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CERTAIN INTEGRALS AND SERIES EXPANSIONS INVOLVING MODIFIED GENERALIZED I-FUNCTION OF PRASAD

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ABSTRACT. In the present paper, we are evaluating two finite single integrals involving the product of Legendre functions, generalized hypergeomeric functions and the modified generalized of multi variable I-function. These integrals are employed to evaluate two finite double integrals. Further these integrals(single and double) has been applied to to establish Fourier series and two Fourier- Legendre series expansions for the modified generalized of multi variable I-function. And finally, at the end of this paper several remarks has been discussed.

1. INTRODUCTION

Raghunayk Mishra, et.al [9, 11], established finite integral and double integrals involving I-function of two variables(IFTV). Later, Raghunayak Mishra [10], applied the same to the drug distribution in the body.

In [2,3,5], B.Satyanarayana, et.al obtained a solution to boundary value problem and also Laplace and Mellin's Transforms of the product of Struve's function and IFTV. In [4], B.Satyanarayana, et.al obtained Euler-Beta transform of IFTV. Y. Pragathi Kumar, et.al [15] obtained transforms of extended general class of polynomials and I-function. Prasad and Singh [8] defined the multivariable H-function. Later Prasad [7] have studied the multivariable I-function. Let us

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assume that $\mathbb{R}$, $\mathbb{C}$ be set of real and complex numbers and $\mathbb{N}$ be the set of positive integers. Also $\mathbb{N}_0 = \{0\} \bigcup \mathbb{N}$. First, we define the generalized modified multivariable I-function. We used the integral representation about this function. We will note it as

$$I[Z_1, ..., Z_r] = \tag{1.1}$$

$$I^{0,n_2;0,n_3;....;0,n_r:\left|R^1:m^1,n^1;....;m^{(r)},n^{(r)}\right.}_{p_2,q_2;p_3,q_3;....;p_r,n_r:\left|R:p^1,q^1;....;p^{(r)},q^{(r)}\right.} \left[\begin{array}{c} Z_1 \\ \vdots \\ Z_r \end{array}\right| \begin{array}{l} \left(a_{2j};\alpha^1_{2j},\alpha^{11}_{2j}\right)_{1,p_2};\left(\alpha_{3j};\alpha^1_{3j},\alpha^{11}_{3j},\alpha^{111}_{3j}\right)_{1,p_3}; \\ \\ \left(b_{2j};\beta^1_{2j},\beta^{11}_{2j}\right)_{1,q_2};\left(\beta_{3j};\beta^1_{3j},\beta^{11}_{3j},\beta^{111}_{3j}\right)_{1,q_3}; \end{array}$$

$$\left.\begin{array}{c} ...;\left(a_{rj};\alpha^1_{rj},....,\alpha^{(r)}_{rj}\right)_{1,p_r}:\left(e_j;u^{'}_j g^{'}_j,....,u^{(r)}_j g^{(r)}_j\right)_{1,R'}:\left(a^{'}_j,\alpha^{'}_j\right)_{1,p^{(1)}},\left(a^{(r)}_j,\alpha^{(r)}_j\right)_{1,p^{(r)}} \\ \vdots \\ ...;\left(b_{rj};\beta^1_{rj},....,\beta^{(r)}_{rj}\right)_{1,q_r}:\left(l_j;U^{'}_j f^{'}_j,....,U^{(r)}_j f^{(r)}_j\right)_{1,R}:\left(b^{'}_j,\beta^{'}_j\right)_{1,q^{(1)}},\left(b^{(r)}_j,\beta^{(r)}_j\right)_{1,q^{(r)}} \end{array}\right]$$

$$= \frac{1}{(2\pi\, w)^r}\int_{L_1}....\int_{L_r}\xi(s_1,....,s_r)\prod_{i=1}^{r}\phi(s_i)\, {z_i}^{s_i} ds_1 ....ds_r\,,$$

where

$$\xi(s_1,....,s_r) = \frac{\prod_{j=1}^{n_2}\Gamma\left(1-a_{2j}+\sum_{i=1}^{2}\alpha^{(i)}_{2j}s_i\right)\prod_{j=1}^{n_3}\Gamma\left(1-a_{3j}+\sum_{i=1}^{3}\alpha^{(i)}_{3j}s_i\right)}{\prod_{j=n_2+1}^{p_2}\Gamma\left(a_{2j}-\sum_{i=1}^{2}\alpha^{(i)}_{2j}s_i\right)\prod_{j=n_3+1}^{p_3}\Gamma\left(a_{3j}-\sum_{i=1}^{3}\alpha^{(i)}_{3j}s_i\right)}$$

