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CONVOLUTION CONDITIONS AND GENERALIZED JANOWSKI-SAKAGUCHI TYPE FUNCTIONS

N. SHILPA[1]

ABSTRACT. In the present paper we obtain convolution conditions for the classes $K(A,B,s,t)$, $S^*(A,B,s,t)$, $K_\lambda(A,B,s,t)$, $S^*_\lambda(A,B,s,t)$ defined by using Janowski class and Sakaguchi type functions.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all analytic univalent functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

defined in the unit disc $\mathcal{U} = \{z : |z| < 1\}$.

For $f, g \in \mathcal{A}$, where f is of the form (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n,\ \ b_n \geq 0$, $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$, is called the convolution or Hadamard product of f and g.

Let $H = \{\omega\,, \omega$ analytic in $\mathcal{U}$, $\omega(0) = 0,\ |\omega(z)| < 1,\ z \in \mathcal{U}\}$.

Let $P(A,B)$ denote the Janowski class [3] containing functions p of the form

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)}\,,\ -1 \leq B < A \leq 1\,,\ \omega \in H.$$

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[1]*corresponding author*

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Now we introduce the following classes of analytic functions

(i) A function $f \in S^*(A,B,s,t)$, if $\dfrac{(s-t)zf'(z)}{f(sz)-f(tz)} \in P(A,B)$.

(ii) $f \in S^*_\lambda(A,B,s,t)$, if $\dfrac{e^{i\lambda}\frac{(s-t)zf'(z)}{f(sz)-f(tz)} - isin\lambda}{cos\lambda} \in P(A,B)$.

The classes $K(A,B,s,t)$ and $K_\lambda(A,B,s,t)$ are defined by $f \in K(A,B,s,t)$, if $zf' \in S^*(A,B,s,t)$ and $f \in K_\lambda(A,B,s,t)$, if $zf' \in S^*_\lambda(A,B,s,t)$ where $s,t \in \mathbb{R}$ with $s \neq t$ for all $z \in \mathcal{U}$ and λ is real and satisfies $|\lambda| < \frac{\pi}{2}$.

Note that for $A = 1-2\alpha,\ B=-1,$ we obtain the classes defined by Frasin [1], when $A = 1-2\alpha,\ B=-1,\ s=1$ we have the classes introduced by Owa [2], for $A = 1-2\alpha,\ B=-1,\ s=1$ and $t=-1$ we have the class introduced and studied by Sakaguchi [4] ans for s=1 we get classes in [5].

2. Main Results

Theorem 2.1. *A function f defined by (1.1) belongs to the class $K(A,B,s,t)$ in $|z| < R \leq 1$ if and only if*

$$\frac{1}{z}\left\{f * \frac{z(1+z)(1+B\rho) - z(1-z)(1+A\rho)u_n(s,t)}{(1-z)^3}\right\} \neq 0, \quad (|z|<R, |\rho|=1),$$

where $u_n(s,t) = \sum_{j=1}^{n-1} s^{n-j}t^{j-1}$

Proof. The function f belongs to $K(A,B,s,t)$ if and only if

$$\frac{(s-t)(zf'(z))'}{f'(sz)-f'(tz)} \in P(A,B).$$

But the function $p(z) = \dfrac{1+A\omega(z)}{1+B\omega(z)}$, $-1 \leq B < A \leq 1, \omega \in H$ is subordinate to $\dfrac{1+Az}{1+Bz}$. They map the unit circle $|z|=1$ onto the boundary of the circle on the line joining $\dfrac{1-A}{1-B}$ and $\dfrac{1+A}{1+B}$ as diameter. When B=-1, the image of the unit circle is the line $\Re\{p(z)\} = \dfrac{1-A}{2}$, $-1 < A \leq 1$. Further $\dfrac{(s-t)(zf'(z))'}{f'(sz)-f'(tz)} = 1$ at $z=0$ and $(1,0)$ lies inside the image circle. The functions $\dfrac{1+A\omega(z)}{1+B\omega(z)}$ are analytic and hence map regions onto regions. Therefore every point in the interior of the unit disc goes over to an interior point of the image disc. Thus $f \in K(A,B,s,t)$ is equivalent to

$$\frac{(s-t)(zf'(z))'}{f'(sz)-f'(tz)} \neq \frac{1+A\rho}{1+B\rho}, \quad (|z|<R, |\rho|=1, B\rho \neq -1) \tag{2.1}$$

