ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **9** (2020), no.8, 5335–5342
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.8.4

OBSERVATION ON SUMS OF POWERS OF INTEGERS DIVISIBLE BY $(2K^2 - 1)$

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ABSTRACT. Sum of powers of integers has been the subject of research for many years. Many new integer sequences are increasing recently. Gulliver (2010) considered the sum of powers of odd integers of the forms $2k-1$ and obtained a simple derivation of some well-known sequences as wellas construction of many new sequences. Several authors have been continued the observation of Guliver recently. This study futhered observe the sum of powers of odd integers of the form $2k^2-1$ which were unknown before. We obtain a simple derivation of some well known sequences as well as construction of many new sequences. We also derive several properties of divisibility of the sequences.

1. INTRODUCTION

Various branches of mathematics discuss integers and many books discuss rows of integers. But in reality, there are still many lines of integers that are still not listed in these books. In 1973, the various rows of unregistered integers were collected by Neil J. A. Sloane [1] in a book called A Handbook of Integer Sequences, which contained 2372 rows of numbers.

After many years the ranks of new integers increased, so that in 1995, [1], he Encyclopedia of Integer Sequences, which contained 5487 sequences of numbers and in 1996 the website of The On-Line of Integer Sequences was launched.

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2010 *Mathematics Subject Classification.* 11Y99.

Key words and phrases. Integer sequences, Divisibility.

Until 2014, there were more than 250000 rows of numbers listed on the website of The On-Line of Integer Sequences.

Gulliver [2] constructs several rows of new numbers by reviewing rowsand the property of its divisibility:

$$\sum_{k=1}^{n}(2k-1)^m ,$$

by looking at the sequences

$$\{\sum_{k=1}^{n}(2k-1)^m\}_{m=1}^{n} .$$

Gulliver reconstructed several known sequences in an easy manner, as well as constructed many new sequences. For example, for $n=2$ and $n=3$ in the sequence formed by $\{\sum_{k=1}^{2}(2k-1)^m\}_{m=1}^{n}$ and $\{\sum_{k=1}^{3}(3k-1)^m\}_{m=1}^{n}$ are 034472 and 074507 in The Online Encyclopedia of Integer Sequences, [3], respectively. Suwarno [4] constructs the sequence of numbers formed by $\sum_{i=1}^{n}(pi-1)^m$ for $p=3,4$ and 5 which can be seen in [2]. Iskandar [5] constructs a line of new integers formed by $\sum_{i=1}^{n}(pi-1)^m$ for $p=6$ and 7. Suprijanto [6] contructs a line of new integers formed by $\sum_{k=1}^{n}(3k-1)^m$. This study aims to construct a series of integers formed based on the rule

$$\sum_{k=1}^{n}(2k^2-1)^m ,$$

where $m,n\in N$ and compare the ranks with those in The On-Line Encyclopedia of Integer Sequences.

By considering sequences of the above form, we succeed to reconstruct several known sequences. We also constructed many new sequences which were unknown to exist in [2] before. We follow the method and observation introduced by Gulliver in [3].

2. Results

In this section we consider the sum of the first n of m-th powers of $2k^2-1$ as follows

$$\sum_{k=1}^{n}(2k^2-1)^m. \tag{2.1}$$

We divide our observation into two cases: n fixed and m fixed.

2.1. **Case** 1: n **fixed.** For $n = 1$, we have $\{1^m\}$, the sequence $A000012$ in [3]. For $n = 2$ to 10, the values are

8	50	344	2402	16808	117650	...
25	339	5257	85923	1436665	24255219	...
56	1300	35048	1009444	30065816	911758900	...
105	3701	152697	6774245	312541065	14753046101	...
176	8742	510608	32185926	2116770116	14285330022	...
273	18151	1423281	120715207	10704110673	975825334951	...
400	34280	3471664	380859848	43742480080	5171698249640	...
561	60201	7644945	1052758089	151918096881	22587972554601	...
760	99802	15525544	2620997290	463997697880	84691813153402	...

