A REVIEW ON MATHEMATICAL MODELLING IN BIOLOGY AND MEDICINE

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ABSTRACT. In recent years, mathematical modelling has been playing an important role in many disciplines. In many fields such as physics, computer science, economics, ecology, biology, mathematics has been used as a tool. We will identify different types of growth models, compartment models and their role in the field of biology and medicine in this review article.

1. INTRODUCTION

Mathematical modelling is a division of mathematical logic or area which helps us in understanding real-life problems and formulating them to the mathematical models and interpreting the solutions to the real world. The action of developing a mathematical model is known as mathematical modelling. It is an interpretation of the system using various concepts and packages. Mathematics has been used in biology as early as the 12th century. Mathematical modelling in the field of biology becomes interdisciplinary research which acquires a knowledge both in mathematics and biology. Nowadays the research has pointed towards interdisciplinary and multidisciplinary research. Mathematical modelling has evolved in many areas such as medicine, dynamics, population biology, ecology, biostatistics, molecular biology, etc... The development

of models in biology is by the combination of both mathematicians and scientists from the field of botany, zoology, chemistry. Modelling can be represented by a diagrammatic way such that it can be easily understood.

2. MODELLING CYCLE

Modelling process can be easily emphasized by a cycle. The Kaiser (1995) and Blum (1996) [13, 14] has formulated a modelling cycle which helps us to understand the model pictorially as shown in Figure 1.
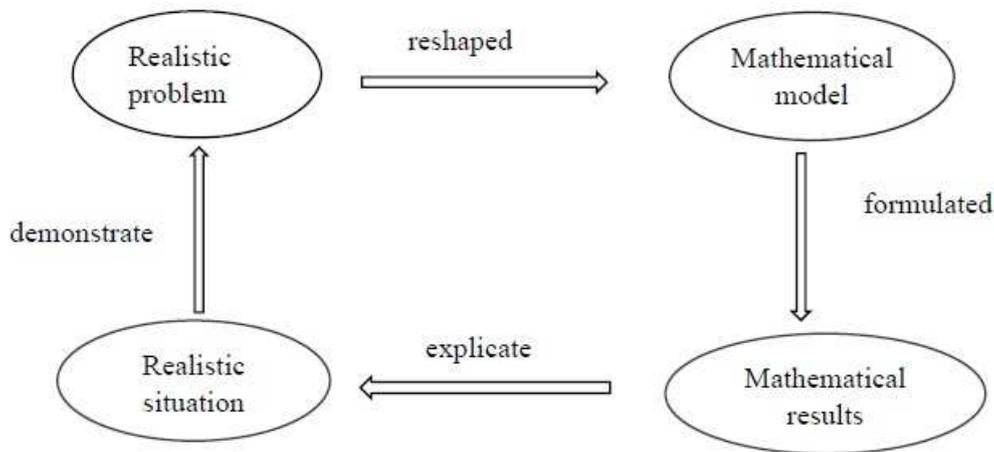

FIGURE 1. Modeling cycle

From the above Figure 1 we can understand that the reality is converted into a mathematical model by using differential equations then solving the equations using various methods and techniques we get a solution and interpret into the real world solution. Models such as various growth models, compartment models are explained below. Population growth models are used to forecast the upcoming population.

3. POPULATION GROWTH MODELS

In the later years of 18th century biologists began to develop many ideas and approaches in population modelling to emphasize dynamics of growing and shrinking ball populations of living organisms. Growth models are used in many