$$\times\frac{...\prod_{j=1}^{n_r}\Gamma\left(1-a_{rj}+\sum_{i=1}^{r}\alpha^{(i)}_{rj}s_i\right)}{\prod_{j=n_r+1}^{p_r}\Gamma\left(a_{rj}-\sum_{i=1}^{r}a^{(i)}_{rj}s_i\right)\prod_{j=1}^{q_2}\Gamma\left(1-b_{2j}+\sum_{i=1}^{2}\beta^{(i)}_{2j}s_i\right)}$$

$$\times\frac{\prod_{j=1}^{R'}\Gamma\left(e_j+\sum_{i=1}^{r}u^{(i)}_j g^{(i)}_j s_i\right)}{\prod_{j=1}^{R}\Gamma\left(l_j+\sum_{i=1}^{r}U^{(i)}_j f^{(i)}_j s_i\right)\prod_{j=1}^{q_3}\Gamma\left(1-b_{3j}+\sum_{i=1}^{3}\beta^{(i)}_{3j}s_i\right)....\prod_{j=1}^{q_r}\Gamma\left(1-b_{rj}+\sum_{i=1}^{r}\beta^{(i)}_{rj}s_i\right)}$$

$$\phi(s_i)=\frac{\prod_{j=1}^{n^{(i)}}\Gamma\left(1-a_j^{(i)}-\alpha_j^{(i)}s_i\right)\prod_{j=1}^{m^{(i)}}\Gamma\left(b_j^{(i)}-\beta_j^{(i)}s_i\right)}{\prod_{j=1+n^{(i)}}^{p^{(i)}}\Gamma\left(a_j^{(i)}-\alpha_j^{(i)}s_i\right)\prod_{j=1+m^{(i)}}^{q^{(i)}}\Gamma\left(1-b_j^{(i)}-\beta_j^{(i)}s_i\right)}$$

$g_j^{(i)}(i=1,...,r;j=1,...,R)$, $f_J^{(i)}(i=1,...,r;j=1,...,R')$, $\alpha_j^{(i)},\beta_j^{(i)},\alpha_{kj}^{(i)},\beta_{kj}^{(i)}(i=1,...,r;k=1,...,r)$ are positive numbers. $e_j(j=1,...,R'),l_j(j=1,...,R)$, $a_j^{(i)},b_j^{(i)}(i=1,...,r);\alpha_{kj}^{(i)},\beta_{kj}^{(i)}(k=2,...,r)$ are complex and here $m^{(i)},n^{(i)},p^{(i)},q^{(i)}$ $(i=1,...,r);m_k,p_k,q_k(k=2,...,r)$ are non-negative integers, where $0\le q_k;0\le m^{(i)}\le q^{(i)};0\le n^{(i)}\le p^{(i)}(i=1,...,r)$ and $0\le n_k\le p_k$.

Here 'i' represents the number of dashes. The contour L_k is in s_k-plane, where $k=1,...,r$ which lies from $\sigma-i\infty$ to $\sigma+i\infty$, where σ is real with the loop. If necessary to ensure that the poles of $\Gamma\left(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k\right)$, for all $j=1,...,n_2$; $\Gamma\left(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k\right)$; for all $j=1,...,n_3$; $\Gamma\left(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k\right)$ for all $j=1,...,n_r$; $\Gamma\left(1-a_j^{(k)}-\alpha_j^{(k)}s_k\right)$: $j=1,...,n^{(k)}$ and $k=1,...,r$ are in left of the contour L_k and the poles of $\Gamma\left(b_j^{(k)}-\beta_j^{(k)}s_k\right)$: $(j=1,...,m^{(k)})$ and $(k=1,...,r)$ are in right of the contour L_k. See [8], for further details and asymptotic expansion of the I-function. Also assume that poles of the integrand are simple and the pole of $\Gamma\left(e_j+\sum_{i=1}^{r}u_j^{(i)}g_j^{(i)}s_i\right)$ lies to the left or right of it according to $u_j^{(i)}$ is positive or negative. The point $Z_i=0$ for all $i=1,...,r$, being tacitly excluded. The contour integral converges absolutely if $|\arg z_i|<\frac{1}{2}\Omega_i\pi$, where

