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SOME NOTES ON CUBIC SPLINE OF PERIODIC FUNCTION

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ABSTRACT. A spline function is a piecewise polynomial function of order m joined smoothly so that it has $m-1$ continuous derivatives. In [1] interpolation of cubic spline function is discussed and this paper extends the results on third order spline of periodic function.

1. INTRODUCTION

In approximation theory spline interpolation, as a part of osculatory interpolation, is a form of interpolation where the interpolant is an adequate piecewise function which represents spline function. This kind of interpolation was introduced by I. J. Schoenberg in 1946 [5].

The spline function avoids the discontinuities in slope that occur with ordinary piecewise functions, except that in the n-th derivative where there is flexibility with respect to continuity.

The aim of this paper is to present for periodic functions belonging to $C^2[0,2\pi]$ the analogues of the recent developments on cubic spline functions and their role in approximation theory.

Initially, we will give some concepts, definitions and notations.

Definition 1.1. *Let $\Delta : a = x_0 < x_1 < x_2 < ... < x_n = b$ be a subdivision of the segment $[a,b]$. A function $S^m_{f\Delta} : [a,b] \to \mathbb{R}, m \in \mathbb{N}$ is called a spline of order m*

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with respect to the subdivision Δ *if* $S^m_{f\Delta} \in C^{m-1}$ *, in other words it has continuous derivatives up to order* $m-1$ *in* $[a,b]$ *and reduces to a polynomial of order smaller or equal to* m *in each of the intervals* $(-\infty, x_1), [x_1, x_2), ..., [x_n, \infty)$.

By S^m_n we denote all splines of order m for a fixed subdivision of segment in n pieces. S^m_n is linear space of dimension $m+n$.

Lemma 1.1. *Let* $f(x) \in C^2[a,b]$*. For each subdivision of the segment* $[a,b]$*,*

$$\Delta : a = x_0 < x_1 < x_2 < ... < x_n = b$$

there exists one and only one spline function in S^3_n *denoted by* $S^3_{f\Delta}$*, such that*

$$S^3_{f\Delta}(x_i) = f(x_i), i = \overline{0,n}$$

and

$$\left(S^3_{f\Delta}\right)''(x_0) = 0 = \left(S^3_{f\Delta}\right)''(x_n).$$

Definition 1.2. *The inner product*

$$\left\langle f, g \right\rangle_n = \int_0^{2\pi} f^{(n)}(t) g^{(n)}(t) dt$$

is defined for functions f *and* g *which have a square-integrable* n*-th derivative on segment* $[0, 2\pi]$*. We define the pseudo-norm* $\|f\|_n = \sqrt{\left\langle f, f \right\rangle_n}$ *on linear spaces* $C^n[0, 2\pi]$ *where* $\|f\|_n = 0$ *iff* $f(t), t \in [0, 2\pi]$ *is polynomial up to order* $n-1$ *[2].*

2. Some examples of splines

Example 1. B^0- *splines. These functions are splines of order* 1 *and* B^1- *splines: these functions are splines of order* 1 *and reach a peak at* $x = x_{i+1}$ *and is upward (downward) sloping for* $x < x_{i+1} (x > x_{i+1})$ *[3].*

$$B^0_i = \begin{cases} 1 & x_i < x < x_{x+1} \\ 0 & elsewhere \end{cases}$$

and

$$B^1_i = \begin{cases} \frac{x - x_i}{x_{i+1} - x_i} & x_i < x < x_{i+1} \\ \frac{x_{i+2} - x}{x_{i+1} - x_i} & x_{i+1} < x < x_{i+2} \\ 0 & elsewhere \end{cases} .$$

Example 2. *Higher order spline functions are defined by the recursion:*

$$B_i^n(x) = \Big(\frac{x - x_i}{x_{i+1} - x_i}\Big) B_i^{n-1}(x) + \Big(\frac{x_{i+n+1} - x}{x_{i+n+1} - x_{i+1}}\Big) B_{i+1}^{n-1}(x).$$

Example 3. *Cubic splines. These functions are splines of order three and its analytic expression is*

$$S^3_{f\Delta} = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

for $x \in [x_i, x_{i+1}]$ where

$$a_i = f(x_i) = f_i, b_i = \frac{f_{i+1} - f_i}{h_{i+1}} - \frac{h_{i+1}}{6}(2\psi_i + \psi_{i+1}), c_i = \frac{\psi_i}{2}, d_i = \frac{\psi_{i+1} - \psi_i}{6h_{i+1}}$$

$\psi_i = \left(S^3_{f\Delta}\right)''(x_i)$ *are solutions of system of linear equations*

$$\begin{cases} \mu_1\psi_0 + 2\psi_1 + \nu_1\psi_2 = \lambda_1 \\ \mu_2\psi_1 + 2\psi_2 + \nu_2\psi_3 = \lambda_2 \\ \vdots \\ \mu_{n-1}\psi_{n-2} + 2\psi_{n-1} + \nu_{n-1}\psi_n = \lambda_{n-1} \end{cases}$$

as amended by the given boundary conditions,

$$h_i = x_i - x_{i-1}, \mu_i = \frac{h_i}{h_i + h_{i+1}}, \nu_i = 1 - \mu_1, \lambda_i = 6f[x_{i-1}, x_i, x_{i+1}],$$

where $f[x_{i-1}, x_i, x_{i+1}]$ is second order divided difference.

