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ASYMPTOTIC DENSITY OF PELLIAN TRIPLETS ASSOCIATED WITH $U^2 - DV^2 = -M$

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ABSTRACT. For positive integers m, D and V, triplet $(-m, D, V)$ is defined to be a Pellian triplet if $-m+DV^2$ is a square. Clearly, for any such triplet we have $U^2-DV^2 = -m$, for some integer U. In this paper, we calculate the asymptotic density of Pellian triplets $(-m, D, V)$ in the cuboid $1 \leq m \leq Z_1, 1 \leq D \leq Z_2$ and $1 \leq V \leq Z_3$, for any given large positive integers Z_1, Z_2, Z_3.

1. INTRODUCTION

Shah [9] defined triplet (m, D, V) to be a Pellian triplet if $m+DV^2$ is a square. He related this triplet with the Pell's equation $U^2 - DV^2 = m$ and obtained the asymptotic value of $F(Z_1, Z_2, Z_3)$, the total number of such triplets in the cuboid $1 \leq m \leq Z_1, 1 \leq D \leq Z_2$ and $1 \leq V^2 \leq Z_3$ for any given positive integers Z_1, Z_2, Z_3 under certain necessary conditions. Shah [10] also obtained the asymptotic value of $F(Z, Z, Z)$ and obtained omega result for its error term.

In the present paper, we modify the above definition and consider the Pellian triplets related with the Pellian equation $U^2 - DV^2 = -m$ and obtain the asymptotic density of the total number of such triplets in the given cuboid.

Definition 1.1. *The triplet $(-m, D, V)$ is defined to be a Pellian triplet if $-m+DV^2$ is a square for positive integers m, D and V.*

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It is easy to see that every Pellian triplet $(-m, D, V)$ is associated with $U^2 - DV^2 = -m$ for some integer U, which justify the name 'Pellian triplet'. For simplicity we write pltp for Pellian triplet. Clearly every square number is represented by at least one pltp; for, it is sufficient to consider V=1. We note that even if (m, D, V) is a pltp, $(-m, D, V)$ may not be pltp. This is because for any given positive integers m, D and V, if the Pellian equation $U^2 - DV^2 = m$ has solution then it need not imply that $U^2 - DV^2 = -m$ also has solution. For completeness we recall that there are many papers which considered different types of Pell's equation. Many authors such as Andreescu et el [1], Burton [3], Kaplan and Williams [4], Le Veque [5], Madni and Shah [6], Matthews [7], Mollin et et [8], Steuding [11], Stevenhagen [12], Telang [13] and others considered some specific Pell equations and their integer solutions.

2. Value of F(Z_1, Z_2, Z_3):

We consider the triad m, D and V-axes and the cuboid $1 \leq m \leq Z_1, 1 \leq D \leq Z_2, 1 \leq V \leq Z_3$. For convenience we select m and D-axes as x and y axes respectively. Thus, in the above cuboid, we have Z_3 planes parallel to mD-plane for each $V \leq Z_3$. In each of these planes, we consider rectangles bounded by m and D-axes for a fixed V. We denote the total number of pltps in each of this rectangles by $F_V(Z_1, Z_2, Z_3)$, whose value will be $F(Z_1, Z_2, Z_3) = \sum_{V \leq Z_3} F_V(Z_1, Z_2, Z_3)$. Here we obtain the value of $F(Z_1, Z_2, Z_3)$ under the condition that $Z_1 \leq Z_2$.

Throughout, by C we mean Euler's constant. The following lemma containing some asymptotic formulas has been stated over here since it is (directly or indirectly) used in the paper.

Lemma 2.1. *(Apostol [2])*

(a) $\sum_{n \leq x} \frac{1}{n} = \log x + C + O(\frac{1}{x})$.

(b) $\sum_{n \leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s}); s > 0, s \neq 1$.

(c) $\sum_{n \leq x} n^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} + O(x^{\alpha})$; *for* $\alpha \geq 0$.