(1.2) $$\Omega_i=\sum_{k=1}^{n^{(i)}}\alpha_k^{(i)}-\sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha_k^{(i)}+\sum_{k=1}^{m^{(i)}}\beta_k^{(i)}-\sum_{k=m^{(i)}+1}^{q^{(i)}}\beta_k^{(i)}+\sum_{k=1}^{n_2}\alpha_{2k}^{(i)}-\sum_{k=n_2+1}^{p_2}\alpha_{2k}^{(i)}$$

$$+\sum_{k=1}^{n_3}\alpha_{3k}^{(i)}-\sum_{k=n_3+1}^{p_3}\alpha_{3k}^{(i)}+....+\sum_{k=1}^{n_r}\alpha_{rk}^{(i)}-\sum_{k=n_r+1}^{p_r}\alpha_{rk}^{(i)}$$

$$-\sum_{k=1}^{q_2}\beta_{2k}^{(i)}-\sum_{k=1}^{q_3}\beta_{3k}^{(i)}\cdots-\sum_{k=1}^{q_r}\beta_{rk}^{(i)}+\sum_{j=1}^{R'}g_j^{(i)}-\sum_{j=1}^{R}f_j^{(i)}>0\text{ (i=1,...,r).}$$

We note

$$\begin{aligned}
\mathrm{A} &= (a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};....;\left(a_{(r-1)j};\alpha'_{(r-1)j},....,\alpha^{r-1}_{(r-1)j}\right)_{1,p_{r-1}}\\
\mathrm{B} &= \left(b_{2j};\beta'_{2j},\beta''_{2j}\right)_{1,q_2};....;\left(b_{(r-1)j};\beta'_{(r-1)j},....,\beta^{r-1}_{(r-1)j}\right)_{1,q_{r-1}}\\
\mathbf{A} &= \left(a_{rj};\alpha'_{rj},....,\alpha^{(r)}_{rj}\right)_{1,p_r};\Im=(a'_j,\alpha'_j)_{1,p'};....;\left(a^{(r)}_j,\alpha^{(r)}_j\right)_{1,p^{(r)}}\\
\mathbf{B} &= \left(b_{rj};\beta'_{rj},....,\beta^{(r)}_{rj}\right)_{1,q_r};\Re=\left(b'_j,\beta'_j\right)_{1,q'};....;\left(b^{(r)}_j,\beta^{(r)}_j\right)_{1,q^{(r)}}\\
\mathrm{E} &= \left(e_j;u'_jg'_j,....,u^{(r)}_jg^{(r)}_j\right)_{1,R'};L=\left(l_j;U'_jf'_j,....,U^{(r)}_jf^{(r)}_j\right)_{1,R}\\
U &= p_2,q_2;p_3,q_3;...;p_{r-1},q_{r-1}\,;\ V=0,n_2;0,n_3;...;0,n_{r-1}\\
Y &= (p',q')\,;....\left(p^{(r)},q^{(r)}\right);\ X=(m',n'):....;\left(m^{(r)},n^{(r)}\right)
\end{aligned}$$

$F_1(z) = {}_UF_V\left[\begin{matrix}(A_U)_s\\(B_V)_s\end{matrix}|Z\right]$, $F_2(z) = {}_{U'}F_{V'}\left[\begin{matrix}(A_{U'})_t\\(B_{V'})_t\end{matrix}|Z\right]$ and $F(s) = \dfrac{(A_U)_sC^s}{(B_V)_ss!}$; $G(t)=\dfrac{(A_{U'})_tD^t}{(B_{V'})_tt!}$. Now, in the following sections we are going to evaluate single and double finite integrals. Later Fourier-Legendre series and double Fourier-Legendre series are also established.

2. SINGLE FINITE INTEGRALS

In this section, it has been evaluated two finite single integrals.

Theorem 2.1.

$$\int_{-1}^{1}(1-x^2)^{\sigma-1}P_N^M(x)\,I\left[z_1\left(1-x^2\right)^{-a_1},....,z_r\left(1-x^2\right)^{-a_r}\right] \tag{2.1}$$

$$\times F_1\left[c\left(1-x^2\right)^h\right]F_2\left[d\left(1-x^2\right)^k\right]dx=\frac{2^M\pi}{\Gamma\left(\frac{N-M}{2}+1\right)\Gamma\left(\frac{1-N-M}{2}\right)}$$

$$\times\sum_{s,t=0}^{\infty}F(s)G(t)I^{V;0,m_r+2,n_r:|R':X}_{U;p_r+2,q_r+2:|R:Y}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\left|\begin{matrix}A;(1+\sigma+\frac{N}{2}+hs+kt;a_1,....,a_r),\\ \cdot\\ B;\mathbf{B},(1+\sigma+\frac{M}{2}+hs+kt;a_1,....,a_r),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(\sigma-\frac{N}{2}+hs+kt;a_1,....,a_r):\mathbf{A}:E:\Im\\ \cdot\\ (\sigma-\frac{M}{2}+hs+kt;a_1,....,a_r):L:\Re\end{matrix}\right]$$

