ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **9** (2020), no.8, 5805–5813
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.8.48

KERNELS IN IDEAL SPACES

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ABSTRACT. In this paper, we introduce and investigate three new notions of locally closed sets and obtain decompositions of $\star$-continuity.

1. INTRODUCTION AND PRELIMINARIES

First, we recall the θ-closure of a subset in a topological space. A point $x \in X$ is called a θ-cluster point of A if $cl(V) \cap A$, ϕ for every open set V containing x. The set of all θ-cluster points of A is called the θ-closure of A and is denoted by cl_θ (A) [5].

Definition 1.1. *A subset A of a topological space (X, τ) is said to be:*

1. *δ-preopen [4] (resp. θ-preopen [6]) if $A \subseteq int(cl_\delta(A))$ (resp. $A \subseteq int(cl_\theta(A))$),*
2. *δ-β-open [1] (resp. θ-β-open [6]) if $A \subseteq cl(int(cl_\delta(A)))$ (resp. $A \subseteq cl(int(cl_\theta(A)))$).*

Definition 1.2. *([3]) A subset A of an ideal topological space (X, τ, I) is said to be $I_{\delta pg}$- closed (resp. $I_{\delta\beta g}$-closed, $I_{\theta\beta g}$-closed) if $A* \subseteq$ whenever $A \subseteq U$ and U is δ-preopen (resp. δ-β-open, θ- β-open) in (X, τ).*

Lemma 1.1. ([3]) Let (X, τ, I) be an ideal space and A a subset of X. We have the following implications:

$$\star - closed \Rightarrow I_{\theta\beta\delta} - closed \Rightarrow I_{\delta\beta g} - closed \Rightarrow I_{\delta pg} - closed.$$

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2010 *Mathematics Subject Classification.* 54A05.

Key words and phrases. $\star$-closed, δ-β-I-LC-set, θ-β-I-LC sets, δ-pre-I-LC sets, $\star$-continuity.

2. Three new forms of locally closed sets

Definition 2.1. *A subset A of an ideal space (X, τ, I) is called*

1. *δ-pre-I-locally closed (briefly, δ-pre-I-LC) if A=U $\cap$ V where U is δ-preopen and V is $\star$-closed.*
2. *δ-β-I-locally closed (briefly, δ-β-I-LC) if A= U $\cap$ V where U is δ-β-open and V is $\star$-closed.*
3. *θ-β-I-locally closed (briefly, θ-β-I-LC) if A= U $\cap$ V where U is θ-β-open and V is$\star$-closed.*

Proposition 2.1. *Let (X, τ, I) be an ideal space and A a subset of X. Then the following hold.*

1. *If A is δ-β-open, then A is a δ-β-I-LC-set.*
2. *If A is $\star$-closed, then A is a δ-β-I-LC-set (resp. a δ-pre-I-LC-set, a θ-β-I-LC-set, an $I_{\delta pg}$-closed set, an $I_{\delta\beta g}$-closed set, an $I_{\theta\beta\delta}$-closed set).*
3. *If A is δ-preopen, then A is a δ-pre-I-LC-set.*
4. *If A is θ-β-open, then A is a θ-β-I-LC-set.*
5. *If A is a δ-β-I-LC-set then A is a θ-β-I-LC-set.*
6. *If A is a δ-pre-I-LC-set, then A is a δ-β-I-LC-set.*

The converses of Proposition 2.1 need not be true as shown by the following Examples.

Example 1. *Let X={a, b, c}, $\tau = \{\phi, \{a\}, \{a, c\}, X\}$ and $I = \{\phi, \{a\}\}$. Then {a, b} is a δ-β-I-LC-set but not δ-β-open.*

Example 2. *Let X={a, b, c}, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $I = \{\phi, \{a\}\}$. Then {a, c} is a δ-β-I-LC-set but neither $\star$-closed nor a δ-pre-I-LC-set.*

Example 3. *Let X={a, b, c}, $\tau = \{\phi, \{a\}, X\}$ and $I = \{\phi\}$. Then {a} is a θ-β-I-LC-set but not a δ-β-I-LC-set.*