This simplifies to

$$(1+B\rho)(zf'(z))' - (1+A\rho)[1+\sum_{n=2}^{\infty} na_n z^{n-1} u_n(s,t)] \neq 0. \tag{2.2}$$

where $u_n(s,t) = \sum_{j=1}^{n-1} s^{n-j}t^{j-1}$. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ we have $zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n$ and $(zf'(z))' = 1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1} = f'(z) * \frac{1}{(1-z)^2}$. Therefore we have

$$f' * \frac{(1+B\rho)-(1-z)(1+A\rho)u_n(s,t)}{(1-z)^2} \neq 0,$$

which is equivalent to

$$\frac{1}{z}\left\{zf' * \frac{[(1+B\rho)-(1+A\rho)u_n(s,t)]z+(1+A\rho)u_n(s,t)z^2}{(1-z)^2}\right\} \neq 0,$$

$(|z|<R, |\rho|=1)$.

$$\frac{1}{z}\left[zf' * \frac{z+\frac{u_n(s,t)(1+A\rho)}{(1+B\rho)-u_n(s,t)(1+A\rho)}z^2}{(1-z)^2}\right] \neq 0. \tag{2.3}$$

Since $zf' * g = f * zg'$, (2.3) becomes

$$\frac{1}{z}\left[f * \frac{z+\frac{(1+B\rho)+u_n(s,t)(1+A\rho)}{(1+B\rho)-u_n(s,t)(1+A\rho)}z^2}{(1-z)^3}\right] \neq 0, \ |z|<R, \ |\rho|=1.$$

□

Theorem 2.2. *A function f defined by (1.1) belongs to the class $S^*(A,B,s,t)$ in $|z|<R\leq 1$ if and only if*

$$\frac{1}{z}\left[f * \frac{z+\frac{u_n(s,t)(1+A\rho)}{(1+B\rho)-u_n(s,t)(1+A\rho)}z^2}{(1-z)^2}\right] \neq 0, \ (|z|<R, \ |\rho|=1),$$

where $u_n(s,t) = \sum_{j=1}^{n-1} s^{n-j}t^{j-1}$.

Proof. Since $f \in S^*(A,B,s,t)$ if and only if $g \in K(A,B,s,t)$ where $g(z) = \int_0^z \frac{f(t)}{t}dt$, we have

$$\frac{1}{z}\left[g * \frac{z + \frac{(1+B\rho)+u_n(s,t)(1+A\rho)}{(1+B\rho)-u_n(s,t)(1+A\rho)}z^2}{(1-z)^3}\right] = \frac{1}{z}\left[f * \frac{z + \frac{u_n(s,t)(1+A\rho)}{(1+B\rho)-u_n(s,t)(1+A\rho)}z^2}{(1-z)^2}\right].$$

Thus result follows immediately from Theorem 2.1. □

As a corollary we can derive coefficient inequalities for the class $S^*(A,B,s,t)$.

Corollary 2.1. *A function $f \in \mathcal{A}$ is in the class $S^*(A,B,s,t)$ if and only if*

$$f(z) = 1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0,$$

where

$$A_n = \frac{(n - u_n(s,t)) + (nB - Au_n(s,t))\rho}{\rho(B-A)} a_n$$

and $u_n(s,t) = \sum_{j=1}^{n-1} s^{n-j}t^{j-1}$.

Proof. A function $f \in S^*(A,B,s,t)$ if and only if

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} \neq \frac{1+A\rho}{1+B\rho}.$$

That is

$$(1+B\rho)(s-t)(zf'(z)) - (1+A\rho)\left[f(sz)-f(tz)\right] \neq 0$$

which implies

$$(B-A)\rho z\left[1 + \sum_{n=2}^{\infty} n(1+B\rho) - u_n(s,t)(1+A\rho)\right] a_n z^n \neq 0.$$