areas of biology. The population growth means the increase in the number of individuals in a population. By using the growth models we can predict the future number of the population. The organism's growth depends on the available resources within the system. The population size grows at an increasing rate when the resources are plentiful and it is modelled as a simple model of exponential growth. This model has a fixed net fertility rate per person. On the other hand, as resources are decreased for expansion, the rate of population growth decreases as population growth rises and is modelled by the logistic growth model. There are two important models of population growth that are based on the propagative process [1, 8], as shown in the Figure 2.
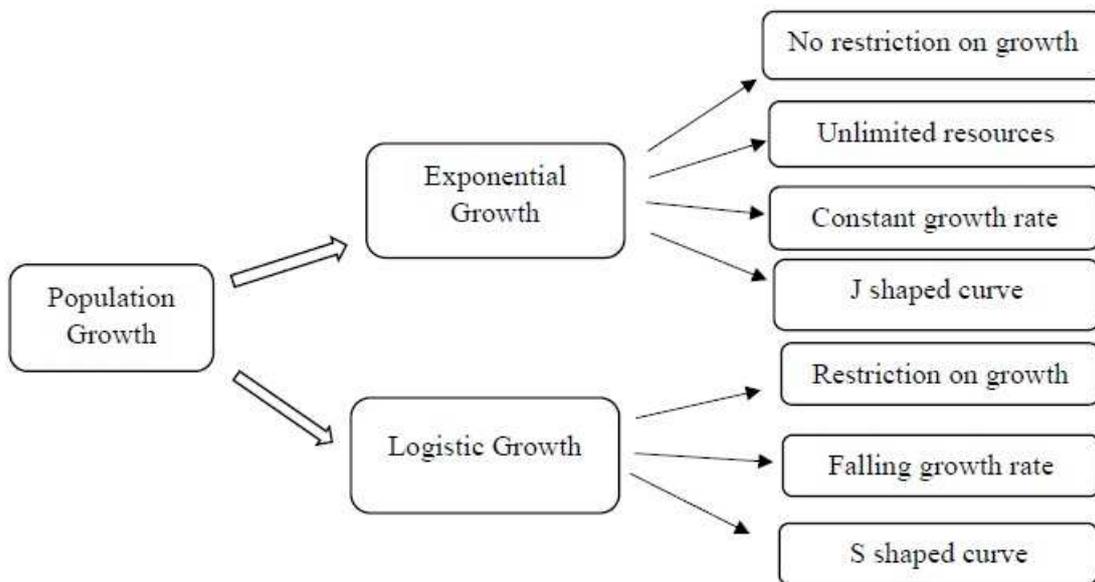

FIGURE 2. Growth Models

From the Figure 2 we can understand the features of the both exponential and logistic model. Now we are going to see some of the growth models.

3.1. **Malthus (Exponential) Growth Model.** In 1798 Thomas Malthus [7, 18, 24] formulated a mathematical model for populationgrowth. The model is based on the assumption that the population grows at a rate proportional to the size of the population. An exponential model has the form

$$\frac{dN}{dt} = rN$$

It has the solution

$$N(t) = N_0 e^{rt}$$

where r is the intrinsic growth rate. The model is used in radioactive decay and the amount of medicine in the bloodstream.

3.2. **Verhulst (Logistic) Growth Model.** In 1845 Verhulst [7, 18, 24] proposed a model which has a self-limiting process. This is applied when the population size is very high. The model is sometimes called a Verhulst model or logistic growth model. The model is given below

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

with solution

$$N(t) = \frac{N_0 K e^{rt}}{[K + N_0(e^{rt} - 1)]}$$

where r and K are desirable constants, and the carrying capacity is K. Here per capita, the birth rate is

