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STRONGLY MULTIPLICATIVE LABELING OF BUTTERFLY NETWORK

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ABSTRACT. A graph $G = (V(G), E(G))$ with p vertices is said to be strongly multiplicative if the vertices of G can be labeled with p distinct integers $1, 2, 3, \ldots, p$ such that the labels induced on the edges by the product of labels of the end vertices are all distinct [1]. In this paper we prove that the Butterfly network $BF(n)$ is strongly multiplicative for all positive integer $n = 4$.

1. INTRODUCTION

Butterfly networks are interconnection networks which play a vital role in memory parallel architecture. They have a good symmetric structure and they are biregular graphs [3].

In this paper, we prove that the Butterfly network $BF(n)$ is strongly multiplicative for all positive integer $n = 4$.

2. STRONGLY MULTIPLICATIVE LABELING OF BUTTERFLY NETWORK

Definition 2.1 (Generalized Butterfly Network $BF(n)$ [2])**.** *The set of nodes V of an n-dimensional Butterfly $BF(n)$ corresponds to the set of pairs $[w, i]$, where i is the dimension or level of a node $(0 \leq i \leq n)$ and w is an n-bit binary number that denotes the row of the node. Two nodes $[w, i]$ and $[w', i']$ are linked by an edge if and only if $i' = i + 1$ and either*

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i) w and w' are identical; or
ii) w and w' differ in precisely the i^{th} bit.

Theorem 2.1. *The Butterfly network $BF(n)$ is strongly multiplicative for $n = 4$.*

Proof. Let the vertex set of the butterfly network be for $1 \leq i \leq 2^{n-2}$,

$$V = \left\{v_{ij}/1 \leq j \leq 12 + 2^3 + 2^4 + ... + 2^n\right\}$$

and the edge set be $E = E_1 \cup E_2 \cup E_3 \cup E_4$, where

$$E_1 = \left\{e_i = (v_{ij}, v_{ij+1}) \cup (v_{i4}, v_{i1}) \cup (v_{i8}, v_{i5})/1 \leq i \leq 2^{n-2}, 1 \leq j \leq 8\right\}.$$

For $1 \leq i \leq 2^{n-2} = 4$, j-odd and $9 \leq j \leq 12$, we have

$$E_2 = \left\{\begin{array}{c} e_i = (v_{ij}, v_{i\,2^3+j}) \cup (v_{ij}, v_{i\,2^3+j+1})\,/, 1 \leq j \leq 4 \\ e_i = (v_{ij}, v_{i\,2^2+j}) \cup (v_{ij}, v_{i\,2^2+j+1})\,/, 5 \leq j \leq 8 \end{array}\right\}.$$

For $1 \leq i \leq 2^{n-2} = 4$, j-even and $13 \leq j \leq 20$, we have

$$E_3 = \left\{\begin{array}{c} e_i = (v_{ij}, v_{i+1\;12+j-1}) \cup (v_{ij}, v_{i+1\;12+j})\,/i-odd, 1 \leq j \leq 8 \\ e_i = (v_{ij}, v_{i\;12+j-1}) \cup (v_{ij}, v_{i\;12+j})\,/i-even, 1 \leq j \leq 8 \end{array}\right\}.$$

For $1 \leq i \leq 2^{n-2} = 4$ and $21 \leq j \leq 36$, we have

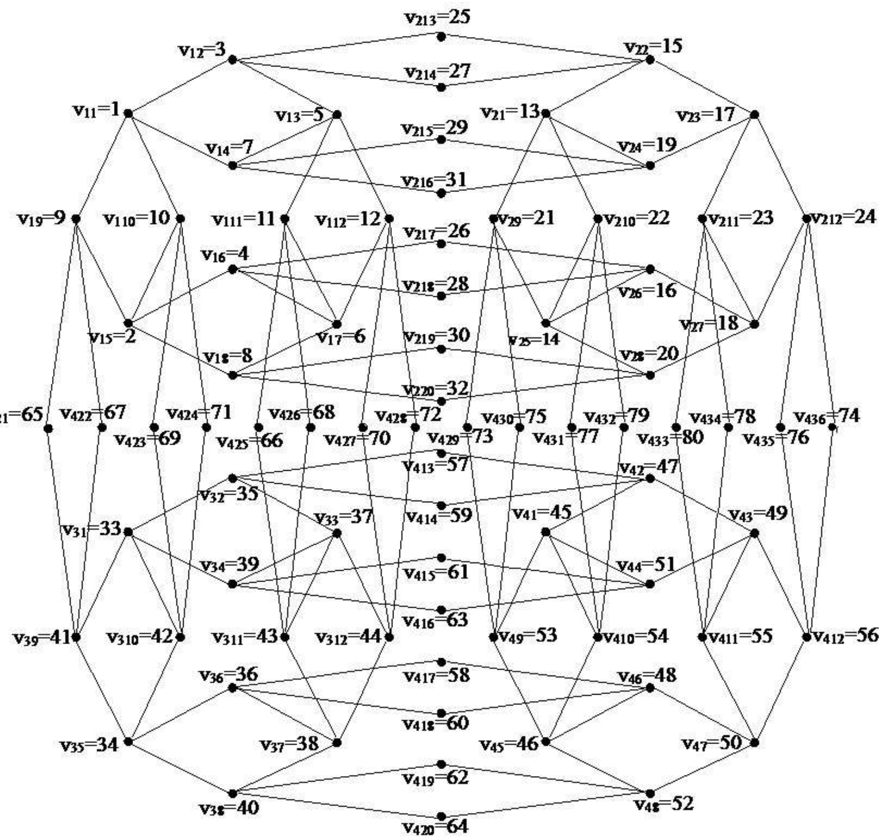
$$E_4 = \left\{\begin{array}{c} e_i = (v_{ij}, v_{4,\,2j+3}) \cup (v_{ij}, v_{4\,2j+4})\;\;/i-odd, 9 \leq j \leq 12 \\ e_i = (v_{ij}, v_{4\;8+2j+3}) \cup (v_{ij}, v_{4\;8+2j+4})\,/i-even, 9 \leq j \leq 12 \end{array}\right\}.$$

We claim that $BF(n)$ is strongly multiplicative for $n = 4$.

Define the vertex labeling of $BF(n)$ as $f : V \to N$ such that $1 \leq i \leq 2^{n-2} = 4$

$$f(v_{ij}) = \left\{\begin{array}{l} (i-1)12 + 2j - 1 + \frac{i-1}{2}\rfloor\, 2^3, 1 \leq j \leq 4 \\ (i-1)12 + 2(j-4) + \frac{i-1}{2}\rfloor\, 2^3, 5 \leq j \leq 8 \\ (i-1)12 + j + \left(\frac{i-1}{2}\right)\rfloor 2^3, 9 \leq j \leq 12 \\ (i-2)12 + 2j - 1 + \left(\frac{i-1}{2}\right)\rfloor 2^3, 13 \leq j \leq 16, i-even \\ (i-2)12 + 2(j-4) + \left(\frac{i-1}{2}\right)\rfloor 2^3, 17 \leq j \leq 20, i-even \\ (i-2)12 + 2(j-4) - 1 + \left(\frac{i-1}{2}\right)\rfloor 2^3, 21 \leq j \leq 24, i-multiple\ of\ 4 \\ (i-2)12 + 2(j-8) + \left(\frac{i-1}{2}\right)\rfloor 2^3, 25 \leq j \leq 28, i-multiple\ of\ 4 \\ (i-2)12 + 2(j-8) - 1 + \left(\frac{i-1}{2}\right)\rfloor 2^3, 29 \leq j \leq 32, i-multiple\ of\ 4 \\ (i-2)12 + j + 3(38-j) + \left(\frac{i-1}{2}\right)\rfloor 2^3, 33 \leq j \leq 36, i-multiple\ of\ 4 \end{array}\right\}$$

To prove all the edge labeling in E are distinct.



