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A NEW SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY OPOOLA DIFFERENTIAL OPERATOR

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ABSTRACT. The main aim of this present research is to introduce a new class $\mathcal{G}_{t,n}^{m}(\gamma,\mu,\beta,\lambda)$ defined by Opoola differential operator involving function $\Im \in \mathcal{A}_n$ and important results are indicated.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{D}_n$ indicate the subclass of the class of function $\mathcal{A}_n$ which is of the form

$$\Im(z) = z + \sum_{\varsigma=n+1}^{\infty} a_\varsigma z^\varsigma, \tag{1.1}$$

consisting of function which are holomorphic in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(\mathbb{U})$ the space of analytic function in $\mathbb{U}$, $n \in \mathbb{N} = \{1,2,3,\cdots\}$. We indicate by $\mathfrak{S}_n^*(\gamma)$, $\mathfrak{R}_n(\gamma)$ and $\mathfrak{K}_n(\gamma)$ the class of function with the class of starlike functions, bounded turning and the class of convex functions, respectively, where

$$\mathfrak{S}_n^*(\gamma) = \left\{\Im \in \mathcal{A}_n : \Re\left(\frac{z\Im'(z)}{\Im(z)}\right) > \gamma\right\},$$

$$\mathfrak{R}_n(\gamma) = \{\Im \in \mathcal{A}_n : \Re\{\Im'(z)\} > \gamma\}$$

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and

$$\mathfrak{K}_n(\gamma) = \left\{ \Im \in \mathcal{A}_n : \Re\left(\frac{z\Im''(z)}{\Im'(z)} + 1 \right) > \gamma \right\},$$

for some $0 \leq \gamma < 1$ and where $z \in \mathbb{U}$.

For a function $\Im(z) \in \mathcal{A}_n$, we define the Opoola differential operator [6] $D^{m,\mu}_{\beta,\lambda}$ as follows

$$D^{0,\mu}_{\beta,\lambda}\Im(z) = \Im(z)$$

$$D^{1,\mu}_{\beta,\lambda}\Im(z) = (1 + (\beta - \mu - 1)\lambda)\Im(z) - z(\mu - \beta)\lambda + z\lambda\Im'(z) = D^{\mu}_{\beta,\lambda}\Im(z), \tag{1.2}$$

$$D^{2,\mu}_{\beta,\lambda}\Im(z) = D^{\mu}_{\beta,\lambda}(D^{1,\mu}_{\beta,\lambda}\Im(z)),$$

$$D^{m,\mu}_{\beta,\lambda}\Im(z) = D^{\mu}_{\beta,\lambda}(D^{m-1,\mu}_{\beta,\lambda}\Im(z)) \quad m \in \mathbb{N}, \tag{1.3}$$

suppose $\Im(z)$ is given by (1.1), then by (1.2) and (1.3), we get

$$D^{m,\mu}_{\beta,\lambda}\Im(z) = z + \sum_{\varsigma=n+1}^{\infty} (1 + (\varsigma + \beta - \mu - 1)\lambda)^m a_\varsigma z^\varsigma \tag{1.4}$$

where $\quad 0 \leq \mu \leq \beta, \lambda \geq 0$ and $m \in \mathbb{N}_0 = \{0, 1, 2, 3 \cdots\}$.

Remark 1.1. *We have the following remarks:*

(1) *When $\mu = \beta = 1$ and $\lambda = n = 1$, $D^m\Im(z)$ is the Salegean differential operator [7].*
(2) *When $\mu = \beta = n = 1$, $D^m_\lambda\Im(z)$ is the Al-Oboudi differential operator [1].*

Remark 1.2. *It follows from the* (1.4) *that*

$$D^{m+1,\mu}_{\beta,\lambda}\Im(z) = (1 + (\beta - \mu - 1)\lambda)D^{m,\mu}_{\beta,\lambda}\Im(z) - z(\mu - \beta)\lambda + z\lambda(D^{m,\mu}_{\beta,\lambda}\Im(z))'$$

for $m \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

We'll need the following lemma to prove our principal theorem.

Lemma 1.1. *[5] Let $s(z)$ be holomorphic in $\mathbb{U}$ with $s(0) = 1$ and if that*

$$\Re\left(1 + \frac{zs'(z)}{s(z)}\right) > \frac{3\gamma - 1}{2\gamma}, \quad z \in \mathbb{U}.$$

Then $\Re(s(z)) > \gamma$ in $\mathbb{U}$ and $\frac{1}{2} \leq \gamma < 1$.