We can establish the asymptotic expansion in a convenient form using the principle of Braaksma([1],p.278) as follows : $a_i > 0; i = 1, ..., r$;

$$\mathrm{Re}(\sigma + hs + kt) - \sum_{i=1}^{r} a_i \min_{\substack{1\le j\le m_i \\ 1\le k\le m^{(i)}}} \mathrm{Re}\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} \frac{b_{hj}}{\beta_{hj}^{h'}} + \frac{b_k^{(i)}}{\beta_k^{(i)}}\right) > |\mathrm{Re}(M)|$$

$\left|\arg(z_i(1-x^2)^{a_i}\right| < \frac{1}{2}\Omega_i\pi$ *where* Ω_i *is given in* (1.2).

Proof. To derive (2.1), expressing the generalized hyper geometric functions in terms of series summation ([12], p.73 Eq.2), and the generalized multivariable I-function as a contour using (1.1) and changing the order of integrations (which is justified by the conditions mentioned above), we get

$$\sum_{s,t=0}^{\infty} F(s)G(t)\frac{1}{(2\pi\, w)^r}\int_{L_1}....\int_{L_r}\xi(s_1,....,s_r)\prod_{k=1}^{r}\varphi_k(s_k) \tag{2.2}$$

$$\times {z_k}^{s_k}\left[\int_{-1}^{1}(1-x^2)^{\sigma+hs+kt-\sum\limits_{i=1}^{r}a_is_i}P_N^M(x)dx\right]ds_1....ds_r\,.$$

Later, evaluating the inner integral by using ([6], eq.16,p.316) and then interchanging the result with (1.1), we obtain the desired result. □

Theorem 2.2.

$$\int_{-1}^{1}(1-x^2)^{\sigma-1}P_N^M(x)F_1\left[c\left(1-x^2\right)^{-h}\right]F_2\left[d\left(1-x^2\right)^{-k}\right] \tag{2.3}$$

$$\times I\left[z_1\left(1-x^2\right)^{a_1},....,z_r\left(1-x^2\right)^{a_r}\right]dx = \frac{2^M\pi}{\Gamma\left(\frac{N-M}{2}+1\right)\Gamma\left(\frac{1-N-M}{2}\right)}$$

$$\times\sum_{s,t=0}^{\infty}F(s)G(t)I_{U;p_r+2,q_r+2:|R:Y}^{V;0,m_r,n_r+2:|R':X}\left[\begin{array}{c|c} z_1 & A;(1-\sigma-\frac{M}{2}+hs+kt;a_1,....,a_r), \\ \vdots & . \\ z_r & B;B,(-\sigma+\frac{N}{2}+hs+kt;a_1,....,a_r), \end{array}\right.$$

$$\left.\begin{array}{c} (1-\sigma+\frac{M}{2}+hs+kt;a_1,....,a_r):A:E:\Im \\ . \\ (-\sigma-\frac{N}{2}+hs+kt;a_1,....,a_r):L:\Re \end{array}\right]$$

Provided $a_i > 0; i = 1, ..., r$*;*

$$\mathrm{Re}(\sigma - hs - kt) + \sum_{i=1}^{r} a_i \min_{\substack{1 \leq j \leq m_i \\ 1 \leq k \leq m^{(i)}}} \mathrm{Re}\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} \frac{a_{hj}-1}{\alpha_{hj}^{h'}} + \frac{a_k^{(i)}-1}{\alpha_k^{(i)}}\right) > |\mathrm{Re}(M)|$$

$\left|\arg(z_i(1-x^2)^{-a_i}\right| < \frac{1}{2}\Omega_i\pi$, *where* Ω_i *is defined by* (1.2). *Similarly,* (2.3) *can be established.*

3. Double Finite Integrals

In this section, we establish two double finite integrals.