To prove that the labelings in edge set E_1 are distinct:
Define an edge induced function $g : E_1 \to N$ such that for all $e_{ij} \in E_1$,
$g(e_{ij}) = f(v_{ij})f(v_{ij+1})$, $1 \le i \le 2^{n-2}, 1 \le j \le 7$.
If e_{ij} and e_{pq} are distinct edges in E_1 then to prove $g(e_{ij}) \ne g(e_{pq})$.

Case 1: For $i \ne p, i, p \le 2, i = 1$ and $p = 2, 1 \le j, q \le 3$.
Assume that $g(e_{ij}) = g(e_{pq})$

$$g(v_{ij}, v_{i\,j+1}) = g(v_{pq}, v_{p\,q+1})$$
$$f(v_{ij})f(v_{i\,j+1}) = f(v_{pq})f(v_{p\,q+1})$$
$[2j-1][2j+1] = [12+2q-1][12+2q+1]$, $j = 6+q$, a contradiction for i.

For $i \ne p, i, p \ge 3, 1 \le j, q \le 3$.
Assume that $g(e_{ij}) = g(e_{pq})$

$$g(v_{ij}, v_{i\,j+1}) = g(v_{pq}, v_{p\,q+1})$$
$$f(v_{ij})f(v_{i\,j+1}) = f(v_{pq})f(v_{p\,q+1})$$
$$[(i-1)12+2j-1+8][(i-1)12+2j+1+8]$$
$$= [(p-1)12+2q-1+8][(p-1)12+2q+1+8]$$

$[12i+2j-5][12i+2j-3] = [12p+2q-5][12p+2q-3]$
$(x-5)(x-3) = (y-5)(y-3), x = 12i+2j$ and $y = 12p+2q$.
$x = 8-y \Rightarrow 12i = 8-(12p+2(q+j))$, a contradiction for i. Hence $g(e_{ij}) \neq g(e_{pq}), \forall\, 1 \leq j, q \leq 4$.

Case 2: For $i \neq p, i, p \leq 2, i = 1$and $p = 2, 5 \leq j, q \leq 7$.
Assume that $g(e_{ij}) = g(e_{pq})$
$$g(v_{ij}, v_{i\,j+1}) = g(v_{pq}, v_{p\,q+1})$$
$$f(v_{ij})f(v_{i\,j+1}) = f(v_{pq})f(v_{p\,q+1})$$
$[2(j-4)][2(j-3)] = [12+2(q-4)][12+2(q-3)]$
$\Rightarrow j = 1-q$, a contradiction for i.
For $i \neq p, i, p \geq 3, 5 \leq j, q \leq 7$.
Assume that $g(e_{ij}) = g(e_{pq})$
$$g(v_{ij}, v_{i\,j+1}) = g(v_{pq}, v_{p\,q+1})$$
$$f(v_{ij})f(v_{i\,j+1}) = f(v_{pq})f(v_{p\,q+1})$$
$[(i-1)12+2(j-4)+8][(i-1)12+2(j-3)+8]$
$= [(p-1)12+2(q-4)+8][(p-1)12+2(q-3)+8]$
$[12i+2j-12][12i+2j-10] = [12p+2q-12][12p+2q-10]$
$(x-12)(x-10) = (y-12)(y-10), x = 12i+2j$ and $y = 12p+2q$.
$x = 22-y \Rightarrow 12i = 22-(12p+2(q+j))$,
a contradiction for i. Hence $g(e_{ij}) \neq g(e_{pq}), \forall\, 5 \leq j, q \leq 7$.

Case 3: For $i \neq p, i, p \geq 3, 1 \leq j \leq 3, 5 \leq q \leq 7$.
Assume that $g(e_{ij}) = g(e_{pq})$
$$g(v_{ij}, v_{i\,j+1}) = g(v_{pq}, v_{p\,q+1})$$
$$f(v_{ij})f(v_{i\,j+1}) = f(v_{pq})f(v_{p\,q+1})$$
$[12i+2j-5][12i+2j-3] = [12p+2q-12][12p+2q-10]$
$(x-5)(x-3) = (y-10)(y-12), x = 12i+2j$ and $y = 12p+2q$.
$x = y-7 \Rightarrow 12i = 12p+2(q-j)-7$, a contradiction for i.
Hence $g(e_{ij}) \neq g(e_{pq}), \forall\, 1 \leq j \leq 4$.
Hence all the edge labeling in E_1 are distinct.
To prove that the labelings in edge set E_2 *are distinct*:
Define an edge induced function $g : E_2 \to N$ such that for all $e_{ij} \in E_2, 1 \leq i \leq 2^{n-2}$,for all $j - odd$ and $1 \leq j \leq 7$.
$g(e_{ij}) = f(v_{ij})f(v_{ij+2^3})/f(v_{ij})f(v_{ij+2^3+1})$, $1 \leq j \leq 4$.
$g(e_{ij}) = f(v_{ij})f(v_{ij+2^2})/f(v_{ij})f(v_{ij+2^2+1})$, $5 \leq j \leq 7$.

If e_{ij} and e_{pq} are distinct edges in E_2 then to prove $g(e_{ij}) \neq g(e_{pq})$.

Case 1: For $i \neq p, i, p \geq 3, 1 \leq j, q \leq 4$, where $j, q - odd.$
when $g(e_{ij}) = f(v_{ij})f(v_{ij+2^3})$, $1 \leq j \leq 4$.
Assume that $g(e_{ij}) = g(e_{pq})$ when $j = 1, q = 3$

$$g(v_{ij}, v_{i\,j+2^3}) = g(v_{pq}, v_{p\,q+2^3})$$
$$f(v_{ij})f(v_{i\,j+2^3}) = f(v_{pq})f(v_{p\,q+2^3})$$

$[(i-1)12 + 2j - 1 + 8][(i-1)12 + 2^3 + j + 8] = [(p-1)12 + 2q - 1 + 8][(p-1)12 + 2^3 + q + 8]$
$[12i + 2j - 5][12i + j + 4] = [12p + 2q - 5][12p + q + 4]$
$(x-3)(x+4) = (y+1)(y+7), x = 12i$ and $y = 12p$.
$x = -1 + \sqrt{(y+4)^2 + 7} \Rightarrow 12i = -1 + \sqrt{(12p+4)^2 + 7}$,
a contradiction for i.
when $g(e_{ij}) = f(v_{ij})f(v_{ij+2^3+1})$, $1 \leq j \leq 4$.
Assume that $g(e_{ij}) = g(e_{pq})$

$$g(v_{ij}, v_{i\,j+2^3+1}) = g(v_{pq}, v_{p\,q+2^3+1})$$
$$f(v_{ij})f(v_{i\,j+2^3+1}) = f(v_{pq})f(v_{p\,q+2^3+1})$$

$[(i-1)12 + 2j - 1 + 8][(i-1)12 + 2^3 + 1 + j + 8] = [(p-1)12 + 2q - 1 + 8][(p-1)12 + 2^3 + 1 + q + 8]$
$[12i + 2j - 5][12i + j + 5] = [12p + 2q - 5][12p + q + 5]$
$(x-3)(x+6) = (y+1)(y+8), j = 1, q = 3, x = 12i$ and $y = 12p$.
$x = -\frac{3}{2} + \sqrt{(y + \frac{9}{2})^2 + 8} \Rightarrow 12i = -\frac{3}{2} + \sqrt{(12p + \frac{9}{2})^2 + 8}$,
a contradiction for i. Hence $g(e_{ij}) \neq g(e_{pq}), \forall\ 1 \leq j, q \leq 4$.