Theorem 2.1. *Uf* $Z_1 \leq Z_2$, *then*

$$F(Z_1, Z_2, Z_3) = \frac{2Z_1^{3/2}}{3}\left(\frac{1}{Z_3} - \zeta(2)\right) + Z_1\sqrt{Z_2}(\log Z_3 + C) + O(\sqrt{Z_1}Z_3) + O\left(\frac{z_1\sqrt{z_2}}{z_3}\right) + O(\sqrt{Z_2}Z_3^2) + O\left(\frac{z_1^2}{z_2^{1/2}}\right) + O\left(\frac{z_1}{z_2^{1/2}}\log Z_3\right).$$

Proof. From the definition of pltp, it is evident that with every pltp $(-m, D, V)$ there is related a Pellian equation $U^2 - DV^2 = -m$. So for fixed $V = \gamma(1 \leq \gamma \leq Z_3)$ and for some positive integer t, we have

$$t^2 = -m + D\gamma^2. \tag{2.1}$$

Here we note that any pltp counted for $F(Z_1, Z_2, Z_3)$ will always represent a square not exceeding $Z_1 + Z_2 Z_3^2$. Now line (2.1) is a linear Diophantine equation in m and D which always has an integral solution. Since $1 \leq m \leq Z_1$ and $1 \leq D \leq Z_2$, we get a rectangle in second quadrant which contains within the lattice points $(-m, D)$ satisfying (2.1). We thus consider a rectangle having vertices at origin $O, R(-Z_1, 0), T(0, Z_2)$ and $S(-Z_1, Z_2)$ as shown in figure 1.

Since $1 \leq V \leq Z_3$, for a fixed value $V = \gamma$ we first count the number of pltps in this rectangle and then sum it over all values of V. We now draw the lines (2.1) for different values of t in mD-plane in the fixed rectangle R_γ. We first consider line for $t = \sqrt{Z_1}$ $(say\ l_1)$ which passes through the point $R(-Z_1, 0)$. We also consider another line for $t = \gamma\sqrt{Z_2}$ (say l_2) which passes through the point $T(0, Z_2)$ as shown in the figure 1.

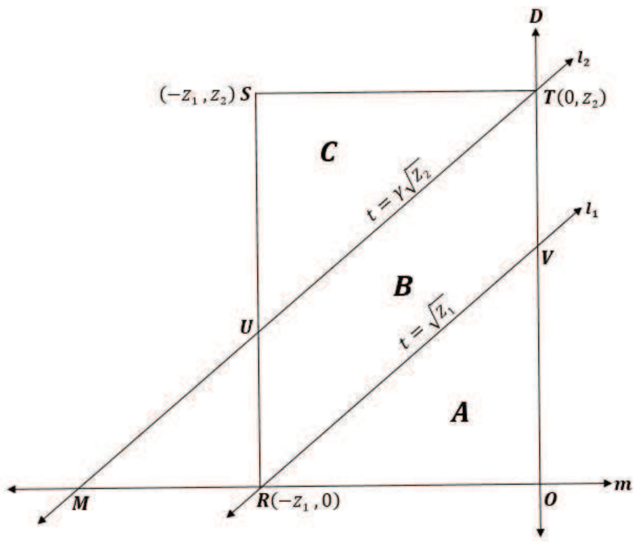

Figure 1

Now these two lines divide the rectangle ORST in three convex regions say A, B and C, where A is triangle VOR, B is parallelogram TURV and C is a triangle UST. For any fixed value γ , we denote the total number of pltps in the region A, B and C by $F_\gamma^A(Z_1, Z_2)$, $F_\gamma^B(Z_1, Z_2)$ and $F_\gamma^C(Z_1, Z_2)$ respectively.

We first consider the ΔVOR (region A) and take any line (2.1) parallel to l_1 in this region as shown in figure 2. Let it intersect m and D-axes in P and Q having coordinates $(-t^2, 0)$ and $(0, \frac{t^2}{\gamma^2})$ respectively. Since P is a lattice point, all the lattice points on $\vec{PQ}$ are given by parametric equation $m = -t^2 + \gamma^2 u$, D=u; for integers u.