Theorem 3.1.

$$\int_{-1}^{1}\int_{-1}^{1} (1-x^2)^{\sigma_1 - 1}(1-y^2)^{\sigma_2-1} P_{N_1}^{M_1}(x) P_{N_2}^{M_2}(y) F_1\left[c\left(1-x^2\right)^h\right] \tag{3.1}$$

$$\times F_2\left[d\left(1-x^2\right)^k\right] F_1\left[c\left(1-y^2\right)^h\right] F_2\left[d\left(1-y^2\right)^k\right]$$

$$\times I\left[z_1\left(1-x^2\right)^{-a_1}\left(1-y^2\right)^{-b_1}, ..., z_r\left(1-x^2\right)^{-a_r}\left(1-y^2\right)^{-b_r}\right] dx\, dy$$

$$= \frac{2^{M_1+M_2}\pi^2}{\Gamma\left(\frac{N_1-M_1}{2}+1\right)\Gamma\left(\frac{1-N_1-M_1}{2}\right)\Gamma\left(\frac{N_2-M_2}{2}+1\right)\Gamma\left(\frac{1-N_2-M_2}{2}\right)}$$

$$\times \sum_{s_1,t_1.s_2,t_2=0}^{\infty} F(s_1)G(t_1)F(s_2)G(t_2) I_{U;p_r+4,q_r+4:|R:Y}^{V;0,m_r+4,n_r:|R':X}\left[\begin{array}{c|c} z_1 & A; A_1(N_1,N_2) : \mathrm{A} : E : \Im \\ \vdots & \cdot \\ z_r & B; \mathrm{B}; B_1(M_1,M_2) : L : \Re \end{array}\right]$$

where

$$A_1(N_1,N_2) \tag{3.2}$$

$$= (1+\sigma_1+\frac{N_1}{2}+hs_1+kt_1; a_1, ..., a_r), (\sigma_1 - \frac{N_1}{2}+hs_1+kt_1; a_1, ..., a_r),$$

$$(1+\sigma_2+\frac{N_2}{2}+hs_2+kt_2; b_1, ..., b_r), (\sigma_2 - \frac{N_2}{2}+hs_2+kt_2; b_1, ..., b_r)$$

$$B_1(M_1,M_2) \tag{3.3}$$

$$= (\sigma_1+\frac{M_1}{2}+hs_1+kt_1; a_1, ..., a_r), (\sigma_1 - \frac{M_1}{2}+hs_1+kt_1; a_1, ..., a_r),$$

$$(\sigma_2+\frac{M_2}{2}+hs_2+kt_2;b_1,....,b_r),(\sigma_2-\frac{M_2}{2}+hs_2+kt_2;b_1,....,b_r)$$

We can establish the asymptotic expansion in a convenient form using the principle of Braaksma([1],p.278) as follows :

$$\mathrm{Re}(\sigma_1+hs_1+kt_1)-\sum_{i=1}^{r}a_i\min_{\substack{1\le j\le m_i\\1\le k\le m^{(i)}}}\mathrm{Re}\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}\frac{b_{hj}}{\beta_{hj}^{h'}}+\frac{db_k^{(i)}}{\beta_k^{(i)}}\right)>|\mathrm{Re}(M_1)|$$

and $\mathrm{Re}(\sigma_2+hs_2+kt_2)-\sum_{i=1}^{r}b_i\min_{\substack{1\le j\le m_i\\1\le k\le m^{(i)}}}\mathrm{Re}\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}\frac{b_{hj}}{\beta_{hj}^{h'}}+\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)>|\mathrm{Re}(M_2)|\ \forall a_i,b_i>0;i=1,....,r;$ $\left|\arg(z_i(1-x^2)^{-a_i}(1-y^2)^{-b_i}\right|<\frac{1}{2}\Omega_i\pi$, *where* Ω_i *is defined by* (1.2).

Proof. To establish (3.1), first evaluating the integral with respect to 'x' with the help of the theorem 1 and changing the order of integration and summation and then evaluating the integral with respect to 'y' with the help of theorem (2.1), we get the desired result (3.1). □

Theorem 3.2.

(3.4)

$$\int_{-1}^{1}\int_{-1}^{1}(1-x^2)^{\sigma_1-1}(1-y^2)^{\sigma_2-1}P_{N_1}^{M_1}(x)P_{N_2}^{M_2}(y)F_1\left[c\ \left(1-x^2\right)^{-h}\right]F_2\left[d\ \left(1-x^2\right)^{-k}\right]$$

$$\times F_2\left[d\ \left(1-x^2\right)^{-k}\right]F_1\left[c\ \left(1-y^2\right)^{-h}\right]F_2\left[d\ \left(1-y^2\right)^{-k}\right]$$