2. Main Result

Definition 2.1. *A function* $\Im(z) \in \mathcal{A}_n$ *is in the class* $\mathcal{G}_{t,n}^{m}(\gamma, \mu, \beta, \lambda)$ *if*

$$\left| \frac{D_{\beta,\lambda}^{m+1,\mu}\Im(z)}{z} \left(\frac{z}{D_{\beta,\lambda}^{m,\mu}\Im(z)} \right)^{t} - 1 \right| < 1 - \gamma \quad (z \in \mathbb{U}),$$

where $0 \le \mu \le \beta, t \ge 0, \lambda \ge 0,$ $m \in \mathbb{N}_0 = \{0, 1, 2, 3 \cdots\}$ *and* $0 \le \gamma < 1.$

Remark 2.1. *The family* $\mathcal{G}_{t,n}^{m}(\gamma, \mu, \beta, \lambda)$ *is a new general class of holomorphic functions which includes several new classes of holormorphic univalent functions along with some important ones. For examples,*

(1) For $m = 0$ and $t = 1$, we have the class

$$\mathcal{G}_{1,n}^{0}(\gamma, \mu, \beta, \lambda) \equiv \mathfrak{S}_n^*(\gamma).$$

(2) For $m = 0$, $\mu = \beta = 1$ and $t = 1$, we have the class

$$\mathcal{G}_{1,n}^{1}(\gamma, 1, 1, 1) \equiv \mathcal{K}_n(\gamma).$$

(3) For $m = 0$ and $t = 0$, we have the class

$$\mathcal{G}_{0,n}^{0}(\gamma, \mu, \beta, \lambda) \equiv \mathcal{R}_n(\gamma).$$

(4) For $\mu = \beta = 1$ and $n = 1$, we have the class

$$\mathcal{G}_{t,n}^{m}(\gamma, 1, 1, \lambda) \equiv \mathcal{G}_{t}^{m}(\gamma, \lambda).$$

introduced by Catas and Lupas [3].

(5) For $\mu = \beta = 1$ and $\lambda = 1$, we have the class

$$\mathcal{G}_{t,n}^{m}(\gamma, 1, 1, 1) \equiv \mathcal{G}_{t,n}^{m}(\gamma).$$

introduced by Catas and Lupas [2].

(6) For $m = 0$ and $n = 1$, the class

$$\mathcal{B}(t, \gamma) = \left\{ \Im \in \mathcal{A} : \left| \Im'(z) \left(\frac{z}{\Im(z)} \right)^{t} - 1 \right| < 1 - \gamma; t \ge 0, 0 \le \gamma < 1,\ z \in \mathbb{U} \right\}$$

intoduced by Frasin and Jahangiri [5].

(7) For $m = 0$, $n = 1$ and $t = 2$, the class

$$\mathcal{B}(\gamma) = \left\{ \Im \in \mathcal{A} : \left| \frac{z^{z}\Im'(z)}{\Im^{2}(z)} - 1 \right| < 1 - \gamma; 0 \le \gamma < 1,\ z \in \mathbb{U} \right\}$$

introduced by Frasin and Darus [4].

Theorem 2.1. *If for the function* $\Im(z) \in \mathcal{A}_n$*,* $\leq \mu \leq \beta, t \geq 0, \lambda \geq 0$*,* $m \in \mathbb{N}_0 = \{0, 1, 2, 3 \cdots\}$ *and* $\frac{1}{2} \leq \gamma < 1$*, we have*

$$\Re\left(\frac{1}{\lambda} \frac{D_{\beta,\lambda}^{m+2,\mu}\Im(z)}{D_{\beta,\lambda}^{m+1,\mu}\Im(z)} - \frac{t}{\lambda} \frac{D_{\beta,\lambda}^{m+1,\mu}\Im(z)}{D_{\beta,\lambda}^{m,\mu}\Im(z)} + \frac{(\mu-\beta)}{D_{\beta,\lambda}^{m+1,\mu}\Im(z)} - \frac{t(\mu-\beta)}{D_{\beta,\lambda}^{m,\mu}\Im(z)} - \frac{(1-t)(1+\lambda\beta-\lambda\mu)}{\lambda} + 1 \right) > \frac{3\gamma-1}{2\gamma},$$

then $\Im(z) \in \mathcal{G}_{t,n}^{m}(\gamma, \mu, \beta, \lambda)$.