Case 2: For $i \neq p, i, p \geq 3, 5 \leq j, q \leq 7$, $j, q - odd.$
Assume that $g(e_{ij}) = g(e_{pq})$,when $j = 5$ and $q = 7$

$$g(v_{ij}, v_{i\,j+2^2}) = g(v_{pq}, v_{p\,q+2^2})$$
$$f(v_{ij})f(v_{i\,j+2^2}) = f(v_{pq})f(v_{p\,q+2^2})$$

$[(i-1)12 + 2(j-4) + 8][(i-1)12 + 4 + j + 8] =$
$[(p-1)12 + 2(q-4) + 8][(p-1)12 + 4 + q + 8]$
$[12i + 2j - 12][12i + j] = [12p + 2q - 12][12p + q]$
$(x-2)(x+5) = (y+2)(y+7), j = 5$,$q = 7, x = 12i$ and $y = 12p$.
$\Rightarrow 12i = -\frac{3}{2} + \sqrt{(12p + \frac{9}{2})^2 - 4}$, a contradiction for i.
Assume that $g(e_{ij}) = g(e_{pq})$,when $j = 5$ and $q = 7$

$$g(v_{ij}, v_{i\,j+2^2+1}) = g(v_{pq}, v_{p\,q+2^2+1})$$
$$f(v_{ij})f(v_{i\,j+5}) = f(v_{pq})f(v_{p\,q+5})$$

$[(i-1)12+2(j-4)+8][(i-1)12+5+j+8]=$
$[(p-1)12+2(q-4)+8][(p-1)12+5+q+8]$
$[12i+2j-12][12i+j+1]=[12p+2q-12][12p+q+1]$
$(x-2)(x+6)=(y+2)(y+8), j=5, q=7, x=12i$ and $y=12p$.
$\Rightarrow 12i=-2+\sqrt{(12p+5)^2+7}$, a contradiction for i.
Hence $g(e_{ij})\neq g(e_{pq}), \forall\, 5\leq j,q\leq 7$.

Case 3: For $i\neq p, i,p\geq 3, 1\leq j\leq 4, 5\leq q\leq 7$.
Assume that $g(e_{ij})=g(e_{pq})$

$$g(v_{ij},v_{i\,j+2^3})=g(v_{pq},v_{p\,q+2^2+1})$$
$$f(v_{ij})f(v_{i\,j+2^3})=f(v_{pq})f(v_{p\,q+2^2+1})$$

$[12i+2j-5][12i+j+4]=[12p+2q-12][12p+q+1]$
$(x+1)(x+7)=(y-2)(y+6), j=3, q=5, x=12i$ and $y=12p$
$x=-4+\sqrt{(y+2)^2-7}\Rightarrow 12i=-4+\sqrt{(12p+2)^2-7}$,
a contradiction for i.
similarly other cases can be proved when
1.$j=1$and $p=7$,2. $j=1$and $p=5$ and 3. $j=3$and $p=7$.
Hence all the edge labeling in E_2 are distinct.
To prove that the labelings in edge set E_3 are distinct:
Define an edge induced function $g:E_3\rightarrow N$ such that for all $e_{ij}\in E_3$,
$1\leq i\leq 2^{n-2}$, $j-even\ 1\leq j\leq 9$.
$g(e_{ij})=f(v_{ij})f(v_{i+1\,12+j-1})/f(v_{ij})f(v_{i+1\,12+j})$,when $i-odd$.
$g(e_{ij})=f(v_{ij})f(v_{i\,12+j-1})/f(v_{ij})f(v_{i\,12+j})$,when $i-even$.
If e_{ij} and e_{pq} are distinct edges in E_3 then to prove $g(e_{ij})\neq g(e_{pq})$.

Case 1: For $i\neq p, i,p\geq 3$ and $odd, 1\leq j,q\leq 4$ and $j,q-even$.
Assume that $g(e_{ij})=g(e_{pq})$

$$g(v_{ij},v_{i+1,12+\,j-1})=g(v_{pq},v_{p+1,12+q-1})$$
$$f(v_{ij})f(v_{i+1,11+\,j})=f(v_{pq})f(v_{p+1,11+\,q})$$

$[(i-1)12+8+(2j-1)][(i+1-2)12+8+(2(j+11)-1)]=$
$[(p-1)12+8+(2q-1)][(p-1)12+8+(2(q+11)-1)]$
$[12i+2j-5][12i+2j+17]=[12p+2q-5][12p+2q+17]$
$(x-5)(x+17)=(y-5)(y+17), x=12i+2j$ and $y=12p+2q$.
$x=-y-12\Rightarrow 12i=-[12+2(j+q)+12p]$, a contradiction for i.
Hence $g(e_{ij})\neq g(e_{pq}), \forall\, 1\leq j,q\leq 4$.
For $i\neq p, i,p\geq 3$ and $odd, 5\leq j,q\leq 8$ and $j,q-even$.

Assume that $g(e_{ij}) = g(e_{pq})$

$$g(v_{ij}, v_{i+1,12+j-1}) = g(v_{pq}, v_{p+1,12+q-1})$$
$$f(v_{ij})f(v_{i+1,11+j}) = f(v_{pq})f(v_{p+1,11+q})$$
$$[(i-1)12+8+2(j-4)][(i+1-2)12+8+2(j+11-4)] =$$
$$[(p-1)12+8+2(q-4)][(p-1)12+8+2(q+11-4)]$$
$$[12i+2j-12][12i+2j+10] = [12p+2q-12][12p+2q+10]$$

$(x-12)(x+10) = (y-12)(y+10), x = 12i+2j$ and $y = 12p+2q$.

$x = 2-y \Rightarrow 12i = 2[1-(j+q)-6p]$, a contradiction for i.

Hence $g(e_{ij}) \neq g(e_{pq}), \forall\, 5 \leq j, q \leq 8$.

Case 2: For $i \neq p, i, p \geq 3$ and $even, 1 \leq j, q \leq 4$ and $j, q - even$.

Assume that $g(e_{ij}) = g(e_{pq})$

$$g(v_{ij}, v_{i,12+j-1}) = g(v_{pq}, v_{p,12+q-1})$$
$$f(v_{ij})f(v_{i,11+j}) = f(v_{pq})f(v_{p,11+q})$$
$$[(i-1)12+8+(2j-1)][(i-2)12+8+2(j+11)-1)] =$$
$$[(p-1)12+8+(2q-1)][(p-2)12+8+2(q+11)-1)]$$
$$[12i+2j-5][12i+2j+5] = [12p+2q-5][12p+2q+5]$$

$(x-5)(x+5) = (y-5)(y+5), x = 12i+2j$ and $y = 12p+2q$.

$x = y \Rightarrow 12i = [2(j+q)+12p]$, a contradiction for i.

Hence $g(e_{ij}) \neq g(e_{pq}), \forall\, 1 \leq j, q \leq 4$.

For $i \neq p, i, p \geq 3$ and $even, 5 \leq j, q \leq 8$ and $j, q - even$.

Assume that $g(e_{ij}) = g(e_{pq})$

$$g(v_{ij}, v_{i,12+j-1}) = g(v_{pq}, v_{p,12+q-1})$$
$$f(v_{ij})f(v_{i,11+j}) = f(v_{pq})f(v_{p,11+q})$$
$$[(i-1)12+8+2(j-4)][(i-2)12+8+2((j+11)-4)] =$$
$$[(p-1)12+8+2(q-4)][(p-2)12+8+2(q+11-4)]$$
$$[12i+2j-12][12i+2j-2] = [12p+2q-12][12p+2q-2]$$

$(x-12)(x-2) = (y-12)(y-2), x = 12i+2j$ and $y = 12p+2q$.