$$\times I\left[z_1\ \left(1-x^2\right)^{-a_1}\ \left(1-y^2\right)^{-b_1},..,z_r\ \left(1-x^2\right)^{-a_r}\ \left(1-y^2\right)^{-b_r}\right]dx\,dy$$

$$=\frac{2^{M_1+M_2}\pi^2}{\Gamma\left(\frac{N_1-M_1}{2}+1\right)\Gamma\left(\frac{1-N_1-M_1}{2}\right)\Gamma\left(\frac{N_2-M_2}{2}+1\right)\Gamma\left(\frac{1-N_2-M_2}{2}\right)}$$

$$\times\sum_{s_1,t_1.s_2,t_2=0}^{\infty}F(s_1)G(t_1)F(s_2)G(t_2)$$

$$\times I_{U;p_r+4,q_r+4:|R:Y}^{V;0,m_r+4,n_r:|R':X}\left[\begin{array}{c|c}z_1 & A;A_2(M_1,M_2):\mathrm{A}:E:\Im\\ \vdots & .\\ z_r & B;\mathrm{B};B_2(N_1,N_2):L:\Re\end{array}\right],$$

where

(3.5)
$$A_2(M_1,M_2)$$

$$=(1-\sigma_1-\frac{M_1}{2}+hs_1+kt_1;a_1,....,a_r),(1-\sigma_1+\frac{M_1}{2}+hs_1+kt_1;a_1,....,a_r),$$

$$(1-\sigma_2-\frac{M_2}{2}+hs_2+kt_2;b_1,....,b_r),(1-\sigma_2+\frac{M_2}{2}+hs_2+kt_2;b_1,....,b_r)$$

$$B_2(N_1,N_2)$$

$$=(-\sigma_1+\frac{N_1}{2}+hs_1+kt_1;a_1,....,a_r),(-\sigma_1-\frac{N_1}{2}+hs_1+kt_1;a_1,....,a_r),$$

$$(-\sigma_2+\frac{N_2}{2}+hs_2+kt_2;b_1,....,b_r),(-\sigma_2-\frac{N_2}{2}+hs_2+kt_2;b_1,....,b_r)$$

We can establish the asymptotic expansion in a convenient form using the principle of Braaksma([1],p.278) as follows :

$$\mathrm{Re}(\sigma_1-hs_1-kt_1)-\sum_{i=1}^{r}a_i\min_{\substack{1\le j\le m_i\\1\le k\le m^{(i)}}}\mathrm{Re}\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+\frac{a_k^{(i)}-1}{\alpha_k^{(i)}}\right)>|\mathrm{Re}(M_1)|$$

and

$$\mathrm{Re}(\sigma_2-hs_2-kt_2)-\sum_{i=1}^{r}b_i\min_{\substack{1\le j\le m_i\\1\le k\le m^{(i)}}}\mathrm{Re}\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+\frac{a_k^{(i)}-1}{\alpha_k^{(i)}}\right)>|\mathrm{Re}(M_2)|$$

$\forall a_i,b_i>0;i=1,....,r;$ $\left|\arg(z_i(1-x^2)^{a_i}(1-y^2)^{b_i}\right|<\frac{1}{2}\Omega_i\pi$, *where* Ω_i *is defined by* (1.2). *Similarly,* (3.4) *can be established.*

4. Fourier-Legendre Series

The Fourier-Legendre series are established here:

Theorem 4.1.

$$\int_{-1}^{1}(1-x^2)^{\sigma-1}P_N^M(x)F_1\left[c\left(1-x^2\right)^{-h}\right]F_2\left[d\left(1-x^2\right)^{-k}\right] \tag{4.1}$$

$$\times I\left[z_1\left(1-x^2\right)^{-a_1},....,z_r\left(1-x^2\right)^{-a_r}\right]dx$$

$$=2^M\pi\sum_{U=0}^{\infty}\frac{(2U+1)(U-M)!}{(U+M)!\Gamma(\frac{U-M}{2}+1)\Gamma(\frac{1-U-M}{2})}P_U^M(x)$$

$$\times\sum_{s,t=0}^{\infty}F(s)G(t)I_{U;p_r+2,q_r+2:|R:Y}^{V;0,m_r+2,n_r:|R':X}$$

$$\left[\begin{array}{c|c} z_1 & A;\mathrm{A},(1+\sigma+\frac{U}{2}+hs+kt;a_1,....,a_r),(\sigma-\frac{U}{2}+hs+kt;a_1,....,a_r):E:\Im \\ \vdots & . \\ z_r & B;(\sigma+\frac{M}{2}+hs+kt;a_1,....,a_r),(\sigma-\frac{M}{2}+hs+kt;a_1,....,a_r),\mathrm{B}:E:\Im \end{array}\right]$$

Provided that $M \leq U$ and the corresponding conditions stated in theorem (2.1) are satisfied.