Theorem 2.1. *Let (X, τ, I) be an ideal space. If A is a δ-β-I-LC-set and B is a $\star$-closed set, then A$\cap$ B is a δ-β-I-LC-set.*

Proof. Let B be $\star$-closed, then A$\cap$ B=(U $\cap$ V) $\cap$ B =U $\cap$ (V$\cap$ B), where V$\cap$ B is $\star$-closed. Hence A$\cap$ B is a δ-β-I-LC-set. □

Remark 2.1. *We have the following implications*

δ-pre-I-LC-set) $\Rightarrow$ *δ-β -I-LC-set)* $\Rightarrow$ *θ-β-I-LC-set.*

Theorem 2.2. *A subset of an ideal topological space (X, τ, I) is $\star$-closed iff it is*

1. *δ-β-I-LC and $I_{\delta\beta g}$-closed.*
2. *θ-β-I-LC and $I_{\delta pg}$-closed.*
3. *δ-pre-I-LC and $I_{\theta\beta\delta}$-closed.*

Proof.
1. Necessity is trivial. We prove only sufficiency. Let A be both δ-β-I-LC-set and $I_{\delta\beta g}$-closed. Since A is δ-β-I-LC, A=U∩ V, where U is δ-β-open and V is $\star$-closed. So, we have $A = U \cap V \subseteq U$. Since A is $I_{\delta\beta g}$-closed, $A* \subseteq U$. Also $A = U \cap V \subseteq V$ and V is $\star$-closed, then $A* \subseteq V$. Consequently, we have $A* \subseteq U \cap V = A$ and hence A is $\star$-closed.
2. This proof is similar to that of (1).
3. This proof is similar to that of (1). □

Theorem 2.3. *For a subset A of an ideal topological space (X, τ, I), the following are equivalent.*

1. *A is $\star$-closed.*
2. *A is δ-pre-I-LC and $I_{\delta pg}$-closed.*
3. *A is δ-pre-I-LC and I$\theta\beta$ δ-closed.*

Theorem 2.4. *For a subset A of an ideal topological space (X, τ, I), the following are equivalent.*

1. *A is$\star$-closed.*
2. *A is δ-β-I-LC and $I_{\delta pg}$-closed.*
3. *A is θ-β-I-LC and $I_{\delta pg}$-closed.*

Theorem 2.5. *For a subset A of an ideal topological space (X, τ, I), the following are equivalent.*

1. *A is $\star$-closed.*
2. *A is δ-pre-I-LC and $I_{\delta\beta g}$-closed.*
3. *A is δ-pre-I-LC and $I_{\theta\beta\delta}$-closed.*

Remark 2.2.

1. *The notions of δ-pre-I-LC sets and $I_{\delta pg}$-closed sets are independent.*
2. *The notions of δ-β-I-LC sets and $I_{\delta pg}$-closed sets are independent.*
3. *The notions of θ-β-I-LC sets and $I_{\delta pg}$-closed sets are independent.*

4. *The notions of δ-pre-I-LC sets and $I_{\delta\beta g}$-closed sets are independent.*
5. *The notions of δ-pre-I-LC sets and $I_{\theta\beta\delta}$-closed sets are independent.*
6. *The notions of δ-β-I-LC sets and $I_{\delta\beta g}$-closed sets are independent.*

Example 4. *Let X={a, b, c}, $\tau = \{\phi, \{c\}, \{a,b\}, X\}$ and $I = \{\phi, \{a\}\}$. Then*

1. *{b} is $I_{\delta pg}$-closed but not δ-β-I-LC.*
2. *{b} is $I_{\delta\beta g}$-closed but not δ-pre-I-LC.*
3. *{b} is $I_{\delta pg}$-closed but not δ-pre-I-LC.*
4. *{b} is $I_{\theta\beta\delta}$-closed but not δ-pre-I-LC.*
5. *{b} is $I_{\delta\beta g}$-closed but not δ-β-I-LC.*
6. *{b} is $I_{\delta pg}$-closed but not θ-β-I-LC.*

Example 5. *Let X = {a, b, c}, $\tau = \{\phi, \{a\}, \{c\}, \{a,c\}, X\}$ and $I = \{\phi, \{a\}\}$. Then*