Example 4. *A cubic periodic spline $S^3_{f\Delta}$ on segment $[0, 2\pi]$ segment is a spline function of order three such that $\left(S^3_{f\Delta}\right)^{(k)}(0) = \left(S^3_{f\Delta}\right)^{(k)}(2\pi), k = 0, 1, 2,$ where $\Delta = \{0 = x_0 < x_1 < ... < x_n = 2\pi\}$ and $f(x) \in C^2[0, 2\pi]$ is $2\pi-$periodic [4].*

Note: The interpolating function $S^3_{f\Delta}$ minimizes the value $\int_0^{2\pi}(g''(t))^2 \mathrm{d}t$ among all functions $g \in C^2$ which coincide with function $f(x)$ at the points $x_i, i = \overline{0, n}$.

3. Main Results

Theorem 3.1. *If $f(x) \in C^2[0, 2\pi]$ is 2π-periodic then for some c intermediate to x_{i-1}, x_i and x_{i+1}*

$$-4.5 - \frac{(f''(c))^2}{2} \le (S^3_{f\Delta})''(x_i) \le 4.5 + \frac{(f''(c))^2}{2}, i = \overline{1, n-1},$$

where $\Delta = \{0 = x_0 < x_1 < ... < x_n = 2\pi\}$.

Proof. Let us take $\psi_k = \max_{1\leq i\leq n-1}|\psi_i|$. Then using Example 1 we get

$$\max_{1\leq i\leq n-1}|\lambda_i| \geq |\lambda_k| \geq 2|\psi_k| - \mu_k|\psi_{k-1}| - \nu_k|\psi_{k+1}| \geq$$

$$\geq 2|\psi_k| - \mu_k|\psi_k| - \nu_k|\psi_k| = |\psi_k|(2-(\mu_k+\nu_k)) =$$

$$= |\psi_k|(2-1) = |\psi_k| = \max_{1\leq i\leq n-1}|\lambda_i|.$$

Consequently

$$\max_{1\leq i\leq n-1}\left|\left(S^3_{f\Delta}\right)''(x_i)\right| \leq \max_{1\leq i\leq n-1}|\lambda_i|.$$

Now from obtained result, the Arithmetic-Geometric inequality and the Mean-Value theorem as applied to the second order difference assure that

$$\max_{1\leq i\leq n-1}\left|\left(S^3_{f\Delta}\right)''(x_i)\right| \leq \max_{1\leq i\leq n-1}\left|6f[x_{i-1},x_i,x_{i+1}]\right| =$$

$$= 6\frac{f''(c)}{2} \leq \frac{9+(f''(c))^2}{2}, i = \overline{1,n-1},$$

where c intermediate to x_{i-1}, x_i and x_{i+1}.
Consequently

$$\left|\left(S^3_{f\Delta}\right)''(x_i)\right| \leq \frac{9+(f''(c))^2}{2}, i = \overline{1,n-1},$$

which yields to our result.

□

Theorem 3.2. *If the Fourier series of periodic function* $f(x) \in C^2[-\pi,\pi]$ *contains only cosine terms, then the Fourier series of the interpolating spline* $S^3_{f\Delta}$*, where*

$$\Delta = \{-\pi = x_{-n} < x_{-(n-1)} < \ldots < x_{-1} < x_0 < x_1 < \ldots < x_{n-1} < x_n = \pi\}$$

such that $|x_{-i}| = |x_i|, i = \overline{1,n}$*, also contains only cosine terms.*

Proof. Let $(S^3_{f\Delta})^* : [-\pi,\pi] \to \mathbb{R}, (S^3_{f\Delta})^* \in C^2[-\pi,\pi]$ be another function such that $S^3_{f\Delta}(-x) = (S^3_{f\Delta})^*(x)$ and $f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$ where $a_k, k = 1,2,..$ are Fourier coefficients of function $f(x) \in C^2[-\pi,\pi]$.
Now we have

$$(S^3_{f\Delta})^*(x_i) = S^3_{f\Delta}(x_{-i}) = f(x_{-i}) = f(-x_i) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k(-x_i) =$$

$$= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx_i = f(x_i), i = \overline{1,n}.$$