$x = 14-y \Rightarrow 12i = 14-[2(j+q)+12p]$, a contradiction for i.

Hence $g(e_{ij}) \neq g(e_{pq}), \forall\, 5 \leq j, q \leq 8$.

Case 3: For $1 \leq j \leq 4, 5 \leq q \leq 8, i \neq p \geq 3, i - odd$ and $p - even$.

Assume that $g(e_{ij}) = g(e_{pq})$,

$$g(v_{ij}, v_{i+1,12+j-1}) = g(v_{pq}, v_{p,12+q-1})$$
$$f(v_{ij})f(v_{i+1,11+j}) = f(v_{pq})f(v_{p,11+q})$$
$$[(i-1)12+8+(2j-1)][(i+1-2)12+8+(2(j+11)-1)] =$$

$[(p-1)12+8+2(q-4)][(p-2)12+8+2(q+11-4)]$
$[12i+2j-12][12i+2j-2]=[12p+2q-12][12p+2q-2]$
$(x-5)(x+17)=(y-12)(y-2), x=12i+2j$ and $y=12p+2q.$
$x+6=\sqrt{(y-7)^2+96} \Rightarrow 12i=-6-2j+\sqrt{(12p+2q-7)^2+96},$
a contradiction for i. Hence $g(e_{ij}) \neq g(e_{pq}), \forall\ 1 \leq j \leq 8.$
similar proof holds $g(e_{ij}) \neq g(e_{pq})$,when $j-even\ 1 \leq j \leq 9$
$g(e_{ij})=f(v_{ij})f(v_{i+1\ 12+j})$,when $i-odd, 1 \leq i \leq 2^{n-2}.$
$g(e_{ij})=f(v_{ij})f(v_{i\ 12+j})$,when $i-even, 1 \leq i \leq 2^{n-2}.$
Therefore all the edge labeling in E_3 are distinct.
To prove that the labelings in edge set E_4 are distinct:
Define an edge induced function $g: E_4 \rightarrow N$ such that for all
$e_{ij} \in E_4, 1 \leq i \leq 2^{n-2}, 9 \leq j \leq 12.$
$g(e_{ij})=f(v_{ij})f(v_{4\ 2j+3})/f(v_{ij})f(v_{4\ 2j+4})$,when $i-odd.$
$g(e_{ij})=f(v_{ij})f(v_{4\ 8+2j+3})/f(v_{ij})f(v_{4\ 8+2j+4})$,when $i-even.$
If e_{ij} and e_{pq} are distinct edges in E_4 then to prove $g(e_{ij}) \neq g(e_{pq}).$

Case 1: For $i \neq p, i, p \geq 3$ and $odd, 9 \leq j, q \leq 10.$
Assume that $g(e_{ij})=g(e_{pq})$ and $j=9$ and $q=10.$
$g(v_{ij}, v_{4,2j+3})=g(v_{pq}, v_{4,2q+3})$
$f(v_{ij})f(v_{4,2\ j+3})=f(v_{pq})f(v_{4,2q+3})$
$[(i-1)12+j+8][(4-2)12+2(2j+3-4)-1+8]=$
$[(p-1)12+q+8][2.12+2(2q+3-4)-1+8]$
$[12i+j-4][24+4j+5]=[12p+q-4][24+4q+5]$
$(x+5)65=(y+6)69, j=9,\ q=10. x=12i$ and $y=12p.$
$\Rightarrow 12i=\frac{69}{65}(12p+6)-5,$ a contradiction for $i.$
Hence $g(e_{ij}) \neq g(e_{pq}), \forall\ 9 \leq j \leq 10.$
For $i \neq p, i, p \geq 3$ and $odd, 11 \leq j, q \leq 12.$
Assume that $g(e_{ij})=g(e_{pq})$ and $j=11$ and $q=12.$
$g(v_{ij}, v_{4,2j+3})=g(v_{pq}, v_{4,2q+3})$
$f(v_{ij})f(v_{4,2\ j+3})=f(v_{pq})f(v_{4,2q+3})$
$[(i-1)12+j+8][(4-2)12+2(2j+3-8)+8]=$
$[(p-1)12+q+8][2.12+2(2q+3-8)+8]$
$[12i+j-4][4j+22]=[12p+q-4][4q+22]$
$(x+7)66=(y+8)70, j=11,p=12, x=12i$ and $y=12p.$
$\Rightarrow 12i=\frac{70}{66}(12p+8)-7,$ a contradiction for $i.$

Hence $g(e_{ij}) \neq g(e_{pq}), \forall\ 11 \leq j \leq 12.$

Case 2: For $i \neq p, i, p \geq 3$ and even, $9 \leq j, q \leq 10.$
Assume that $g(e_{ij}) = g(e_{pq})$ and $j = 9$ and $q = 10.$

$$g(v_{ij}, v_{4,8+2j+3}) = g(v_{pq}, v_{4,8+2q+3})$$
$$f(v_{ij})f(v_{4,2\,j+11}) = f(v_{pq})f(v_{4,2q+11})$$
$$[(i-1)12 + j + 8][2.12 + 2(2j + 11 - 8) - 1 + 8] =$$
$$[(p-1)12 + q + 8][2.12 + 2(2q + 11 - 8) - 1 + 8]$$
$$[12i + j - 4][37 + 4j] = [12p + q - 4][37 + 4q]$$
$$(x + 5)73 = (y + 6)77, j = 9,\ q = 10, x = 12i \text{ and } y = 12p.$$
$\Rightarrow 12i = \frac{77}{73}(12p + 6) - 5$, a contradiction for i.

Hence $g(e_{ij}) \neq g(e_{pq}), \forall\ 9 \leq j \leq 10.$
For $i \neq p, i, p \geq 3$ and even, $11 \leq j, q \leq 12.$
Assume that $g(e_{ij}) = g(e_{pq})$ and $j = 11$ and $q = 12.$

$$g(v_{ij}, v_{4,8+2j+3}) = g(v_{pq}, v_{4,8+2q+3})$$
$$f(v_{ij})f(v_{4,2\,j+11}) = f(v_{pq})f(v_{4,2q+11})$$
$$[(i-1)12 + j + 8][2.12 + 2j + 11 + 3(36 - (2j + 11) + 2) + 8] =$$
$$[(p-1)12 + q + 8][2.12 + 2q + 11 + 3(36 - (2q + 11) + 2) + 8]$$
$$[12i + j - 4][43 + 2j + 3(27 - 2j)] = [12p + q - 4][43 + 2q + 3(27 - 2q)]$$
$$[12i + 7]80 = [12p + 8]76, \text{when } j = 11, q = 12$$
$\Rightarrow 12i = \frac{74}{80}(12p + 8) - 7$, a contradiction for i.

Hence $g(e_{ij}) \neq g(e_{pq}), \forall\ 11 \leq j, q \leq 12.$

Case 3: For $i \neq p, i, p \geq 3, i - odd$ and $p - even.$
Assume that $g(e_{ij}) = g(e_{pq}), 9 \leq j \leq 10, 11 \leq q \leq 12.$

$$g(v_{ij}, v_{4,2j+3}) = g(v_{pq}, v_{4,8+2q+3})$$
$$f(v_{ij})f(v_{4,2\,j+3}) = f(v_{pq})f(v_{4,2q+11})$$
$$[(i-1)12 + j + 8][2.12 + 2(2j + 3 - 4) - 1 + 8] =$$
$$[(p-1)12 + q + 8][2.12 + 2q + 11 + 3(36 - (2q + 11) + 2) + 8]$$
$$[12i + j - 4][29 + 4j] = [12p + q - 4]74, j = 9 \text{ and } q = 12$$
$$[12i + 5]65 = [12p + 8]76$$
$\Rightarrow 12i = \frac{76}{65}(12p + 8) - 5$, a contradiction for i.