$$r\left(1 - \frac{N}{K}\right)$$

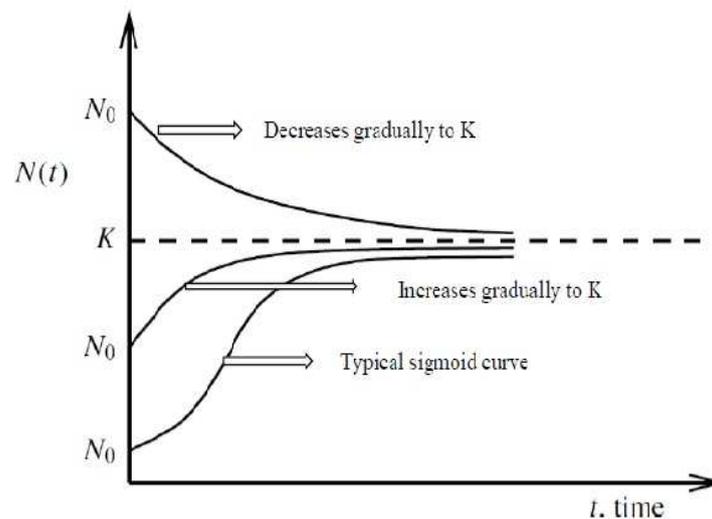When $t \to \infty$ in the solution we get $N(t) \to K$.

FIGURE 3. Logistic Growth

The performance of the exponential and logistic growth is represented in a pictorial form.
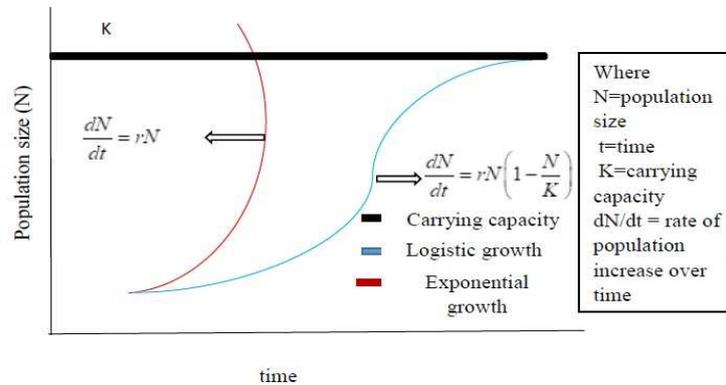
FIGURE 4. Exponential Vs Logistic

From the above Figure 5 we can identify the behaviour of both models. The exponential model grows exponentially, there is no restriction and the resources are abundant whereas in the logistic model the resources are limited. The initial part of the logistic curve is exponential and in the middle the growth rate decreases and at the end, it attains asymptote K which is the carrying capacity. This model is used in various fields in medicine modelling the growth of tumours, in chemistry reaction models, in physics fermi distribution, in agriculture modelling crop.

The following various growth models are based on the logistic growth model. We see how the models are formulated by the logistic growth equation.

3.3. **Torner (Generic) Growth Model.** Torner et al (1976) [7, 23] have proposed a model which is the modified form of the logistic growth model. The model is known as the generic growth model. It is represented as follows:

$$\frac{dN}{dt} = rN^{1+\beta(1-\gamma)}\left[1 - \left(\frac{N}{K}\right)^{\beta}\right]^{\gamma}$$

whereβ ,γ are positive and $\gamma < 1 + \frac{1}{\beta}$. Solving the above equation we get

$$N(t) = \frac{N}{\left[1 + \left[(\gamma-1)\beta r K^{\beta(1-\gamma)}t + \left[\left(\frac{K}{N_0}\right) - 1\right]^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\right]^{\frac{1}{\beta}}}$$

This model is used in the field of molecular biology for the growth of the bacteria namely Escherichia coli K-12.

3.4. **Von Bertalanffy Growth Model.** Von Bertalanffy [3, 7] proposed a model for the growth of fish which is the modified logistic growth equation. The proposed model is a kind of Bernoulli differential equation which is given as follows:

$$\frac{dN}{dt} = rN^{\frac{2}{3}}\left[1 - \left(\frac{N}{K}\right)^{\frac{1}{3}}\right]^{\frac{1}{3}}$$

The above equation leads to the solution

$$N(t) = K\left[1 + \left[1 - \left(\frac{N_0}{K}\right)^{\frac{1}{3}}\right]e^{\frac{1}{3}rtK^{\frac{1}{3}}}\right]^3$$

This model is commonly applied in ecology especially in modelling of fish growth.