1. *{a, c} is δ-β-I-LC but not $I_{\delta pg}$-closed.*
2. *{a, c} is δ-pre-I-LC but not $I_{\theta\beta\delta}$-closed.*
3. *{a, c} is δ-β-I-LC but not $I_{\delta\beta g}$-closed.*

Example 6. *Let X={a, b, c}, $\tau = \{\phi, \{a\}, \{b\}, \{a,b\}, X\}$ and $I = \{\phi, \{c\}\}$. Then*

1. *{a} is θ-β-I-LC but not $I_{\delta pg}$-closed.*
2. *{a} is δ-pre-I-LC but not $I_{\delta pg}$-closed.*
3. *{a} is δ-pre-I-LC but not $I_{\delta\beta g}$-closed.*

3. ON NEW SUBSETS OF τ^*

Definition 3.1. *Let A be a subset of a topological space (X,τ). Then the θ-β kernel of the set A, is denoted by θ-β-ker(A), is the intersection of all θ-β-open supersets of A.*

Definition 3.2. *A subset A of a topological space (X,τ) is called a $\wedge$ θ-β-set if A = θ-β-ker(A).*

Definition 3.3. *A subset A of an ideal topological space (X,τ,I) is called λ-θ-β-I-closed if A=L$\cap$ F where L is a $\wedge$ θ-β-set and F is $\star$-closed.*

Lemma 3.1.

1. *Every θ-β-I-LC-set is λ-θ-β-I-closed.*
2. *Every $\star$-closed set is λ-θ-β-I-closed but not conversely.*

Example 7. *Let X= {a, b, c}, $\tau = \{\phi, \{c\}, X\}$ and $I = \{\phi\}$. Then {c} is λ-θ-β-I-closed but not $\star$-closed.*

Lemma 3.2. *For a subset A of an ideal topological space (X, τ, I), the following conditions are equivalent.*

1. *A is λ-θ-β-I-closed.*
2. *A=L∩ cl*(A) where L is a $\wedge$ θ-β-set.*
3. *A= θ-β-ker(A)∩ cl*(A).*

Lemma 3.3. *A subset $A \subset (X, \tau, I)$ is $I_{\delta pg}$-closed if and only if cl*(A) $\subset$ θ-β-ker(A).*

Theorem 3.1. *For a subset A of an ideal topological space (X, τ, I), the following conditions are equivalent.*

1. *A is $\star$-closed.*
2. *A is $I_{\delta pg}$-closed and θ-β-I-LC.*
3. *A is $I_{\delta pg}$-closed and λ-θ-β-I-closed.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Obvious.
(3) $\Rightarrow$ (1) Since A is $I_{\delta pg}$-closed, so by Lemma 3.3, cl*(A)$\subset$ θ-β-ker(A). Since A is λ-θ-β-I-closed, so by Lemma 3.2, A=θ-β-ker(A)∩ cl*(A)=cl*(A). Hence A is $\star$-closed.
The following two Examples show that the concepts of $I_{\delta pg}$-closed sets and λ-θ-β-I-closed sets are independent. □

Example 8. *Let X ={a, b, c}, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $I = \{\phi, \{a\}\}$. Then {a, c} is λ-θ-β-I-closed but not $I_{\delta pg}$-closed.*

Example 9. *Let X = {a, b, c}, $\tau = \{\phi, \{c\}, \{a, b\}, X\}$ and $I = \{\phi, \{a\}\}$. Then {b} is $I_{\delta pg}$-closed but not λ-θ-β-I-closed.*

Definition 3.4. *Let A be a subset of a topological space (X, τ). Then*

1. *The δ-pre-kernel of the set A, denoted by δ-pre-ker(A), is the intersection of all δ-preopen supersets of A.*
2. *The δ-β-kernel of the set A, denoted by δ-β-ker(A), is the intersection of all δ-β-open supersets of A.*

Definition 3.5. *A subset A of a topological space (X, τ) is called*

1. *a $\wedge$ δ-pre-set if A = δ-pre-ker(A).*
2. *a $\wedge$ δ-β-set if A = δ-β-ker(A).*

Definition 3.6. *A subset A of an ideal topological space (X, τ, I) is called*

1. *λ-δ-pre-I-closed if A=L$\cap$ F where L is a $\wedge$ δ-pre-set and F is $\star$-closed.*
2. *λ-δ-β-I-closed if A=L$\cap$ F where L is a $\wedge$ δ-β-set and F is $\star$-closed.*

Lemma 3.4.