Hence $g(e_{ij}) \neq g(e_{pq}), \forall\ 9 \leq j \leq 10, 11 \leq q \leq 12.$
Similar proof holds to show $g(e_{ij}) \neq g(e_{pq}),$
$g(e_{ij}) = f(v_{ij})f(v_{4,2j+4})$, when $i - odd.$
$g(e_{ij}) = f(v_{ij})f(v_{4,8+2j+4})$, when $i - even.$

Therefore all the edge labeling in E_4 are distinct.

To prove that the labelings in edge set E_1 *and* E_2 *are distinct*:

If $e_{ij} \in E_1$ and $e_{pq} \in E_2$ are distinct edges, then to prove $g(e_{ij}) \neq g(e_{pq})$ and assuming $i, p \geq 3$.

For $i \neq p, i \leq 3, 1 \leq i, p \leq 2^{n-2}$, $1 \leq j, q \leq 8$ and $q - odd.$

Case 1: For $1 \leq j, q \leq 4$,Assume that $g(e_{ij}) = g(e_{pq})$.

$g(v_{ij}, v_{i,j+1}) = g(v_{pq}, v_{p,q+2^3})$

$f(v_{ij})f(v_{i,\,j+1}) = f(v_{pq})f(v_{p,2^3+q})$

$[(i-1)12 + (2j-1) + 8][(i-1)12 + 2j + 1 + 8] =$

$[(p-1)12 + (2q-1) + 8][(p-1)12 + q + 2^3 + 8]$

$[12i + 2j - 5][12i + 2j - 3] = [12p + 2q - 5][12p + q + 4]$

$(x-5)(x-3) = (y-3)(y+5)$, where $q = 1$, $x = 12i + 2j, y = 12p$

$x = 4 + \sqrt{(y+1)^2 - 15}, 12i = 4 - 2j + \sqrt{(12p+1)^2 - 15},$

a contradiction to i.

Assume that $g(e_{ij}) = g(e_{pq})$.

$g(v_{ij}, v_{i,j+1}) = g(v_{pq}, v_{p,q+2^3+1})$

$f(v_{ij})f(v_{i,\,j+1}) = f(v_{pq})f(v_{p,2^3+q+1})$

$[(i-1)12 + (2j-1) + 8][(i-1)12 + 2j + 1 + 8] =$

$[(p-1)12 + (2q-1) + 8][(p-1)12 + q + 2^3 + 9]$

$[12i + 2j - 5][12i + 2j - 3] = [12p + 2q - 5][12p + q + 5]$

$(x-5)(x-3) = (y-3)(y+6)$, where $q = 1$, $x = 12i + 2j, y = 12p$

$x = 4 + \sqrt{y^2 + 3y - 19} \Rightarrow 12i = 4 - 2j + \sqrt{y^2 + 3y - 19},$

a contradiction to i.Similar proof holds when $q = 3$.

Hence $g(e_{ij}) \neq g(e_{pq}), \forall\ 1 \leq j, q \leq 4$

Case 2: For $5 \leq j, q \leq 8, q - odd.$

Assume that $g(e_{ij}) = g(e_{pq})$.

$g(v_{ij}, v_{i,j+1}) = g(v_{pq}, v_{p,q+2^2})$

$f(v_{ij})f(v_{i\,j+1}) = f(v_{pq})f(v_{p,2^2+q})$

$[(i-1)12 + 2(\,j-4) + 8][(i-1)12 + 2(j-3) + 8] =$

$[(p-1)12 + 2(q-4) + 8][(p-1)12 + q + 2^2 + 8]$

$(12i + 2j - 12)(12i + 2j - 10) = (12p + 2q - 12)(12p + q)$

$(x-12)(x-10) = (y-2)(y+5), q = 5, x = 12i + 2j$ and $y = 12p.$

$x = 11 + \sqrt{y^2 + 3y - 9}, \Rightarrow 12i = -2j + 11 + \sqrt{y^2 + 3y - 9},$

a contradiction for i.similar proof holds when $q = 7$.

Assume that $g(e_{ij}) = g(e_{pq})$.

$g(v_{ij}, v_{i,j+1}) = g(v_{pq}, v_{p,q+2^2+1})$

$f(v_{ij})f(v_{i\,j+1}) = f(v_{pq})f(v_{p,2^2+q+1})$

$[(i-1)12 + 2(\,j-4) + 8][(i-1)12 + 2(j-3) + 8] =$

$[(p-1)12 + 2(q-4) + 8][(p-1)12 + q + 2^2 + 9]$

$(12i + 2j - 12)(12i + 2j - 10) = (12p + 2q - 12)(12p + q + 1)$

$(x-12)(x-10) = (y-2)(y+6), q = 5, x = 12i + 2j$ and $y = 12p$.

$x = 11 + \sqrt{y^2 + 4y - 13}, \Rightarrow 12i = -2j + 11 + \sqrt{y^2 + 3y - 13},$

a contradiction for i.similar proof holds when $q = 7$.

Case 3: For $5 \le q \le 8, q - odd$ and $1 \le j \le 4$.

Assume that $g(e_{ij}) = g(e_{pq})$.

$f(v_{ij})f(v_{i\,j+1}) = f(v_{pq})f(v_{p,2^2+q})$

$[(i-1)12 + 2\,j - 1 + 8][(i-1)12 + 2j + 1 + 8] =$

$[(p-1)12 + 2(q-4) + 8][(p-1)12 + q + 2^2 + 8]$

$[12i + 2j - 5][12i + 2j - 3] = [12p + 2q - 12][12p + q]$

$(x-5)(x-3) = (y-2)(y+5), x = 12i + 2j, y = 12p$ and $q = 5$.

$x = 4 + \sqrt{y^2 + 3y - 9} \Rightarrow 12i = -2j + 4 + \sqrt{y^2 + 3y - 9},$

a contradiction to i. Hence $g(e_{ij}) \neq g(e_{pq}), \forall\, 1 \le j \le 4$.

Thus all the edge labeling in E_1 and E_2 are distinct.

To prove that the labelings in edge set E_2 and E_3 are distinct:

If $e_{ij} \in E_2$ and $e_{pq} \in E_3$ are distinct edges, then to prove $g(e_{ij}) \neq g(e_{pq})$.

For $i \neq p, 1 \le i, p \le 2^{n-2}$, $1 \le j, q \le 8$ and $q - odd$.

Case 1: For $1 \le j, q \le 4, j, p - odd$ and $q - even$ the following cases

Assume that $g(v_{ij}, v_{i,2^3+j}) = g(v_{pq}, v_{p+1\;12+q-1})$

$f(v_{ij})f(v_{i\,j+2^3}) = f(v_{pq})f(v_{p+1,12+q-1})$

$[(i-1)12 + 2\,j - 1 + 8][(i-1)12 + 2^3 + j + 8] =$

$[(p-1)12 + 2q - 1 + 8][(p-1)12 + 2(11+q) - 1 + 8]$

$[12i + 2j - 5][12i + j + 4] = [12p + 2q - 5][12p + 2q + 17]$

$(x+1)(x+7) = (y-5)(y+17), x = 12i, y = 12p + 2q$ and $j = 3$.

$x = -4 + \sqrt{(y+6)^2 - 112} \Rightarrow 12i = -4 + \sqrt{(12p + 2q + 6)^2 - 112},$

a contradiction to i.