3.5. **Blumberg Growth Model.** Blumberg (1968) [3, 5, 7] proposed the concept of growth that is the modification of the logistic model. He found that its turning point is the key restriction in the logistic tangent. He regarded the natural growth rate function as a time-independent function so that this trend would eventually reach a growth rate of zero, thereby making this curve an ultra-logistic curve. Thus, the proposed model he called as a hyper logistic function and the equation is as follows:

$$\frac{dN}{dt} = rN^{\alpha}\left[1 - \frac{N}{K}\right]^{\gamma}$$

This model is used for approximating data from population dynamics, biosatistics, Debugging theory and computer viruses propagation theory.

3.6. **Gompertz Growth Model.** Gompertz [3,7] proposed equation for the population dynamics and is given as follows:

$$\frac{dN}{dt} = \frac{r}{\beta\gamma}N\left[1 - \left(\frac{N}{K}\right)^{\beta}\right]^{\gamma}$$

Solving the Bernoulli differential equation we get

$$N(t) = K\exp\left\{\left[\ln\left(\frac{N_0}{K}\right)\right]^{1-\gamma} + \frac{r}{K^{\beta\gamma}}(-1)^{\gamma}(1-\gamma)t\right\}^{\frac{1}{1-\gamma}}$$

This model is used in the field of population biology for the growth of the tumour.

3.7. **Richard Growth Model.** Richard (1959) [3, 7] propound his model for population growth using the equation given below

$$\frac{dN}{dt} = rN \left[1 - \left(\frac{N}{K} \right)^{\beta} \right]$$

Solving this Bernoulli differential we have the solution as follows:

$$N(t) = K \left[1 - e^{\beta rt} \left[1 - \left(\frac{N_0}{K} \right)^{-\beta} \right] \right]^{\frac{1}{\beta}}$$

This model is often applied applied to fit empirical data on forest growth that is in the growth of plant.

The above types of growth models are obtained by the generalisation of the logistic growth model with different values of the parameter α, β, γ [1,7,8,22]. The pictorial representation of the above models with a generalisation of the logistic model is given as below.

D.Hathout (2013) [10] have proposed a new model known as the Hyperbolic growth model in which the accuracy of the real data in past gives good result. The author developed this model using the exponential growth model and says that whenever the model is developed it should be tested with real data such that the models behaviour can be known. The author concludes that the growth models are used to predict the population growth in future.

4. Compartment models

A compartment is a segment of the whole system. In Pharmacokinetics the compartment model is the mathematical description of the body where the body is represented as a series of compartments sorted either in series or in parallel. In Epidemiology [15] the behaviour of the infectious disease is mostly found on their compartment structure. It was initially formulated by Kermack and McKendrick.

M. A. Khanday (2016) [12, 16, 17] has formulated the compartment models for drug diffusion through oral and intravenous modes. The author says that depending upon the condition of patient and fact of a disease the drug can be
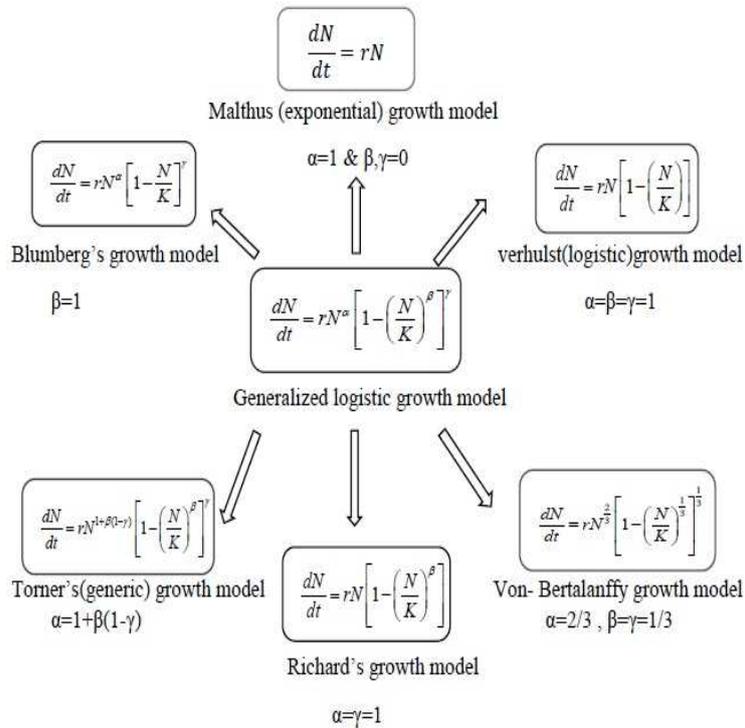
FIGURE 5. Generalized logistic growth model