1. *Every $\star$-closed set is λ-δ-pre-I-closed.*
2. *Every δ-β-I-LC-set is λ-δ-β-I-closed.*
3. *Every $\star$-closed set is λ-δ-β-I-closed.*
4. *Every δ-pre-I-LC-set is λ-δ-pre-I-closed.*

Example 10. *Let X = {a, b, c}, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $I = \{\phi, \{c\}\}$. Then*

1. *{a} is λ-δ-pre-I-closed but not $\star$-closed.*
2. *{a} is λ-δ-β-I-closed but not $\star$-closed.*

Lemma 3.5. *For a subset A of an ideal topological space (X, τ, I), the following are equivalent.*

1. *(a) A is λ-δ-pre-I-closed.*
   (b) A=L$\cap$ cl(A) where L is a$\wedge$ δ-pre-set.*
   (c) A=δ-pre-ker(A)$\cap$ cl(A).*
2. *(a) A is λ-δ-β-I-closed.*
   (b) A=L$\cap$ cl(A) where L is a$\wedge$ δ-β-set.*
   (c) A=δ-β-ker(A)$\cap$ cl(A).*

Lemma 3.6. *A subset $A \subset (X, \tau, I)$ is $I_{\delta\beta g}$-closed if and only if cl*(A)$\subset$ δ-β-ker(A). A subset $A \subset (X, \tau, I)$ is $I_{\theta\beta\delta}$-closed if and only if cl*(A)$\subset$ δ-pre-ker(A).*

Theorem 3.2. *For a subset A of an ideal topological space (X, τ, I), the following are equivalent.*

1. *(a) A is $\star$-closed.*
   (b) A is $I_{\delta\beta g}$-closed and δ-β-I-LC.
   (c) A is $I_{\delta\beta g}$-closed and λ-δ-β-I-closed.
2. *(a) A is $\star$-closed.*
   (b) A is $I_{\theta\beta\delta}$-closed and δ-pre-I-LC.
   (c) A is $I_{\theta\beta\delta}$-closed and λ-δ-pre-I-closed.

By Examples 3.11, 3.12 and 3.13 we realize that the following concepts are independent.

1. *λ-δ-β-I-closed sets and $I_{\delta\beta g}$-closed sets.*
2. *λ-δ-pre-I-closed sets and $I_{\theta\beta\delta}$-closed sets.*

Example 11. *Let X ={a, b, c}, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $I = \{\phi, \{a\}\}$. Then {a, c} is λ-δ-β-I-closed but not $I_{\delta\beta g}$-closed.*

Example 12. *Let X ={a, b, c}, $\tau = \{\phi, \{c\}$, {a, b}, X} and $I = \{\phi, \{a\}\}$. Then*
1. *{b} is $I_{\delta\beta g}$-closed but not λ-δ-β-I-closed.*
2. *{b} is $I_{\theta\beta\delta}$-closed but not λ-δ-pre-I-closed.*

Example 13. *Let X ={a, b, c}, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $I = \{\phi, \{c\}\}$. Then {a} is λ-δ-pre-I-closed but not $I_{\theta\beta\delta}$-closed.*

4. Decompositions of $\star$-continuity

Definition 4.1. *A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be $\star$-continuous [2] (resp. $I_{\delta\beta g}$-continuous, $I_{\theta\beta\delta}$-continuous, $I_{\delta pg}$-continuous, δ-β-I-LC-continuous, θ-β-I-LC- continuous, δ-pre-I-LC-continuous, λ-θ-β-I-continuous, λ-δ-pre-I-continuous, λ-δ-β-I-continuous) if f^{-1}(A) is $\star$-closed (resp. $I_{\delta\beta g}$-closed, $I_{\theta\beta\delta}$-closed, $I_{\delta pg}$-closed, δ-β-I-LC, θ-β-I-LC, δ-pre-I-LC, λ-θ-β-I-closed, λ-δ-pre-I-closed, λ-δ-β-I-closed) in (X, τ, I) for every closed set A of (Y, σ).*