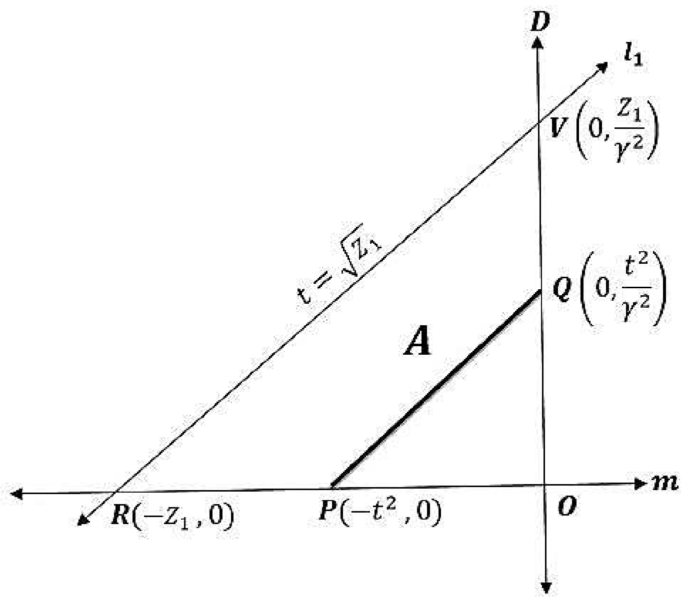

Figure 2

We restrict the value of D such that $0 < D \leq \frac{t^2}{\gamma^2}$, i.e. $0 < u \leq \frac{t^2}{\gamma^2}$. Then it is easily seen that there are $[\frac{t^2}{\gamma^2}]$ lattice points on $\bar{PQ}$. Also value of $-t^2$ ranges from $-Z_1$ to 0. Thus

$$F_{\gamma}^{A}(Z_1, Z_2) = \sum_{t \leq \sqrt{Z_1}} [\frac{t^2}{\gamma^2}] \tag{2.2}$$

We next consider region B (which is a parallelogram TURV). Here we note that line $t = \gamma\sqrt{Z_2}$ (i.e. l_2) intersect m-axis at the point, say M having coordinate $(-\gamma^2 Z_2, 0)$. We take any line (2.1) parallel to l_1(or l_2) in the region as shown in figure 3. Let it intersect m and D-axes in $P_1^{'}$ and $Q^{'}$ respectively. Also let it intersect $\bar{UR}$ in $P^{'}$. Then the coordinates of $P^{'}$ and $Q^{'}$ are $(-Z_1, \frac{t^2 - Z_1}{\gamma^2}$ and $(0, \frac{t^2}{\gamma^2})$ respectively. Also the coordinate of $P_1^{'}$ is $(-t^2, 0)$. Since $P_1^{'}$ is a lattice point, all the lattice points on $\vec{P_1^{'}Q'}$ are given by $m = -t^2 + \gamma^2 u$, $D = u$; $u \in Z$.

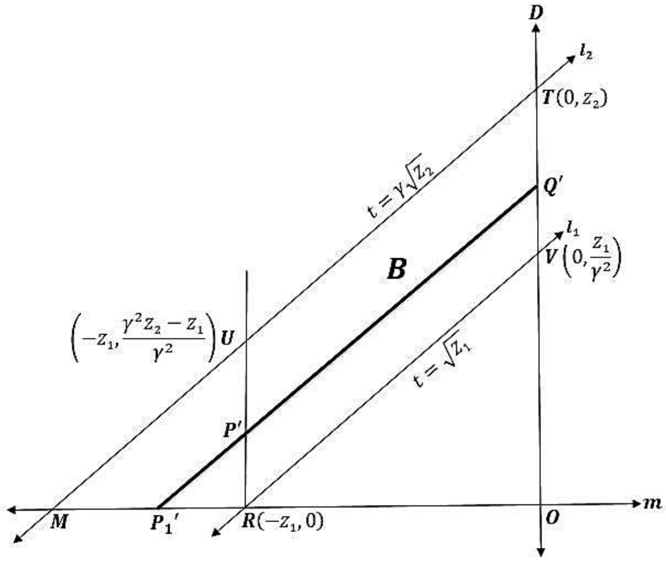

Figure 3

Using the coordinates of P' and Q', it is easily observed that there are $[\frac{Z_1}{\gamma^2}]$ lattice points on $\bar{P'Q'}$. Also the value of $-t^2$ ranges from $-\gamma^2 Z_2$ to $-Z_1$. Thus we have

$$F_\gamma^B(Z_1, Z_2) = \sum_{\sqrt{Z_1} < t \leq \gamma} \sqrt{Z_2}[\frac{Z_1}{\gamma^2}]. \tag{2.3}$$