Proof. To obtain (4.1), let

$$f(x) = (1-x^2)^{\sigma-1} I\left[z_1\left(1-x^2\right)^{-a_1},....,z_r\left(1-x^2\right)^{-a_r}\right] \tag{4.2}$$

$$\times F_1\left[c\left(1-x^2\right)^{-h}\right] F_2\left[d\left(1-x^2\right)^{-k}\right] = \sum_{U=0}^{\infty} C_U P_U^M(x)$$

Clearly, the above equation is valid, as it is bounded and continuous in (-1,1). On both sides of (4.2) multiplying with and integrating with respect to x from -1 to 1, using the theorem (2.1) and using the orthogonality of Legendre function ([6], p.279), we obtain the value of C_U in (4.2), and then we get the desired result. □

Theorem 4.2.

$$\int_{-1}^{1} (1-x^2)^{\sigma-1} P_N^M(x) F_1\left[c\left(1-x^2\right)^{-h}\right] F_2\left[d\left(1-x^2\right)^{-k}\right] \tag{4.3}$$

$$\times I\left[z_1\left(1-x^2\right)^{a_1},....,z_r\left(1-x^2\right)^{a_r}\right] dx$$

$$= 2^{M-1}\pi \sum_{U=0}^{\infty} \frac{(2U+1)(U-M)!}{(U+M)!\Gamma(\frac{U-M}{2}+1)\Gamma(\frac{1-U-M}{2})} P_U^M(x) \sum_{s,t=0}^{\infty} F(s)G(t) I_{U;p_r+2,q_r+2:|R:Y}^{V;0,m_r,n_r+2:|R':X}$$

$$\left[\begin{array}{c|c} z_1 & A;(1-\sigma-\frac{M}{2}+hs+kt;a_1,....,a_r),(1-\sigma+\frac{M}{2}+hs+kt;a_1,....,a_r), \\ : & . \\ z_r & B;\mathrm{B},(-\sigma-\frac{U}{2}+hs+kt;a_1,....,a_r),(-\sigma+\frac{U}{2}+hs+kt;a_1,....,a_r): \end{array}\right.$$
$$\left.\begin{array}{c} \mathrm{A}:E:\Im \\ . \\ L:\Re \end{array}\right]$$

Provided that $M \leq U$ and the corresponding conditions stated in theorem (2.2) are satisfied. Similarly, (4.3) *can be established.*

5. Double Fourier-Legendre Series

Theorem 5.1.

$$\int_{-1}^{1}\int_{-1}^{1}(1-x^2)^{\sigma_1-1}(1-y^2)^{\sigma_2-1}P_{N_1}^{M_1}(x)P_{N_2}^{M_2}(y)F_1\left[c\left(1-x^2\right)^h\right] \tag{5.1}$$

$$\times F_2\left[d\left(1-x^2\right)^k\right]F_1\left[c\left(1-y^2\right)^h\right]F_2\left[d\left(1-y^2\right)^k\right]$$

$$\times I\left[z_1\left(1-x^2\right)^{-a_1}\left(1-y^2\right)^{-b_1},..,z_r\left(1-x^2\right)^{-a_r}\left(1-y^2\right)^{-b_r}\right]dx\,dy=2^{M_1+M_2-2}\pi^2$$

$$\times\sum_{U_1,U_2=0}^{\infty}\frac{(2U_1+1)(2U_2+1)(U_1-M_1)!}{\Gamma\left(\frac{U_1-M_1}{2}+1\right)\Gamma\left(\frac{1-U_1-M_1}{2}\right)(U_1+M_1)!\Gamma\left(\frac{U_2-M_2}{2}+1\right)}$$

$$\times\frac{(U_2-M_2)!}{\Gamma\left(\frac{1-U_2-M_2}{2}\right)(U_2+M_2)!}\sum_{s_1,t_1.s_2,t_2=0}^{\infty}F(s_1)G(t_1)F(s_2)G(t_2)$$

$$\times I_{U;p_r+4,q_r+4:|R:Y}^{V;0,m_r+4,n_r:|R':X}\left[\begin{array}{c|c} z_1 & A;\mathrm{A},A_1(U_1,U_2):E:\Im \\ \vdots & . \\ z_r & B;B_1(M_1,M_2),\mathrm{B}:L:\Re \end{array}\right]P_{U_1}^{M_1}(x)P_{U_2}^{M_2}(y)$$

where $A_1(...)$, $B_1(...)$ are defined respectively by (3.2) *and* (3.3). *Provided that $M_1 \le U_1$, $M_2 \le U_2$ and the corresponding conditions stated in theorem (3.1) are satisfied.*

