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INFINITE SERIES OF FRACTIONAL ORDER OF FIBONACCI DELTA OPERATOR AND ITS SUM

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ABSTRACT. In this paper, we derive numerical and closed form solutions of fractional order Fibonacci difference equation and Generalized infinite series of fractional Fibonacci summation formula using forward Fibonacci delta operator with several parameters and its inverse on real valued functions. Suitable examples are provided to illustrate our findings.

1. INTRODUCTION

In [6], Miller and Rose develop the discrete type of Riemann-Liouville fractional derivative and give some properties of fractional difference operator. In [7], Jerzy Popenda develop a special type of alpha difference operator as $\Delta_\alpha f(t) = f(t+1) - \alpha f(t)$. The authors in [8], have extended the delta alpha operator to generalized α-difference operator as $\Delta_\alpha f(t) = f(t+\ell) - \alpha f(t)$, for any real valued function $f(t)$. The authors in [9] introduced the q-difference operator as $\Delta_q f(t) = f(tq) - f(t)$ and the difference operator $\Delta_{t(\ell)}$ with variable coefficients as $\Delta_{t(\ell)} f(t) = f(t+\ell) - tf(t)$.

These operators motivate us to develop the following forward Fibonacci delta operator. The generalized forward Fibonacci delta operator with several parameters $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ on a real valued function is defined by

$$\Delta_{(\alpha)\ell} f(t) = f(t+n\ell) - \alpha_1 f(t+(n-1)\ell) - \alpha_2 f(t+(n-2)\ell) - \cdots - \alpha_n f(t) \tag{1.1}$$

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To develop the higher order forward Fibonacci difference and its inverse difference (sum), we consider the following basic definitions of generalized α-difference operators and its inverses. For more study on generalized difference operator, one can refer [1–5] and their references.

2. Preliminaries

In the theory of difference equations, usually the domain space of given function is a countable set. But, here we take uncountable domain set J_ℓ, which is defined below, and present basic definition of alpha delta operator and its inverse.

Definition 2.1. *Let $\ell > 0,\ J_\ell$ be any subset of R such that $t \in J_\ell$ implies $t \pm \ell \in J_\ell$ and $f, g : J_\ell \to R^m$ be a function. Then the generalized alpha delta operator $\Delta_{(\alpha)\ell}$ on $f(t)$ is defined as*

$$\Delta_{(\alpha)\ell} f(t) = f(t+\ell) - \alpha f(t),\ t \in J_\ell, \tag{2.1}$$

$$\Delta_{(\alpha)-\ell} f(t) = f(t-\ell) - \alpha f(t),\ t \in J_\ell. \tag{2.2}$$

If there exists a function $g : J_\ell \to R^m$ such that $\Delta_{(\alpha)\ell} y(t) = x(t),\ t \in J_\ell$, then we write $y(t) + c = \Delta^{-1}_{\alpha(\ell)} x(t),\ c$ is constant, and

$$y(t)||_a^k = \Delta^{-1}_{(\alpha)\ell} x(t)||_a^k = y(k) - \alpha y(a) = \Delta^{-1}_{(\alpha)\ell} x(k) - \alpha \Delta_\ell^{-1} x(a),\ a, k \in J_\ell.$$

Similarly, if $\Delta_{(\alpha)-\ell} y(t) = x(t),\ t \in J_\ell$, then $\Delta^{-1}_{(\alpha)-\ell} x(t) = y(t) + c$ and

$$\Delta_{(\alpha)-\ell} x(t)||_a^k = \Delta^{-1}_{(\alpha)-\ell} x(k) - \alpha \Delta^{-1}_{(\alpha)-\ell} x(a) = y(k) - \alpha y(b),\ a, k \in J_\ell.$$