We finally consider region C, which is a triangle STU. We consider any line through S parallel to l_2 and suppose it intersect m-axis at the point say N. Then it can be observed that coordinate of N and U are $(-Z_1 - \gamma^2 Z_2, 0)$ and $(-Z_1, \frac{\gamma^2 Z_2 - Z_1}{\gamma^2)}$ respectively. Here too we take any line (2.1) parallel to l_2 in ΔSTU as shown in figure 4. Let it intersect m-axis at P_2' and $\bar{ST}$ at Q''. Then for Q'' we have $D = Z_2, m = \gamma^2 Z_2 - t^2$. Also for P_2' we have D=0, $m = -t^2$. Thus the coordinates of Q'' and P_2' are $(\gamma^2 Z_2 - t^2, Z_2)$ and $(-t^2, 0)$ respectively. Also $P'' \equiv (-Z_1, \frac{t^2 - Z_1}{\gamma^2})$. Since P_2' is a lattice point, all the lattice points on $\vec{P''Q''}$ are given by $m = -t^2 + \gamma^2 u$, $D = u$; $u \in Z$.

We next calculate the number of lattice points on $\bar{P''Q''}$. For that using the coordinates of P'' and Q'', we observe that there are $[Z_2 - \frac{t^2 - Z_1}{\gamma^2}]$ lattice points on $P''\bar{Q}''$. Also the value of $-t^2$ ranges from $-Z_1 - \gamma^2 Z_2$ to $-\gamma^2 Z_2$. This gives

(2.4) $$F_\gamma^C(Z_1, Z_2) = \sum_{\gamma\sqrt{Z_2}<t\leq\sqrt{Z_1+\gamma^2 Z_2}} [Z_2 - \frac{t^2 - Z_1}{\gamma^2}]$$

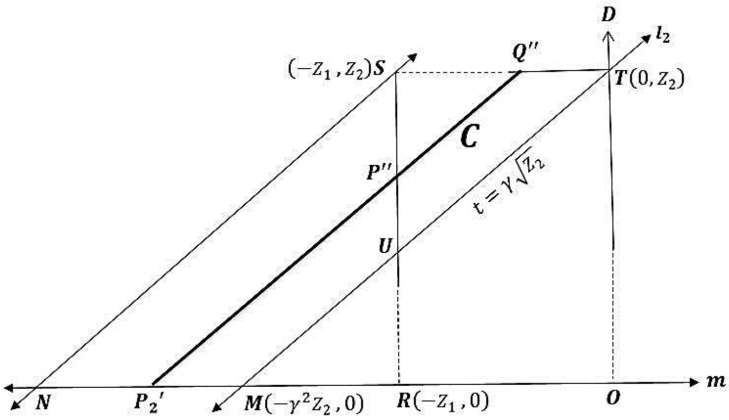

Figure 4

Hence for the rectangle R_γ, by (2.2), (2.3) and (2.4) the total number of pltps are given by

$$F_\gamma^A(Z_1, Z_2) + F_\gamma^B(Z_1, Z_2) + F_\gamma^C(Z_1, Z_2)$$
$$= \sum_{t\leq\sqrt{Z_1}}[\tfrac{t^2}{\gamma^2}] + \sum_{\sqrt{Z_1}<t\leq\gamma\sqrt{Z_2}}[\tfrac{Z_1}{\gamma^2}] + \sum_{\gamma\sqrt{Z_2}<t\leq\sqrt{Z_1+\gamma^2 Z_2}}[Z_2 - \tfrac{t^2-Z_1}{\gamma^2}].$$

Hence by total number of required pltps is

(2.5)
$$F(Z_1, Z_2, Z_3)$$
$$= \sum_{\gamma\leq Z_3}\{\sum_{t\leq\sqrt{Z_1}}[\frac{t^2}{\gamma^2}] + \sum_{\sqrt{Z_1}<t\leq\gamma\sqrt{Z_2}}[\frac{Z_1}{\gamma^2}] + \sum_{\gamma\sqrt{Z_2}<t\leq\sqrt{Z_1+\gamma^2 Z_2}}[Z_2 - \frac{t^2 - Z_1}{\gamma^2}]\}$$
$$= S_1 + S_2 + S_3,$$

(say), where $S_1 = \sum_{\gamma\leq Z_3}\{\sum_{t\leq\sqrt{Z_1}}[\frac{t^2}{\gamma^2}]$, $S_2 = \sum_{\sqrt{Z_1}<t\leq\gamma\sqrt{Z_2}}[\frac{Z_1}{\gamma^2}]$ and

$$S_3 = \sum_{\gamma\sqrt{Z_2}<t\leq\sqrt{Z_1+\gamma^2 Z_2}} [Z_2 - \frac{t^2 - Z_1}{\gamma^2}].$$