Proof. To obtain (5.1), let

$$f(x,y)=(1-x^2)^{\sigma_1-1}(1-y^2)^{\sigma_2-1}F_1\left[c\left(1-x^2\right)^h\right]F_2\left[d\left(1-x^2\right)^k\right] \tag{5.2}$$

$$\times F_1\left[c\left(1-y^2\right)^h\right]F_2\left[d\left(1-y^2\right)^k\right]$$

$$\times I\left[z_1\left(1-x^2\right)^{-a_1}\left(1-y^2\right)^{-b_1},...,z_r\left(1-x^2\right)^{-a_r}\left(1-y^2\right)^{-b_r}\right]$$

$$=\sum_{U_1,U_2=0}^{\infty}C_{U_1,U_2}P_{U_1}^{M_1}(x)P_{U_2}^{M_2}(y)$$

The above equation is valid, since f(x,y) is continuous and bunded in $(-1,1)\times(-1,1)$. Multiplying both sided of (5.2) by $P_{U_1}^{M_1}(x)P_{U_2}^{M_2}(y)$ and integrating with respect to x and y respectively from -1 to 1, using theorem (3.1) and the orthogonality property of Legendre function ([6], p.279), we obtain the value of C_{U_1,U_2} . Substituting the value of C_{U_1,U_2} in (5.2), we get the desired result. □

Theorem 5.2.

$$\int_{-1}^{1}\int_{-1}^{1}(1-x^2)^{\sigma_1-1}(1-y^2)^{\sigma_2-1}P_{N_1}^{M_1}(x)P_{N_2}^{M_2}(y)F_1\left[c\left(1-x^2\right)^{-h}\right] \tag{5.3}$$

$$\times F_2\left[d\left(1-x^2\right)^{-k}\right]F_1\left[c\left(1-y^2\right)^{-h}\right]F_2\left[d\left(1-y^2\right)^{-k}\right]$$

$$\times I\left[z_1\left(1-x^2\right)^{a_1}\left(1-y^2\right)^{b_1},..,z_r\left(1-x^2\right)^{a_r}\left(1-y^2\right)^{b_r}\right]dx\,dy=2^{M_1+M_2-2}\pi^2$$

$$\times\sum_{U_1,U_2=0}^{\infty}\frac{(2U_1+1)(2U_2+1)(U_1-M_1)!}{\Gamma\left(\frac{U_1-M_1}{2}+1\right)\Gamma\left(\frac{1-U_1-M_1}{2}\right)(U_1+M_1)!\Gamma\left(\frac{U_2-M_2}{2}+1\right)}$$

$$\times\frac{(U_2-M_2)!}{\Gamma\left(\frac{1-U_2-M_2}{2}\right)(U_2+M_2)!}\sum_{s_1,t_1.s_2,t_2=0}^{\infty}F(s_1)G(t_1)F(s_2)G(t_2)I_{U;p_r+4,q_r+4:|R:Y}^{V;0,m_r,n_r+4:|R':X}$$

$$\left[\begin{array}{c|c} z_1 & A;A_2(M_1,M_2),\mathrm{A}:E:\Im \\ \vdots & . \\ z_r & B;\mathrm{B},B_2(U_1,U_2):L:\Re \end{array}\right]P_{U_1}^{M_1}(x)P_{U_2}^{M_2}(y)$$

Where $A_2(...)$,$B_2(...)$ *are defined respectively by* (3.4) *and* (3.5). *Provided that* $M_1\leq U_1$, $M_2\leq U_2$ *and the corresponding conditions stated in theorem (3.2) are satisfied. On applying the above method,* (5.3) *can be established.*

6. Remark

We can obtain the same type of double integrals and Fourier-Legenddre, Fourier series H-function given by Prasad and Singh [8] and also for the I-function given by Prasad [7], reflecting the respective notations and validity conditions. These functions of several variables are extensions of the multivariable H-function defined by Srivastava and Panda [13, 14].

7. Conclusion

The I-function presented in this paper, is quite basic in its nature. Therefore, on specializing the parameters of this function, one may obtain different single and double Fourier-Legendre series expansions concerning a large variety of special functions of single variable and multi variables.

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