specified in any mode. These models were formulated on Ficks perfusion principle, first-order kinetics and balance law. Here they have used two-compartment model for drug diffusion for both oral and intravenous route.

V. H. Badshah et.al (2013) [4, 6, 11] have demonstrated the historical feature of deterministic modelling over the infectious disease in population dynamics. The author discussed many aspects of infectious diseases and explained various model with and without the latent period. The author concludes that this study correlates the mathematical modelling and dynamical behaviour of infectious diseases.

K. Ergen, A. Cilli, N. Yahnioglu (2014) [9] have used a SIR model to predict some of the epidemic disease such as TB (tuberculosis), HIV, CCHF (Crimean - congohemorraghic fever). The author says by knowing the previous year data and by calculating the infection rate we can predict the number of infected ones. The author concludes that by using SIR model the results are obtained in short time.
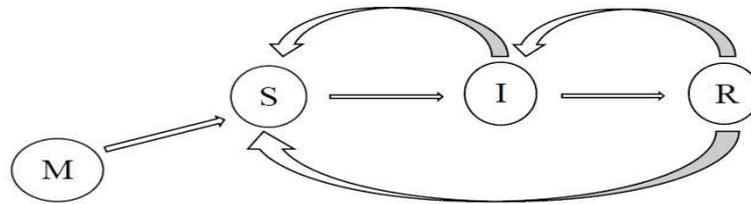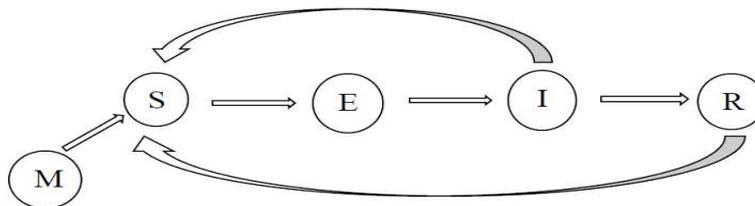
FIGURE 6. Models without latent period

FIGURE 7. Models with latent period

Putani Pongsumpun (2018) [19, 20] has formulated the SIR model for the effect of vaccination for the dengue disease. The author says that the conditions for the stability of disease are free and endemic equilibrium states are analysed using basic reproductive rate. The author concludes that the consumption of vaccination can reduce the ailment of disease.

G. Shabbir, H. Khanand and M. A. Sadiq (2010) [2, 12, 15, 16, 21] have developed an exact solution to the models such as SIS, SIR. The author says that for some models with first-order differential equations are non-integrable but they have formulated the analytic solution using integrable tools. The author concludes that an exact solution predicts the behaviour of the models

5. CONCLUSION

Mathematical modeling in many areas of research has now grown. Mathematical models are formulated and systematically solved based on the type of problem developed in the real-world. We have seen in this study the different types of growth models, compartment models and their application in the fields of medicine, ecology, population biology, molecular biology. Models of growth are used in the area of biology to assess the organism's survival. In the field

of medicine the Compartment models are used to assess the medication for the various diseases.

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