We now calculate the values of S_1, S_2 and S_3 separately. First,

$$S_1 = \sum_{\gamma\leq Z_3}\{\sum_{t\leq\sqrt{Z_1}}[\tfrac{t^2}{\gamma^2}] = \sum_{\gamma\leq Z_3}\sum_{t\leq\sqrt{Z_1}}\{\tfrac{t^2}{\gamma^2} + O(1)\}$$
$$= (\tfrac{(\sqrt{Z_1})^3}{3} + O(Z_1))\sum_{\gamma\leq Z_3}\tfrac{1}{\gamma^2} + O(Z_1^{1/2} Z_3)$$

$= (\frac{(\sqrt{Z_1})^3}{3} + O(Z_1))(\frac{-1}{Z_3} + \zeta(2) + O(\frac{1}{Z_3^2})) + O(Z_1^{1/2} Z_3)$.

Thus

$$S_1 = \frac{1}{3}(\zeta(2) - \frac{1}{Z_3})Z_1^{3/2} + O(\sqrt{Z_1}Z_3) + O(\frac{Z_1^{3/2}}{Z_3^2}) + O(Z_1). \tag{2.6}$$

Next

$S_2 = \sum_{\gamma \le Z_3} \sum_{\sqrt{Z_1} < t \le \gamma\sqrt{Z_2}} [\frac{Z_1}{\gamma^2}] = \sqrt{Z_2} \sum_{\gamma \le Z_3} \{\frac{Z_1}{\gamma} + O(\gamma)\}$ - $\sqrt{Z_1} \sum_{\gamma \le Z_3} \{\frac{Z_1}{\gamma^2} + O(1)\}$
$= Z_1\sqrt{Z_2}(\log Z_3 + C + O(\frac{1}{Z_3})$ - $Z_1^{3/2}(\zeta(2) - \frac{1}{Z_3} + O(\frac{1}{Z_3^2}))$ + $O(\sqrt{Z_2}Z_3^2)$ + $O(\sqrt{Z_1}Z_3)$.

Thus

$$\begin{aligned} S_2 &= Z_1\sqrt{Z_2}(\log Z_3 + C) - Z_1^{3/2}(\zeta(2) - \frac{1}{Z_3}) \\ &+ O(\frac{Z_1\sqrt{Z_2}}{Z_3}) + O(\frac{Z_1^{3/2}}{Z_3^2}) + O(\sqrt{Z_2}Z_3^2). \end{aligned} \tag{2.7}$$

Finally, we estimate S_3. For any γ, clearly we have $\sqrt{Z_1} \le \sqrt{Z_2} \le \gamma\sqrt{Z_2}$. Then

$\sqrt{Z_1 + \gamma^2 Z_2} = \gamma\sqrt{Z_2}(1 + \frac{Z_1}{\gamma^2 Z_2})^{1/2}$
$= \gamma\sqrt{Z_2}\{1 + \frac{Z_1}{2\gamma^2 Z_2} - \frac{Z_1^2}{8\gamma^4 Z_2^2} + O(\frac{Z_1^3}{\gamma^6 Z_2^3})\}$.

Thus $\sqrt{Z_1 + \gamma^2 Z_2} = \gamma\sqrt{Z_2} + \frac{Z_1}{2\gamma\sqrt{Z_2}}$ - $\frac{Z_1^2}{8\gamma^3 Z_2^{3/2}} + O(\frac{Z_1^3}{\gamma^5 Z_2^{5/2}}) = \delta$ (say). This gives

$$\delta - \gamma\sqrt{Z_2} = \frac{Z_1}{2\gamma\sqrt{Z_2}} - \frac{Z_1^2}{8\gamma^3 Z_2^{3/2}} + O(\frac{Z_1^3}{\gamma^5 Z_2^{5/2}}).$$