Example 2.2. *Let $\ell > 0,\ \ell Z = 0, \pm\ell, \pm 2\ell, \cdots$, and take $J_\ell = R - \ell Z$. Define $f(t) : J_\ell \to R^3$ by*

$$f(t) = (2^t, \frac{1}{2^t}, (1+\ell)^{\frac{t}{\ell}}),\ t \in J_\ell. \tag{2.3}$$

Component wise applying $\Delta_{(\alpha)\ell}$ leads to define

$$\Delta_{(\alpha)\ell} 2^t = 2^{t+\ell} - \alpha 2^t = 2^t(2^\ell - \alpha),$$

$$\Delta_{(\alpha)\ell} \frac{1}{2^t} = \frac{1}{2^{t+\ell}} - \alpha \frac{1}{2^t} = \frac{1}{2^t}\Big[\frac{1}{2^\ell} - \alpha\Big] = \frac{1}{2^t}(2^{-\ell} - \alpha)$$

and

$$\Delta_{(\alpha)\ell} (1+\ell)^{\frac{t}{\ell}} = (1+\ell)^{\frac{t+\ell}{\ell}} - \alpha (1+\ell)^{\frac{t}{\ell}} = (1+\ell)^{\frac{t}{\ell}}\Big[(1+\ell) - \alpha\Big].$$

yields

$$\Delta_{(\alpha)\ell} f(t) = \Big(2^t(2^\ell - \alpha), \frac{1}{2^t}(2^{-\ell} - \alpha), (1+\ell)^{\frac{t}{\ell}}((1+\ell) - \alpha)\Big) \tag{2.4}$$

and

$$\Delta_{(\alpha)-\ell} f(t) = \Big(2^t(2^{-\ell} - \alpha), \frac{1}{2^t}(2^\ell - \alpha), (1+\ell)^{\frac{t}{\ell}}((1+\ell)^{-1} - \alpha)\Big), \tag{2.5}$$

where $J_{-\ell} = R - (-\ell)\, Z = R - \ell\, Z$.

From (2.4) *and* (2.5), *we find that*

$$\Delta^{-1}_{(\alpha)\ell}\Big(2^t, \frac{1}{2^t}, (1+\ell)^{\frac{t}{\ell}}\Big) = \Big(\frac{2^t}{(2^\ell - \alpha)}, \frac{1}{2^t}\frac{1}{(2^{-\ell} - \alpha)}, \frac{(1+\ell)^{\frac{t}{\ell}}}{((1+\ell) - \alpha)}\Big). \tag{2.6}$$

It is clear that, if $f(t) = \Big(f_1(t), f_2(t), f_3(t), \cdots, f_n(t)\Big) \in R^n$ and

$$g(t) = \Big(g_1(t), g_2(t), g_3(t), \cdots, g_n(t)\Big) \in \overset{n}{R}$$

such that $\Delta_{(\alpha)\pm\ell} g(t) = f(t),\ t \in J_\ell$, then

$$\Delta^{-1}_{(\alpha)\pm\ell} f(t)||_{t=a}^{k} = g(t)||_{t=a}^{k} = \Big(g_1(t)||_a^k, g_2(t)||_a^k, \cdots, g_n(t)||_a^k\Big).$$

The above expression states that taking component wise inverse principle is sufficient for vector valued functions. Hence, we deals with real valued functions.

Next, we extend the alpha difference operator to its inverse operator.

Definition 2.3. *If* $\Delta_{(\alpha)\ell} g(t)$, *then* $g(t) = \Delta^{-1}_{(\alpha)\ell} f(t)$ *and*

$$\Delta^{-1}_{(\alpha)\ell} f(t)\Big|\Big|_t^{t-m\ell} := \Delta^{-1}_{\alpha(\ell)} f(t) - \alpha^m \Delta^{-1}_{\alpha(\ell)} f(t - m\ell) = \sum_{r=1}^{m} \alpha^{r-1} f(t - r\ell), \tag{2.7}$$

where by convention $\sum\limits_{t=c}^{c-k} := 0$ *whenever* $k \in N$.