TABLE 1

The sequence is named $A034491$ in [3], and the other sequence is a new line. The first column is called $A131422$ in [3]. If you pay attention to $n = 2, 4, 6, 8, 10$ all the values are even numbers for the next n sequence. We obtained:

$$2 \mid \sum_{k=1}^{n}(2k^2-1)^m, \exists n, m \in N\,.$$

The unit numbers on each line repeat every four elements. To show more clearly, the values above are presented in modulo 10 in the Table 2.

The remainder of division 10 in integers $1, 7, 17, 31, 49, 71, 97, 127, 161$ and 99 if they are raised are presented in Table 3.

The periods of these residues are:

$1, 31, 71, 161$: period 1 $49, 199$: period 2 $7, 17, 97, 127$: period 3

which are all factors of $\theta(10) = 4$, where θ is an Euler-θ function. These values show that the period of the unit digits of numbers 2 must be 4.

To prove Table 3 can be solved by doing mathematical induction. Taking the value (1.2) of modulo 3, for $n = 3$ and $n = 6$, with an even number m, it is obtained:

$$3 \mid \sum_{k=1}^{n}(2k^2-1)^m\,.$$

Power m	1	2	3	4	5	6	7	8	9	10	...
1	1	8	5	6	5	6	3	0	1	0	...
2	1	0	9	0	1	2	1	0	1	2	...
3	1	4	7	8	7	8	1	4	5	4	...
4	1	2	3	4	5	6	7	8	9	0	...
5	1	8	5	6	5	6	3	0	1	0	...
6	1	0	9	0	1	2	1	0	1	2	...
7	1	4	7	8	7	8	1	4	5	4	...
8	1	2	3	4	5	6	7	8	9	0	...
9	1	8	5	6	5	6	3	0	1	0	...
10	1	0	9	0	1	2	1	0	1	2	...
...	...	...	...	...	...	...	...	...	...	...	...

TABLE 2. The residues modulo 10 of the powers of integers of the form $(2k^2-1)^m$.

Power m	1	7	17	31	49	71	97	127	161	199	...
1	1	7	7	1	9	1	7	7	1	9	...
2	1	9	9	1	1	1	9	9	1	1	...
3	1	3	3	1	9	1	3	3	1	9	...
4	1	1	1	1	1	1	1	1	1	1	...
5	1	7	7	1	9	1	7	7	1	9	...
6	1	9	9	1	1	1	9	9	1	1	...
...	...	...	...	...	...	...	...	...	...	...	...

TABLE 3. The residues modulo 10 of integers $1, 7, 17, 31, 49, 71, 97, 127, 161$ and 199.

The remainder of the modulo 3 division of odd-numbered integers $(2k^2-1)^m$ that the period of the remainder of division by 3 is one of 1 or 2, because $\theta(3) = 2$. For $n = 5$ and 7 with odd m, we get:

$$3 \mid \sum_{k=1}^{n} (2k^2-1)^m \,.$$

For $n = 9$, we get:

$$3 \mid \sum_{k=1}^{n} (2k^2 - 1)^m .$$

Another interesting point is obtained when (1.2) is displayed by taking the values at the remainder of the division in modulo 4, for $n = 2, 6$ and 10 with odd m obtained:

$$4 \mid \sum_{k=1}^{n} (2k^2 - 1)^m ,$$

and also for $n = 4$ and 8, we get:

$$4 \mid \sum_{k=1}^{n} (2k^2 - 1)^m .$$

With the remainder of the modulo division of 4 of the odd number rank $(2k^2 - 1)^m$ that period of the remainder of the division by 4 is 1 or 2, because $\theta(4) = 2$. Also if (3.3) is displayed by taking the values at the remainder of the division of modulo 8, for $n = 2, 4, 6$ and 10 with odd m is obtained:

$$8 \mid \sum_{k=1}^{n} (2k^2 - 1)^m ,$$

and for $n = 8$, we obtained:

$$8 \mid \sum_{k=1}^{n} (2k^2 - 1)^m .$$

The remainder of the modulo division of 8 of the odd number rank $(2k^2 - 1)^m$ that the period of the remainder of the division by 8 is 1 or 2, because $\theta(8) = 4$.

2.2. **Case** 2**:** m **fixed.** Finally, we consider the sequences from the columns in (2.1). As we mentioned above, for $m = 1, 2$ and 3.