Also, for the above value of δ, we have $\delta^2 = \gamma^2 Z_2\{1 + \frac{Z_1}{\gamma^2 Z_2} + O(\frac{Z_1^3}{\gamma^6 Z_2^3})\}$ and $\delta^3 = \gamma^3 Z_2^{3/2}\{1 + \frac{3Z_1}{2\gamma^2 Z_2} + \frac{3Z_1^2}{8\gamma^4 Z_2^2} + O(\frac{Z_1^3}{\gamma^6 Z_2^3)}\}$. Thus

$$\begin{aligned} S_3 &= \textstyle\sum_{\gamma \le Z_3} \sum_{\gamma\sqrt{Z_2} < t \le \sqrt{Z_1 + \gamma^2 Z_2}} [Z_2 - \frac{t^2 - Z_1}{\gamma^2}] \\ &= \textstyle\sum_{\gamma \le Z_3} \sum_{\gamma\sqrt{Z_2} < t \le \delta} \{Z_2 - \frac{t^2 - Z_1}{\gamma^2} + O(1)\} \\ &= Z_2 \textstyle\sum_{\gamma \le Z_3} \sum_{\gamma\sqrt{Z_2} < t \le \delta} 1 - \sum_{\gamma \le Z_3} \frac{1}{\gamma^2} \sum_{\gamma\sqrt{Z_2} < t \le \delta} (t^2 - Z_1) \\ &\quad + O(\textstyle\sum_{\gamma \le Z_3} (\delta - \gamma\sqrt{Z_2})) \\ &= Z_2 \textstyle\sum_{\gamma \le Z_3} (\delta - \gamma\sqrt{Z_2}) - \sum_{\gamma \le Z_3} \frac{1}{\gamma^2} \{\frac{2\delta^3 + 3\delta^2 + \delta}{6} - \frac{2\gamma^3 Z_2^{3/2} + 3\gamma^2 Z_2 + \gamma Z_2^{1/2}}{6} \\ &\quad - Z_1(\delta - \gamma\sqrt{Z_2})\} + O(\textstyle\sum_{\gamma \le Z_3} \{\frac{Z_1}{2\gamma\sqrt{Z_2}} - \frac{Z_1^2}{8\gamma^3 Z_2^{3/2}} + O(\frac{Z_1^3}{\gamma^5 Z_2^{5/2}})\}) \\ &= Z_2 \textstyle\sum_{\gamma \le Z_3} \{\frac{Z_1}{2\gamma\sqrt{Z_2}} - \frac{Z_1^2}{8\gamma^3 Z_2^{3/2}} + O(\frac{Z_1^3}{\gamma^5 Z_2^{5/2})}\} \\ &\quad - \textstyle\sum_{\gamma \le Z_3} \frac{1}{6\gamma^2} \{2\gamma^3 Z_2^{3/2} + 3Z_1\sqrt{Z_2}\gamma + \frac{3Z_1^2}{4\gamma\sqrt{Z_2}} + O(\frac{Z_1^3}{\gamma^3 Z_2^{3/2}}) \end{aligned}$$