Assume that $g(v_{ij}, v_{i,2^3+j+1}) = g(v_{pq}, v_{p+1\;12+q-1})$

$f(v_{ij})f(v_{i,\,j+2^3+1}) = f(v_{pq})f(v_{p+1,12+q-1})$

$[(i-1)12 + 2\,j - 1 + 8][(i-1)12 + 2^3 + j + 1 + 8] =$

$[(p-1)12+2q-1+8][(p-1)12+2(11+q)-1+8]$
$[12i+2j-5][12i+j+5]=[12p+2q-5][12p+2q+17]$
$(x+1)(x+8)=(y-5)(y+17), x=12i, y=12p+2q$ and $j=3$.
$y=-6+\sqrt{x^2+9x+129} \Rightarrow 12p=-2q-6+\sqrt{x^2+9x+129}$,
a contradiction to p.
Assume that $g(v_{ij}, v_{i,2^3+j}) = \ g(v_{pq}, v_{p+1\ 12+q})$
$$f(v_{ij})f(v_{i\ j+2^3}) = f(v_{pq})f(v_{p+1,12+q})$$
$[(i-1)12+2\ j-1+8][(i-1)12+2^3+j+8]=$
$[(p-1)12+2q-1+8][(p-1)12+2(12+q)-1+8]$
$[12i+2j-5][12i+j+4]=[12p+2q-5][12p+2q+19]$
$(x+1)(x+7)=(y-5)(y+19), x=12i, y=12p+2q$ and $j=3$.
$x=-4+\sqrt{(y+7)^2-135} \Rightarrow 12i=-4++\sqrt{(y+7)^2-135}$,
a contradiction to i.
Assume that $g(v_{ij}, v_{i,2^3+j+1}) = \ g(v_{pq}, v_{p+1\ 12+q})$
$$f(v_{ij})f(v_{i\ j+2^3+1}) = f(v_{pq})f(v_{p+1,12+q})$$
$[(i-1)12+2\ j-1+8][(i-1)12+2^3+j+1+8]=$
$[(p-1)12+2q-1+8][(p-1)12+2(12+q)-1+8]$
$[12i+2j-5][12i+j+5]=[12p+2q-5][12p+2q+19]$
$(x+1)(x+8)=(y-5)(y+19), x=12i, y=12p+2q$ and $j=3$.
$y=-7+\sqrt{x^2+9x+152}, 12p=-7-2q+\sqrt{x^2+9x+152}$,
a contradiction to p. Hence $g(e_{ij}) \neq g(e_{pq}), \forall\ 1 \leq j \leq 4$,when $p-$ odd.

Case 2: For $1 \leq i \leq 2^{n-2}, 1 \leq j, q \leq 4, j-odd, p-even$ and $q-even.$
Assume that $g(v_{ij}, v_{i,2^3+j}) = \ g(v_{pq}, v_{p\ 12+q-1})$
$$f(v_{ij})f(v_{i\ j+2^3}) = f(v_{pq})f(v_{p,12+q-1})$$
$[(i-1)12+2\ j-1+8][(i-1)12+2^3+j+8]=$
$[(p-1)12+2q-1+8][(p-2)12+2(11+q)-1+8]$
$[12i+2j-5][12i+j+4]=[12p+2q-5][12p+2q+5]$
$(x+1)(x+7)=(y-5)(y+5), x=12i, y=12p+2q$ and $j=3$.
$y=\sqrt{(x+4)^2+16} \Rightarrow 12p=-2q+\sqrt{(12i+4)^2+16}$,
a contradiction to p.
Assume that $g(v_{ij}, v_{i,2^3+j+1}) = \ g(v_{pq}, v_{p\ 12+q-1})$
$$f(v_{ij})f(v_{i,\ j+2^3+1}) = f(v_{pq})f(v_{p,12+q-1})$$
$[(i-1)12+2\ j-1+8][(i-1)12+2^3+j+1+8]=$
$[(p-1)12+2q-1+8][(p-2)12+2(11+q)-1+8]$

$[12i+2j-5][12i+j+5] = [12p+2q-5][12p+2q+5]$
$(x+1)(x+8) = (y-5)(y+5), x = 12i, y = 12p+2q$ and $j = 3$.
$y = \sqrt{x^2+9x+33} \Rightarrow 12p = -2q + \sqrt{x^2+9x+33}$,
a contradiction to p.
Assume that $g(v_{ij}, v_{i,2^3+j}) = \quad g(v_{pq}, v_{p\ 12+q})$
$f(v_{ij})f(v_{i,\ j+2^3}) = f(v_{pq})f(v_{p,12+q})$
$[(i-1)12+2\ j-1+8][(i-1)12+2^3+j+8] =$
$[(p-1)12+2q-1+8][(p-2)12+2(12+q)-1+8]$
$[12i+2j-5][12i+j+4] = [12p+2q-5][12p+2q+7]$
$(x+1)(x+7) = (y-5)(y+7), x = 12i, y = 12p+2q$ and $j = 3$.
$x = -4 + \sqrt{(y+1)^2-27} \Rightarrow 12i = -4 + \sqrt{(y+1)^2-27}$,
a contradiction to i.
Assume that $g(v_{ij}, v_{i,2^3+j+1}) = \quad g(v_{pq}, v_{p\ 12+q})$
$f(v_{ij})f(v_{i\ j+2^3+1}) = f(v_{pq})f(v_{p,12+q})$
$[(i-1)12+2\ j-1+8][(i-1)12+2^3+j+1+8] =$
$[(p-1)12+2q-1+8][(p-2)12+2(12+q)-1+8]$
$[12i+2j-5][12i+j+5] = [12p+2q-5][12p+2q+7]$
$(x+1)(x+8) = (y-5)(y+7), x = 12i, y = 12p+2q$ and $j = 3$.
$y = -1 + \sqrt{x^2+9x+44}, 12p = -1 + 2q + \sqrt{x^2+9x+44}$,
a contradiction to p.
Hence $g(e_{ij}) \neq g(e_{pq}), \forall\ 1 \leq j \leq 4$,when $p - even$.

Case 3: For $5 \leq j, q \leq 8, j, p - odd$ and $q - even$.
Assume that $g(v_{ij}, v_{i,2^2+j}) = \quad g(v_{pq}, v_{p+1\ 12+q-1})$
$f(v_{ij})f(v_{i\ j+2^2}) = f(v_{pq})f(v_{p+1,12+q-1})$
$[(i-1)12+2(j-4)+8][(i-1)12+2^2+j+8] =$
$[(p-1)12+2(q-4)+8][(p-1)12+2(12+q-4)+8]$
$[12i+2j-12][12i+j] = [12p+2q-12][12p+2q+12]$
$(x-2)(x+5) = (y-12)(y+12), x = 12i, y = 12p+2q$ and $j = 3$.
$y = -1 + \sqrt{x^2+3x+134} \Rightarrow 12p = -1 - 2q + \sqrt{x^2+3x+134}$,
a contradiction to p.
Assume that $g(v_{ij}, v_{i,2^2+j}) = \quad g(v_{pq}, v_{p+1,12+q})$
$f(v_{ij})f(v_{i\ j+2^2}) = f(v_{pq})f(v_{p+1,12+q})$
$[(i-1)12+2(j-4)+8][(i-1)12+2^2+j+8] =$
$[(p-1)12+2(q-4)+8][(p-1)12+2(12+q-4)+8]$