Proof. Consider

$$s(z) = \frac{D_{\beta,\lambda}^{m+1,\mu}\Im(z)}{z} \left(\frac{z}{D_{\beta,\lambda}^{m,\mu}\Im(z)} \right)^{t},$$

then $s(z)$ is holomorphic in $\mathbb{U}$ with $s(0) = 1$, by simplification we have

$$\ln(s(z)) = \ln(D_{\beta,\lambda}^{m+1,\mu}\Im(z)) - \ln(z) + t\ln(z) - t\ln(D_{\beta,\lambda}^{m,\mu}\Im(z)).$$

A simple differentiation yields

$$\frac{zs'(z)}{s(z)} = \frac{1}{\lambda} \frac{D_{\beta,\lambda}^{m+2,\mu}\Im(z)}{D_{\beta,\lambda}^{m+1,\mu}\Im(z)} - \frac{t}{\lambda} \frac{D_{\beta,\lambda}^{m+1,\mu}\Im(z)}{D_{\beta,\lambda}^{m,\mu}\Im(z)} + \frac{(\mu-\beta)}{D_{\beta,\lambda}^{m+1,\mu}\Im(z)} - \frac{t(\mu-\beta)}{D_{\beta,\lambda}^{m,\mu}\Im(z)} - \frac{(1-t)(1+\lambda\beta-\lambda\mu)}{\lambda}.$$

By the hypothesis of the Theorem 2.1, we get

$$\Re\left(1 + \frac{zs'(z)}{s(z)}\right) > \frac{3\gamma-1}{2\gamma}.$$

Hence, by Lemma 1.1, we have

$$\Re\left(\frac{D_{\beta,\lambda}^{m+1,\mu}\Im(z)}{z} \left(\frac{z}{D_{\beta,\lambda}^{m,\mu}\Im(z)} \right)^{t} \right) > \gamma$$

therefore, $\Im(z) \in \mathcal{G}_{t,n}^{m}(\gamma, \mu, \beta, \lambda)$ by Definition 2.1. □

We have the following corollaries as consequences of the above theorem.

Choosing $m = 1$, $t = 1$, $\gamma = \frac{1}{2}$ and $\beta = \lambda = \mu = 1$, we have:

Corollary 2.1. *Suppose* $\Im(z) \in \mathcal{A}_n$ *and*

$$\Re\left(\frac{2z\Im''(z) + z^2\Im'''(z)}{z\Im''(z) + \Im'(z)} - \frac{z\Im''(z)}{\Im'(z)} \right) > -\frac{1}{2}, \quad z \in \mathbb{U}.$$

Then

$$\Re\left(1+\frac{z\Im''(z)}{\Im'(z)}\right)>\frac{1}{2}.$$

That is $\Im(z)$ is convex of order $\frac{1}{2}$.

Choosing $m=0$, $t=0$, $\gamma=\frac{1}{2}$ and $\beta=\lambda=\mu=1$, we have:

Corollary 2.2. *Suppose $\Im(z)\in\mathcal{A}_n$ and*

$$\Re\left(1+\frac{z\Im''(z)}{\Im'(z)}\right)>\frac{1}{2},\quad z\in\mathbb{U}.$$

Then

$$\Re\left[\Im'(z)\right]>\frac{1}{2}.$$

Also, if the function $\Im$ is convex of order $\frac{1}{2}$ then $\Im\in\mathcal{G}_{0,n}^{0}(\gamma,1,1,1)\equiv\mathfrak{R}_n(\frac{1}{2})$.

Choosing $m=0$, $t=1$, $\gamma=\frac{1}{2}$ and $\beta=\lambda=\mu=1$, we have:

Corollary 2.3. *Suppose $\Im(z)\in\mathcal{A}_n$ and*

$$\Re\left(\frac{z\Im''(z)}{\Im'(z)}-\frac{z\Im'(z)}{\Im(z)}\right)>-\frac{3}{2},\quad z\in\mathbb{U}.$$

Then

$$\Re\left[\frac{z\Im'(z)}{\Im(z)}\right]>\frac{1}{2}.$$

That is $\Im(z)$ is starlike of order $\frac{1}{2}$.

Choosing $m=1$, $t=0$, $\gamma=\frac{1}{2}$ and $\beta=\lambda=\mu=1$, we have:

Corollary 2.4. *Suppose $\Im(z)\in\mathcal{A}_n$ and*

$$\Re\left(\frac{2z\Im''(z)+z^3\Im'''(z)}{z^2\Im''(z)+z\Im'(z)}\right)>-\frac{1}{2},\quad z\in\mathbb{U}.$$

Then

$$\Re\left[z\Im''(z)+\Im'(z)\right]>\frac{1}{2}.$$

3. Conclusions

In this research, we describe new subclass of holomorphic functions applying Opoola differential operator, and some of its properties were created. The obtained results include the properties of certain subclasses of holomorphic functions.

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