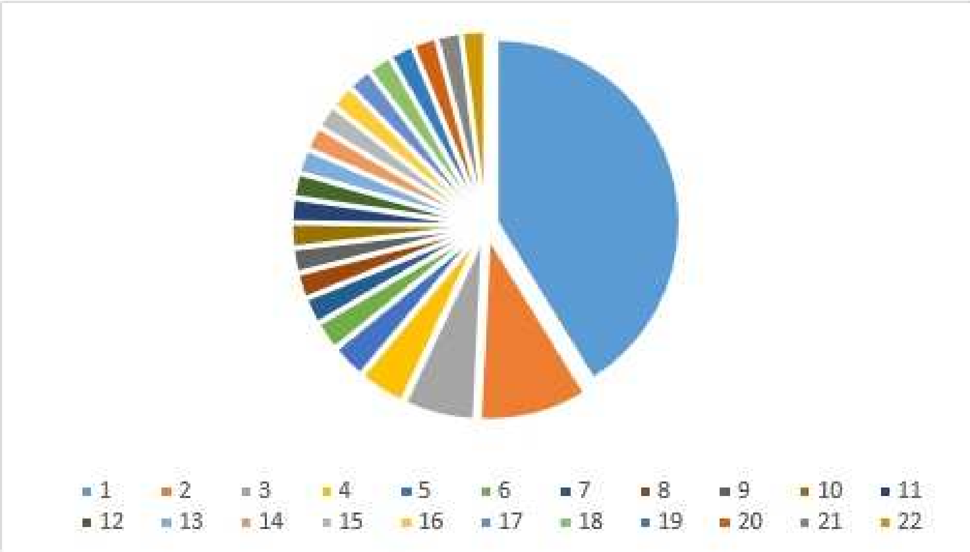

Fig. 3.1: Graphical representation of convergence speed

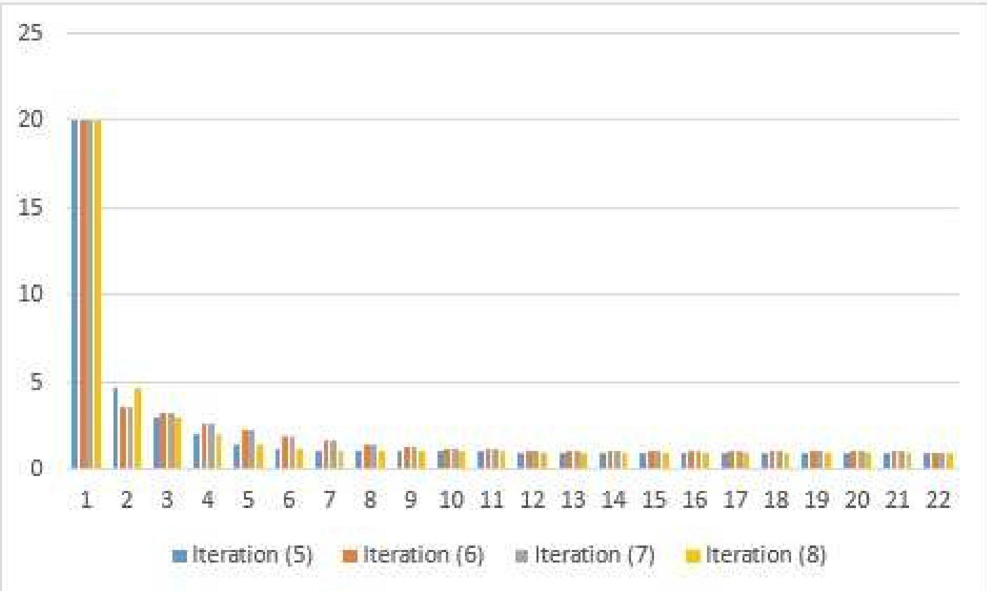

Fig. 2: Convergence Behaviour Iteration Scheme

4. CONCLUSION

The above analysis of convergence speed of particular iteration process, it is investigated that the convergence rate of such iterative processes vary efficiently when the coefficients involved in these methods are interchanged with its respective iterative schemes. By numerical consideration, clearly the iterative process (5) converges with a rate faster than the iterations (6)-(8). Moreover, it is clear from the graphical presentation that the iterative procedure (7) moves quicker than the iteration (6) and the iteration (8) approaches to its fixed point better that the iterations (6) and (7). So all in all, we say that the iterative plan (5) moves more rapid than all the cases of fixed point iterative process considered in this manuscript.Therefore, it is analyzed that the coefficients of these iterative processes have a notable effect on the convergence rate on these schemes. Following is the convergence behaviour of this iterative procedure:

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