$[12i + 2j - 12][12i + j] = [12p + 2q - 12][12p + 2q + 12]$
$(x - 2)(x + 5) = (y - 12)(y + 12), x = 12i, y = 12p + 2q$ and $j = 5$.
$y = \sqrt{x^2 + 3x + 134} \Rightarrow 12p = -2q + \sqrt{x^2 + 3x + 134}$,
a contradiction to p.
Assume that $g(e_{ij}) = g(e_{pq})$
$g(v_{ij}, v_{i,2^2+j+1}) = \quad g(v_{pq}, v_{p+1\ 12+q-1})$
$f(v_{ij})f(v_{i\ j+2^2+1}) = f(v_{pq})f(v_{p+1,12+q-1})$
$[(i - 1)12 + 2(j - 4) + 8][(i - 1)12 + 2^2 + j + 1 + 8] =$
$[(p - 1)12 + 2(q - 4) + 8][(p - 1)12 + 2(12 + q - 1 - 4) + 8]$
$[12i + 2j - 12][12i + j + 1] = [12p + 2q - 12][12p + 2q + 10]$
$(x - 2)(x + 6) = (y - 12)(y + 10), x = 12i, y = 12p + 2q$ and $j = 5$.
$y = 1 + \sqrt{x^2 + 4x + 109} \Rightarrow 12p = 1 - 2q + \sqrt{x^2 + 3x + 109}$,
a contradiction to p.
Assume that $g(e_{ij}) = g(e_{pq})$
$g(v_{ij}, v_{i,2^2+j+1}) = \quad g(v_{pq}, v_{p+1\ 12+q})$
$f(v_{ij})f(v_{i\ j+2^2+1}) = f(v_{pq})f(v_{p+1,12+q})$
$[(i - 1)12 + 2(j - 4) + 8][(i - 1)12 + 2^2 + j + 1 + 8] =$
$[(p - 1)12 + 2(q - 4) + 8][(p - 1)12 + 2(12 + q - 4) + 8]$
$[12i + 2j - 12][12i + j + 1] = [12p + 2q - 12][12p + 2q + 12]$
$(x - 2)(x + 6) = (y - 12)(y + 12), x = 12i, y = 12p + 2q$ and $j = 5$.
$y = \sqrt{x^2 + 4x + 132} \Rightarrow 12p = -2q + \sqrt{x^2 + 4x + 132}$,
a contradiction to p.
Hence $g(e_{ij}) \neq g(e_{pq}), \forall\ 5 \leq j \leq 8$,when $p - odd$.
Similarly we can prove the case when $1 \leq i \leq 2^{n-2}, 5 \leq j, q \leq 8, j - odd, p - even$ and $q - even$.
Hence $g(e_{ij}) \neq g(e_{pq}), \forall\ 1 \leq j \leq 8$ and $p - even$.
Therefore all the edge labeling in E_2 and E_3 are distinct.
To prove that the labelings in edge set E_3 and E_4 are distinct:
If $e_{ij} \in E_3$ and $e_{pq} \in E_4$ are distinct edges, then to prove $g(e_{ij}) \neq g(e_{pq})$.

Case 1: For $i \neq p, i, p - odd, 1 \leq j \leq 8, j -$ even and $9 \leq q \leq 12$.
For $1 \leq j \leq 4$ and $9 \leq q \leq 10$.
Assume that $g(v_{ij}, v_{i+1,12+j-1}) = \quad g(v_{pq}, v_{4,\ 2q+3})$
$f(v_{ij})f(v_{i+1,\ j+11}) = f(v_{pq})f(v_{42q+3})$
$[(i - 1)12 + 2j - 1 + 8][(i - 1)12 + 2(11 + j) - 1 + 8] =$

$[(p-1)12+q+8][(4-2)12+2((2q+3)-4)-1+8]$
$[12i+2j+17][12i+2j-5]=[12p+q-4][4q+29]$
$(x-5)(x+17)=(32+q)(4q+29), x=12i+2j$ and $p=3$.
$x=-6+\sqrt{4q^2+157q+1049} \Rightarrow 12i=-6-2j+\sqrt{4q^2+157q+1049}$,
a contradiction to i.
For $1 \leq j \leq 4$ and $11 \leq q \leq 12$.
Assume that $g(v_{ij}, v_{i+1,12+j-1}) = \; g(v_{pq}, v_{4\;2q+3})$
$f(v_{ij})f(v_{i+1,\,j+11}) = f(v_{pq})f(v_{42q+3})$
$[(i-1)12+2j-1+8][(i-1)12+2(11+j)-1+8] =$
$[(p-1)12+q+8][(4-2)12+2((2q+3)-8)+8]$
$[12i+2j+17][12i+2j-5]=[12p+q-4][4q+22]$
$(x-5)(x+17)=(32+q)(4q+22), x=12i+2j$ and $p=3$.
$x=-6+\sqrt{4q^2+150q+825} \Rightarrow 12i=-6-2j+\sqrt{4q^2+150q+825}$,
a contradiction to i.
For $5 \leq j \leq 8$ and $9 \leq q \leq 10$.
Assume that $g(v_{ij}, v_{i+1,12+j-1}) = \; g(v_{pq}, v_{4\;2q+3})$
$f(v_{ij})f(v_{i+1,\,j+11}) = f(v_{pq})f(v_{42q+3})$
$[(i-1)12+2(j-4)+8][(i-1)12+2(11+j-4)+8] =$
$[(p-1)12+q+8][(4-2)12+2((2q+3)-4)-1+8]$
$[12i+2j-12][12i+2j+10]=[12p+q-4][4q+29]$
$(x-12)(x+10)=(32+q)(4q+29), x=12i+2j$ and $p=3$.
$x=1+\sqrt{4q^2+157q+1049} \Rightarrow 12i=1-2j+\sqrt{4q^2+157q+1049}$,
a contradiction to i.
For $5 \leq j \leq 8$ and $11 \leq q \leq 12$.
Assume that $g(v_{ij}, v_{i+1,12+j-1}) = \; g(v_{pq}, v_{4,\,2q+3})$
$f(v_{ij})f(v_{i+1,\,j+11}) = f(v_{pq})f(v_{4,2q+3})$
$[(i-1)12+2(j-4)+8][(i-1)12+2(11+j-4)+8] =$
$[(p-1)12+q+8][(4-2)12+2((2q+3)-8)+8]$
$[12i+2j-12][12i+2j+10]=[12p+q-4][4q+22]$
$(x-12)(x+10)=(32+q)(4q+22), x=12i+2j$ and $p=3$.
$x=1+\sqrt{4q^2+150q+825} \Rightarrow 12i=1-2j+\sqrt{4q^2+150q+825}$,
a contradiction to i. Hence $g(e_{ij}) \neq g(e_{pq}), \forall\, 1 \leq j \leq 8$.

Case 2: For $i \neq p, i, p - odd, 1 \leq j \leq 8, j -$ even and $9 \leq q \leq 12$.
For $1 \leq j \leq 4$ and $9 \leq q \leq 10$.

Assume that $g(v_{ij}, v_{i+1,12+j}) = \quad g(v_{pq}, v_{4,\, 2q+4})$

$$f(v_{ij})f(v_{i+1\; j+12}) = f(v_{pq})f(v_{4,2q+4})$$

$[(i-1)12+2j-1+8][(i-1)12+2(12+j)-1+8] =$

$[(p-1)12+q+8][(4-2)12+2((2q+4)-4)-1+8]$

$[12i+2j-5][12i+2j+19] = [12p+q-4][4q+31]$

$(x-5)(x+19) = (32+q)(4q+31), x = 12i+2j$ and $p = 3$.

$x = -7+\sqrt{4q^2+159q+1136} \Rightarrow 12i = -7-2j+\sqrt{4q^2+157q+1136}$,

a contradiction to i.

For $1 \leq j \leq 4$ and $11 \leq q \leq 12$.