Consequently $(S^3_{f\Delta})^* : [-\pi, \pi] \to \mathbb{R}$ represents a spline for function $f(x)$ and from the uniqueness of the interpolating spline we get that$(S^3_{f\Delta})^* = S^3_{f\Delta}$ respectively $S^3_{f\Delta}(x_{-i}) = S^3_{f\Delta}(-x_i) = S^3_{f\Delta}(x_i), i = \overline{1,n}$, which implies that the Fourier series of the spline contains only cosine terms. □

Theorem 3.3. *If the Fourier series of periodic function $f(x) \in C^2[-\pi, \pi]$ contains only sine terms, then the Fourier series of the interpolating spline $S^3_{f\Delta}$, where*

$$\Delta = \{-\pi = x_{-n} < x_{-(n-1)} < ... < x_{-1} < x_0 < x_1 < ... < x_{n-1} < x_n = \pi\}$$

such that $|x_{-i}| = |x_i|, i = \overline{1,n}$, also contains only sine terms.

Proof. Let $(S^3_{f\Delta})^* : [-\pi, \pi] \to \mathbb{R}, (S^3_{f\Delta})^* \in C^2[-\pi, \pi]$ be another function such that $(S^3_{f\Delta})^*(x) = -S^3_{f\Delta}(-x)$ and $f(x) = \sum_{k=1}^{\infty} a_k \sin kx$ where $a_k, k = 1, 2, ..$ are Fourier coefficients of function $f(x) \in C^2[-\pi, \pi]$.
Now we have

$$-(S^3_{f\Delta})^*(x_i) = S^3_{f\Delta}(x_{-i}) = f(x_{-i}) = f(-x_i) = \sum_{k=1}^{\infty} a_k \sin k(-x_i) =$$

$$= -\sum_{k=1}^{\infty} a_k \sin kx_i = -f(x_i), i = \overline{1,n}.$$

Consequently $(S^3_{f\Delta})^* : [-\pi, \pi] \to \mathbb{R}$ represents a spline for function $f(x)$ and from the uniqueness of the interpolating spline we get that $(S^3_{f\Delta})^* = S^3_{f\Delta}$ respectively $S^3_{f\Delta}(x_{i-1}) = S^3_{f\Delta}(-x_i) = -S^3_{f\Delta}(x_i), i = \overline{1,n}$, which implies that the Fourier series of the spline contains only sine terms. □

Theorem 3.4. *Let $f(x) \in C^2[0, 2\pi]$ and the spline $s \in S^n_3$, be its interpolant, i.e. $s(x_i) = f(x_i), i = \overline{1,n}$. If f and s satisfy the boundary conditions $s'(0) = f'(0), s'(2\pi) = f'(2\pi), s''(0) = f''(0)$ and $s''(2\pi) = f''(2\pi)$, then*

$$(\|f - s\|_2)^2 = (\|f\|_2 - \|s\|_2)(\|f\|_2 + \|s\|_2).$$

Proof. We have that $(\|f - s\|_2)^2 = (\|f\|_2 - \|s\|_2)(\|f\|_2 + \|s\|_2) - 2\left\langle f - s, s\right\rangle_2$.
Since $f(x) \in C^2[0, 2\pi]$ and $s \in C^2[a, b]$ has continuous derivatives of order 2, by successive integrations, using the boundary conditions and since $s'''(x) = 0$ we

find that

$$\begin{aligned}\Big\langle f-s,s\Big\rangle_2 &= -\int_0^{2\pi}\Big(f(x)-s(x)\Big)s'''(x)\mathrm{d}x\\ &= -\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\Big(f(x)-s(x)\Big)s'''(x)\mathrm{d}x\\ &= -\sum_{i=1}^{n}\Big(f(x)-s(x)\Big)s'''(x)\mathrm{d}x\Big|_{x_{i-1}}^{x_i}=0.\end{aligned}$$

Therefore

$$(\|f-s\|_2)^2=(\|f\|_2-\|s\|_2)(\|f\|_2+\|s\|_2).$$

□

Theorem 3.5. *Let* $f(x)=0$ *and the spline* $s\in S_3^n$ *be its interpolant, i.e.* $s(x_i)=f(x_i), i=\overline{1,n}$. *If* f *and* s *satisfy the boundary conditions* $s'(0)=f'(0)$, $s'(2\pi)=f'(2\pi), s''(0)=f''(0)$ *and* $s''(2\pi)=f''(2\pi)$, *then* $s=0$.

Proof. For $f(x)=0$, from Theorem 3.4, we have:

$$(\|0-s\|_2)^2=0-(\|s\|_2)^2\Rightarrow(\|s\|_2)^2=0\Rightarrow\|s\|_2=0.$$

Now from the boundary conditions since $s^{(i)}(0)=s^{(i)}(2\pi)=0,\ i=1,2\Rightarrow s=0$. □

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