This simplifies into

$$1 + \sum_{n=2}^{\infty} \frac{(n-u_n(s,t))(nB - Au_n(s,t))\rho}{\rho(B-A)} a_n z^{n-1} \neq 0,$$

which completes the proof. □

Theorem 2.3. *For* $|z| < R \leq 1$, λ *real with* $|\lambda| < \frac{\pi}{2}$ *and* $|\rho| = 1$, *we have* $f \in K_\lambda(A, B, s, t)$ *if and only if*

$$\frac{1}{z}\left[f * \frac{z + \frac{(1+B\rho)+u_n(s,t)(1+\eta\rho)}{(1+B\rho)-u_n(s,t)(1+\eta\rho)}z^2}{(1-z)^3}\right] \neq 0,\ |z| < R,\ |\rho| = 1,$$

where $\eta = (A\cos\lambda + iB\sin\lambda)\, e^{-i\lambda}$ *and* $u_n(s,t) = \sum_{j=1}^{n-1} s^{n-j}t^{j-1}$.

Proof. $f \in K_\lambda(A, B, s, t)$ in $|z| < R \leq 1$ if and only if

$$\frac{e^{i\lambda}\dfrac{(s-t)(zf')'}{f'(sz)-f'(tz)} - isin\lambda}{cos\lambda} \neq \frac{1+A\rho}{1+B\rho}, \quad (|z| < R, |\rho| = 1, B\rho \neq -1).$$

This simplifies to

$$(s-t)\left(zf'(z)\right)'(1+B\rho) - (1+\eta\rho)\left(f'(sz) - f'(tz)\right) \neq 0. \tag{2.4}$$

with $\eta = (Acos\lambda + iBsin\lambda)e^{-i\lambda}$.

Now proceeding exactly as in Theorem 2.1 and replacing A by η the result follows. □

Theorem 2.4. *For* $|z| < R \leq 1$, λ *real with* $|\lambda| < \frac{\pi}{2}$ *and* $|\rho| = 1$, *we have* $f \in S^*_\lambda(A, B, s, t)$ *if and only if*

$$\frac{1}{z}\left[f * \frac{z + \frac{u_n(s,t)(1+\eta\rho)}{(1+B\rho)-u_n(s,t)(1+\eta\rho)}z^2}{(1-z)^2}\right] \neq 0,\ |z| < R,\ |\rho| = 1,$$

where $\eta = (A\cos\lambda + iB\sin\lambda)\, e^{-i\lambda}$ *and* $u_n(s,t) = \sum_{j=1}^{n-1} s^{n-j}t^{j-1}$.

Proof. The result follows from Theorem 2.3 in the same way as Theorem 2.2 followed from Theorem 2.1. □

As a corollary we can derive coefficient inequalities for the class $S^*_\lambda(A, B, s, t)$.

Corollary 2.2. *A function* $f \in \mathcal{A}$ *is in the class* $S^*_\lambda(A, B, s, t)$ *if and only if*

$$f(z) = 1 + \sum_{n=2}^{\infty} d_n z^{n-1} \neq 0,$$

where $d_n = \frac{(n-u_n(s,t))+(nB-\gamma u_n(s,t))\rho}{\rho(B-A)}a_n$ *and* $\gamma = (A\cos\lambda + iB\sin\lambda)\, e^{-i\lambda}$.

Proof. A function $f \in S^*_\lambda(A, B, s, t)$ if and only if

$$\frac{e^{i\lambda} \frac{(s-t)zf'(z)}{f(sz)-f(tz)} - i \sin \lambda}{\cos \lambda} \neq \frac{1 + A\rho}{1 + B\rho}.$$

That is, $(1 + B\rho)(s - t)(zf'(z)) - (1 + \gamma\rho)\left[f(sz) - f(tz)\right] \neq 0$. The rest of the proof follows as in Corollary 2.1. □

Remark 2.1. *When $s = 1$ and $t = 0$ we get Convolution conditions as in [3] for the classes $K(A, B)$, $S^*(A, B)$, $K_\lambda(A, B)$, $S^*_\lambda(A, B)$ and for $s = 1$ we get the results in [5].*

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PG DEPARTMENT OF MATHEMATICS
JSS COLLEGE OF ARTS, COMMERCE AND SCIENCE, OOTY ROAD
MYSORE - 570 025
INDIA
Email address: drshilpamaths@gmail.com