Assume that $g(v_{ij}, v_{i+1,12+j}) = \quad g(v_{pq}, v_{4\; 2q+4})$

$$f(v_{ij})f(v_{i+1,\; j+12}) = f(v_{pq})f(v_{42q+4})$$

$[(i-1)12+2j-1+8][(i-1)12+2(12+j)-1+8] =$

$[(p-1)12+q+8][(4-2)12+2((2q+4)-8)+8]$

$[12i+2j-5][12i+2j+19] = [12p+q-4][4q+24]$

$(x-5)(x+19) = (32+q)(4q+24), x = 12i+2j$ and $p = 3$.

$x = -7+\sqrt{4q^2+152q+912} \Rightarrow 12i = -7-2j+\sqrt{4q^2+152q+912}$,

a contradiction to i.

For $5 \leq j \leq 8$ and $9 \leq q \leq 10$.

Assume that $g(v_{ij}, v_{i+1,12+j}) = \quad g(v_{pq}, v_{4\; 2q+4})$

$$f(v_{ij})f(v_{i+1\; j+12}) = f(v_{pq})f(v_{42q+4})$$

$[(i-1)12+2(j-4)+8][(i-1)12+2(12+j-4)+8] =$

$[(p-1)12+q+8][(4-2)12+2((2q+4)-4)-1+8]$

$[12i+2j-12][12i+2j+12] = [12p+q-4][4q+31]$

$(x-12)(x+12) = (32+q)(4q+31), x = 12i+2j$ and $p = 3$.

$x = \sqrt{4q^2+159q+1136} \Rightarrow 12i = -2j+\sqrt{4q^2+159q+1136}$,

a contradiction to i.

For $5 \leq j \leq 8$ and $11 \leq q \leq 12$.

Assume that $g(v_{ij}, v_{i+1,12+j}) = \quad g(v_{pq}, v_{4,\, 2q+4})$

$$f(v_{ij})f(v_{i+1,\; j+12}) = f(v_{pq})f(v_{4,2q+4})$$

$[(i-1)12+2(j-4)+8][(i-1)12+2(12+j-4)+8] =$

$[(p-1)12+q+8][(4-2)12+2((2q+4)-8)+8]$

$[12i+2j-12][12i+2j+12] = [12p+q-4][4q+24]$

$(x-12)(x+12) = (32+q)(4q+24), x = 12i+2j$ and $p = 3$.

$x = \sqrt{4q^2+152q+912} \Rightarrow 12i = -2j+\sqrt{4q^2+152q+912}$,

a contradiction to i.

Case 3: For $i \neq p, i, p - odd, 1 \leq j \leq 8, j -$ even and $9 \leq q \leq 12$.
For $1 \leq j \leq 4$ and $9 \leq q \leq 10$.
Assume that $g(v_{ij}, v_{i+1,12+j-1}) = \quad g(v_{pq}, v_{4,\,2q+4})$
$$f(v_{ij})f(v_{i+1,\,j+11}) = f(v_{pq})f(v_{4,2q+4})$$
$[(i-1)12+2j-1+8][(i-1)12+2(11+j)-1+8] =$
$[(p-1)12+q+8][(4-2)12+2((2q+4)-4)-1+8]$
$[12i+2j+17][12i+2j-5] = [12p+q-4][4q+31]$
$(x-5)(x+17) = (32+q)(4q+31), x = 12i+2j$ and $p = 3$.
$x = -6+\sqrt{4q^2+159q+1113} \Rightarrow 12i = -2j-6+\sqrt{4q^2+159q+1113}$,
a contradiction to i.
For $1 \leq j \leq 4$ and $11 \leq q \leq 12$.
Assume that $g(v_{ij}, v_{i+1,12+j-1}) = \quad g(v_{pq}, v_{4,\,2q+4})$
$$f(v_{ij})f(v_{i+1,\,j+11}) = f(v_{pq})f(v_{4,2q+4})$$
$[(i-1)12+2j-1+8][(i-1)12+2(11+j)-1+8] =$
$[(p-1)12+q+8][(4-2)12+2((2q+4)-8)+8]$
$[12i+2j+17][12i+2j-5] = [12p+q-4][4q+24]$
$(x+17)(x-5) = (32+q)(4q+24), x = 12i+2j$ and $p = 3$.
$x = -6+\sqrt{4q^2+152q+889} \Rightarrow 12i = -2j-6+\sqrt{4q^2+152q+889}$,
a contradiction to i.
For $5 \leq j \leq 8$ and $9 \leq q \leq 10$.
Assume that $g(v_{ij}, v_{i+1,12+j-1}) = \quad g(v_{pq}, v_{4\,2q+4})$
$$f(v_{ij})f(v_{i+1,\,j+11}) = f(v_{pq})f(v_{42q+4})$$
$[(i-1)12+2(j-4)+8][(i-1)12+2(11+j-4)+8] =$
$[(p-1)12+q+8][(4-2)12+2((2q+4)-4)-1+8]$
$[12i+2j-12][12i+2j+10] = [12p+q-4][4q+31]$
$(x+12)(x+10) = (32+q)(4q+31), x = 12i+2j$ and $p = 3$.
$x = 1+\sqrt{4q^2+159q+1113} \Rightarrow 12i = -2j+1+\sqrt{4q^2+159q+1113}$,
a contradiction to i.
For $5 \leq j \leq 8$ and $11 \leq q \leq 12$.
Assume that $g(v_{ij}, v_{i+1,12+j-1}) = \quad g(v_{pq}, v_{4,\,2q+4})$
$$f(v_{ij})f(v_{i+1,\,j+11}) = f(v_{pq})f(v_{4,2q+4})$$
$[(i-1)12+2(j-4)+8][(i-1)12+2(11+j-4)+8] =$
$[(p-1)12+q+8][(4-2)12+2((2q+4)-8)+8]$
$[12i+2j-12][12i+2j+10] = [12p+q-4][4q+24]$
$(x-12)(x+10) = (32+q)(4q+24), x = 12i+2j$ and $p = 3$.

$x = 1 + \sqrt{4q^2 + 152q + 889} \Rightarrow 12i = -2j + 1 + \sqrt{4q^2 + 152q + 889}$,
a contradiction to i. Hence $g(e_{ij}) \neq g(e_{pq}), \forall\ 5 \leq j \leq 8$.
Hence $g(e_{ij}) \neq g(e_{pq}), \forall\ 1 \leq j \leq 8$ and $i, p - odd$.
Similar proof holds when i and p even and $i-$ odd and $p-$ even.
Therefore all the edge labeling in E_3 and E_4 are distinct.
Thus all edge labeling in E_1, E_2, E_3, E_4, E_1 and E_2, E_2 and E_3,andE_3 and E_4 are distinct. Hence Butterfly network $BF(n = 4)$is strongly multiplicative. □

3. Conclusion

In this paper we have proved that the Butterfly graph $BF(n = 4)$ is strongly multiplicative.Finding strongly multiplicative labeling for other interconnection networks is quite challenging.

References

[1] B.D. Acharya, K.A. Germina, V. Ajitha: *Multiplicatively Indexable Graphs, in Labeling of Discrete Structures and Applications*, Narosa Publishing House, New Delhi, 29-40, 2008.

[2] I. Rajasingh, B. Rajan, S. Tresa Arockiamary: *Total edge irregularity strength of Butterfly networks,* International Journal of Computer Applications, **49**(3)(2012), 1–09.

[3] K. Karunagaran, P. Sumathi, V. Nandakumar: *Theory and Application of Butterfly Network in Graph Theory*, International Journal of Innovtive Research in Science, Engineering and Technology **14**(6) (2015), 1–10.

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