For $m = 1$

$$\sum_{k=1}^{n} (2k^2 - 1)^m = \frac{1}{3} n(n+2)(2n-1)$$

showing:

$$5 \mid \sum_{k=1}^{n}(2k^2-1)^m, \text{ for } n = 5r \text{ and } 5r-2 \text{ with } r \geq 1$$

$$\sum_{k=1}^{n}(2k^2-1)^m = \frac{1}{3}n(n+2)(n-1)$$

$$2 \mid \sum_{k=1}^{n}(2k^2-1), \text{ for } n = 2r, r \geq 1\,.$$

The above statement can be indicated by the explanation below:

$$2 \mid \sum_{k=1}^{n}(2k^2-1) = 2 \mid \frac{1}{3}2r(2r+2)(4r-1)\,.$$

Clearly that $2 \mid 2r(2r+2)(4r-1)$, showed that $3 \mid r(2r+2)(4r-1), r \geq 1$.

(1) $r = 3\alpha, \alpha \in Z^+$ obtained $3 \mid 3\alpha(6\alpha+2)(12\alpha-1)$.
So, $3 \mid r(2r+2)(4r-1)$ for $r = 3\alpha$.
(2) $r = 3\alpha+1, \alpha \in Z^+$ obtained $3 \mid 3(3\alpha+1)(6\alpha+4)(4\alpha+1)$.
So, $3 \mid r(2r+2)(4r-1)$ for $r = 3\alpha+1$.
(3) $r = 3\alpha+2, \alpha \in Z^+$ obtained $3 \mid 3(3\alpha+2)(2\alpha+2)(12\alpha+7)$.
So, $3 \mid r(2r+2)(4r-1)$ for $r = 3\alpha+2$.

Therefore, for $n = 2r$ then $2 \mid 2r(2r+2)(4r-1)$. Mean while for $n = 2r-1$ is obtained

$$2 \mid \sum_{k=1}^{n}(2k^2-1) = 2 \mid \frac{1}{3}(2r-1)(2r+1)(4r-3)\,.$$

Also note that

$$\sum_{k=1}^{n}(2k^2-1) = \frac{1}{3}n(n+2)(2n-1)\,,$$

showing

$$4 \mid \sum_{k=1}^{n}(2k^2-1), \text{ for } \; n = 2r \text{ and } 4r-2, r \geq 1\,.$$

For $m = 2$

$$\sum_{k=1}^{n}(2k^2-1)^2 = \frac{1}{5}n(4n^4+10n^3-10n+1)\,,$$

showing:

$$2 \mid \sum_{k=1}^{n}(2k^2-1)^2, \text{ for } n = 2r \text{ and } n = 2r-1 \text{ with } r \geq 1\,.$$

Also note that

$$\sum_{k=1}^{n}(2k^2-1)^2 = \frac{1}{5}n(4n^4+10n^3-10n+1)\,,$$

then

$$3 \mid \sum_{k=1}^{n}(2k^2-1)^2, \text{ for } n = 3r \text{ with } r \geq 1\,,$$

and

$$\sum_{k=1}^{n}(2k^2-1)^2 = \frac{1}{5}n(4n^4+10n^3-10n+1)\,,$$

then

$$4 \mid \sum_{k=1}^{n}(2k^2-1)^2, \text{ for } n = 4r \text{ with } r \geq 1\,.$$

For $m = 3$

$$\sum_{k=1}^{n}(2k^2-1)^3 = \frac{1}{105}n(n+2)(120n^5+180n^4-192n^3-216n^2+142n+31)\,,$$

showing

$$2 \mid \sum_{k=1}^{n}(2k^2-1)^3, \text{ for } n = 2r \text{ with } r \geq 1\,.$$

3. REMARKS

As we showed above, what we did in this paper is an observation on the sequences of the form (2.1) and their related properties. We reconstructed several known sequences as listed in [2] in an elementary way as well as constructed many new sequences. We are now working on rigorous proofs for the above results and conjectures. The paper, which is now in preparation, will be published elsewhere in a separate paper.

Acknowledgment

The authors would like to thank the anonymous referee(s) for careful reading and many helpful comments.

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