$$+3\gamma^2 Z_2 + 3Z_1 + O(\frac{Z_1^3}{\gamma^4 Z_2^2}) + \gamma\sqrt{Z_2} + \frac{Z_1}{2\gamma\sqrt{Z_2}} - \frac{Z_1^2}{8\gamma^3 Z_2^{3/2}}$$
$$+O(\frac{Z_1^3}{\gamma^5 Z_2^{5/2}}) - 2\gamma^3 Z_2^{3/2} - 3\gamma^2 Z_2 - \gamma\sqrt{Z_2} - \frac{3Z_1^2}{\gamma\sqrt{Z_2}} + \frac{3Z_1^3}{4\gamma^3 Z_2^{3/2}}$$
$$+O(\frac{Z_1^4}{\gamma^5 Z_2^{5/2}})\} + O(\frac{Z_1}{Z_2^{1/2}}(\log Z_3 + C + O(\frac{1}{Z_3})))$$
$$+O(\frac{Z_1^2}{Z_2^{3/2}}(-\frac{1}{2Z_3^2} + \zeta(3) + O(Z_3^{-3})))$$
$$= (\frac{Z_1\sqrt{Z_2}}{2} - \frac{Z_1\sqrt{Z_2}}{2})\sum_{\gamma\le Z_3}\frac{1}{\gamma} - \frac{Z_1}{2}\sum_{\gamma\le Z_3}\frac{1}{\gamma^2}$$
$$+(-\frac{Z_1^2}{8Z_2^{3/2}} - \frac{Z_1^2}{8\sqrt{Z_2}} - \frac{Z_1}{12\sqrt{Z_2}} + \frac{Z_1^2}{2\sqrt{Z_2}})\sum_{\gamma\le Z_3}\frac{1}{\gamma^3}$$
$$+(\frac{Z_1^2}{48Z_2^{3/2}} - \frac{Z_1^3}{8Z_2^{3/2}})\sum_{\gamma\le Z_3}\frac{1}{\gamma^5} + O(\frac{Z_1^3}{Z_2^{3/2}}\sum_{\gamma\le Z_3}\frac{1}{\gamma^5})$$
$$+O(\frac{Z_1^4}{Z_2^{5/2}}\sum_{\gamma\le Z_3}\frac{1}{\gamma^7}) + O(\frac{Z_1}{Z_2^{1/2}}\log Z_3) + O(\frac{Z_1^2}{Z_2^{3/2}})$$
$$= -\frac{Z_1}{2}\{-\frac{1}{Z_3} + \zeta(2) + O(\frac{1}{Z_3^2})\} + (-\frac{Z_1}{12\sqrt{Z_2}} + \frac{Z_1^2(3Z_2-1)}{8Z_2^{3/2}})$$
$$\{-\frac{1}{2Z_3^2} + \zeta(3) + O(Z_3^{-3})\} + (\frac{Z_1^2(1-6Z_1)}{48Z_2^{3/2}})\{-\frac{1}{4Z_3^4} + \zeta(5) + O(Z_3^(- 5))\}$$
$$+O(\frac{Z_1^3}{Z_2^{3/2}}) + O(\frac{Z_1^4}{Z_2^{5/2}}) + O(\frac{Z_1}{Z_2^{1/2}}\log Z_3) + O(\frac{Z_1^2}{Z_2^{3/2}}).$$

Since $Z_1 \le Z_2$, we get

$$S_3 = \frac{1}{2}Z_1(\frac{1}{Z_3} - \zeta(2)) + (-\frac{Z_1}{12Z_2^{1/2}} + \frac{3Z_1^2}{8Z_2^{1/2}})(\zeta(3) - \frac{1}{2Z_3^2})$$
$$+O(\frac{Z_1}{Z_3^2}) + O(\frac{Z_1^2}{Z_2^{1/2}}) + O(\frac{Z_1}{Z_2^{1/2}}\log Z_3).$$

This gives

$$S_3 = \frac{1}{2}Z_1(\frac{1}{Z_3} - \zeta(2)) + O(\frac{Z_1}{Z_3^2}) + O(\frac{Z_1^2}{Z_2^{1/2}}) + O(\frac{Z_1}{Z_2^{1/2}}\log Z_3). \quad (2.8)$$

Finally using (2.6), (2.7) and (2.8) in (2.5) we get the asymptotic value for the total number of Pellian triplets in the given cuboid as

$$F(Z_1, Z_2, Z_3) = \frac{2Z_1^{3/2}}{3}(\frac{1}{Z_3} - \zeta(2)) + Z_1\sqrt{Z_2}(\log Z_3 + C)$$
$$+ O(\sqrt{Z_1}Z_3) + O(\frac{Z_1\sqrt{Z_2}}{Z_3} + O(\sqrt{Z_2}Z_3^2)$$
$$+O(\frac{Z_1^2}{Z_2^{1/2}}) + O(\frac{Z_1}{Z_2^{1/2}}\log Z_3). \qquad \square$$

The following result gives the asymptotic density of the total number of Pellian triplets in the given cuboid.

Corollary 2.1. *In the cuboid* $1 \le m \le Z_1, 1 \le D \le Z_2$ *and* $1 \le V \le Z_3$*, the Pellian triplets* $(-m, D, V)$ *has the asymptotic density*

$$\frac{2Z_1^{1/2}}{3Z_2Z_3}(\frac{1}{Z_3} - \zeta(2)) + \frac{1}{Z_2^{1/2}Z_3}(\log Z_3 + C) + O(\frac{1}{Z_1^{1/2}Z_2})$$
$$+O(\frac{1}{Z_2^{1/2}Z_3}) + O(\frac{Z_3}{Z_1Z_2^{1/2}}) + O(\frac{\log Z_3}{Z_2^{3/2}Z_3}).$$

Proof. The result follows easily from the value of $F(Z_1, Z_2, Z_3)$ and the fact that $Z_1 \leq Z_2$. □

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