Theorem 4.1. *A function $f : (X, \tau, I) \to (Y, \sigma)$ is $\star$-continuous if and only if it is*
1. *θ-β-I-LC-continuous and $I_{\delta pg}$-continuous.*
2. *δ-β-I-LC-continuous and $I_{\delta\beta g}$-continuous.*
3. *δ-pre-I-LC-continuous and $I_{\theta\beta\delta}$-continuous.*

Proof. It is an immediate consequence of Theorem 2.2. □

Theorem 4.2. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following are equivalent.*
1. *f is $\star$-continuous.*
2. *f is δ-pre-I-LC-continuous and $I_{\delta pg}$-continuous.*
3. *f is δ-pre-I-LC-continuous and $I_{\theta\beta\delta}$-continuous.*

Proof. It is an immediate consequence of Theorem 2.3 □

Theorem 4.3. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following are equivalent.*
1. *f is $\star$-continuous.*
2. *f is δ-β-I-LC-continuous and $I_{\delta pg}$-continuous.*
3. *f is θ-β-I-LC-continuous and $I_{\delta pg}$-continuous.*

Proof. It is an immediate consequence of Theorem 2.4. □

Theorem 4.4. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following are equivalent.*

1. *f is $\star$-continuous.*
2. *f is δ-pre-I-LC-continuous and $I_{\delta\beta g}$-continuous.*
3. *f is δ-pre-I-LC-continuous and $I_{\theta\beta\delta}$-continuous.*

Proof. It is an immediate consequence of Theorem 2.4. □

Theorem 4.5. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following are equivalent.*

1. *f is $\star$-continuous.*
2. *f is $I_{\delta pg}$-continuous and θ-β-I-LC-continuous.*
3. *f is $I_{\delta pg}$-continuous and λ-θ-β-I-continuous.*

Proof. It is an immediate consequence of Theorem 3.1. □

Theorem 4.6. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following are equivalent.*

1. *(a) f is $\star$-continuous.*
   (b) f is $I_{\delta\beta g}$-continuous and δ-β-I-LC-continuous.
   (c) f is $I_{\delta\beta g}$-continuous and λ-δ-β-I-continuous.
2. *(a) f is $\star$-continuous.*
   (b) f is $I_{\theta\beta\delta}$-continuous and δ-pre-I-LC-continuous.
   (c) f is $I_{\theta\beta\delta}$-continuous and λ-δ-pre-I-continuous.

Proof. It is an immediate consequence of Theorem 3.2. □

5. CONCLUSION

The notions of sets and functions in topological spaces, ideal topological spaces, minimal spaces and ideal minimal spaces are extensively developed and used in many engineering problems, information systems, particle physics, computational topology and mathematical sciences. By researching generalizations of closed sets in various fields in general topology, some new separation axioms have been founded and they turn out to be useful in the study of digital topology. Therefore, all functions defined in this paper will have many possibilities of applications in digital topology and computer graphics.

REFERENCES

[1] E. HATIR, T. NOIRI: *Decompositons of continuity and complete continuity*, Acta Math. Hungar., **113** (2006), 281-287.

[2] V. INTHUMATHI, S. KRISHNAPRAKASH, M. RAJAMANI: *Strongly I locally closed sets and decompositions of $\star$-continuity*, Acta Math. Hungar., **130**(4) (2011), 358-362.

[3] T. NOIRI, V. POPA: *Between $\star$-closed sets and Ig-closed sets in ideal topological spaces*, Rendiconti del Circolo Matematico di Palermo, **59**(2010), 251-260.

[4] S. RAYCHAUDHURI, M. N. MUKHERJEE: *On δ-almost continuity and δ- preopen sets*, Bull. Inst. Math. Acad. Sinica, **21** (1993), 357-366.

[5] N. V. VELICKO: *H-closed topological spaces*, Amer. Math. Soc. Transl., **78**(2) (1968), 103-118.

[6] T. NOIRI, V. POPA: *On m-almost continuous multi functions*, (submitted)

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