Example 2.4. *Applying inverse principle to equation* (2.7), *taking* $f(t) = 3^t$ *and*

$$\Delta^{-1}_{(\alpha)\ell} 3^t\Big|_t^{t-m\ell} := \Delta^{-1}_{(\alpha)\ell} 3^t - \alpha^m \Delta^{-1}_{(\alpha)\ell} 3^{t-m\ell} = \sum_{r=1}^{m} \alpha^{r-1} 3^{t-m\ell}. \tag{2.8}$$

Using (2.6) *in L.H.S of* (2.8) *and then expanding the R.H.S of* (2.8), *we get*

$$\frac{1}{3^\ell - \alpha}\Big[3^t - \alpha^m 3^{t-m\ell}\Big] = 3^{t-\ell} + \alpha 3^{t-2\ell} + \alpha^2 3^{t-3\ell} + \cdots + \alpha^{m-1} 3^{t-m\ell}.$$

Take $t = 4,\ m = 2,\ \alpha = 1$ *and* $\ell = 1$

$$\frac{1}{2}\Big[3^4 - 3^2\Big] = 3^3 + 3^2 \Rightarrow \frac{1}{2}\Big[81 - 9\Big] = 27 + 9 = \frac{1}{2}(72) = 36.$$

Thus, the Inverse principle satisfies the relation (2.8).

Corollary 2.5. *If* $\Delta_{(\alpha)\ell} g(t)$ *and* $g(-\infty) = 0$*, then* $g(t) = \Delta^{-1}_{(\alpha)\ell} f(t)$ *and*

$$\Delta^{-1}_{(\alpha)\ell} f(t) = \sum_{r=1}^{\infty} \alpha^{r-1} f(t - r\ell). \tag{2.9}$$

Proof. The proof follows by applying $\lim\limits_{m\to\infty}$ in (2.7). □

3. Basic concepts of Fibonacci delta operator

Through out this paper, J_ℓ is a subset of R such that $t \in J_\ell$ implies $(t+\ell) \in J_\ell$. When $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n),\ n \geq 2$, then the corresponding alpha difference operator is called as Fibonacci delta operator.

Theorem 3.1. *For any positive integer* m*,* $(\alpha_1, \alpha_2) = \alpha$ *and* $\ell \neq 0$*, we have*

$$\Delta^m_{(\alpha)\ell} f(t) = \begin{cases} \frac{n^{(m)}}{m!}\alpha_1^m - \frac{n^{(m-1)}}{(m-2)!}\alpha_1^{m-2}\alpha_2 + \cdots (-1)^{\frac{m}{2}} \frac{n^{(\frac{m}{2})}}{(\frac{m}{2})!}\alpha_2^{\frac{m}{2}},\ if\ m\ is\ even \\ -\frac{n^{(m)}}{m!}\alpha_1^m + \frac{n^{(m-1)}}{(m-2)!}\alpha_1^{m-2}\alpha_2 - \cdots (-1)^{\frac{m+1}{2}} \frac{n^{(\frac{m+1}{2})}}{(\frac{m-1}{2})!}\alpha_1\alpha_2^{\frac{m-1}{2}},\ if\ m\ is\ odd \end{cases} \tag{3.1}$$

Proof. The proof follows by applying $\Delta_{(\alpha)\ell}$ repeatedly on $f(t)$ up-to m times. □

Example 3.2. *Take* $m = 4$ *in* (3.1)*, we get the values for first four series and remaining series becomes zero. Thus, the* 4^{th} *order* $\Delta_{(\alpha)\ell}$ *operator is*

$$\begin{aligned}\Delta^4_{(\alpha)\ell} f(t) &= f(t-\ell) - \frac{4^{(1)}}{1!}\alpha_1 f(t-2\ell) + (\frac{4^{(2)}}{2!}\alpha_1^2 - \frac{4^{(1)}}{1!}\alpha_2) f(t-3\ell) \\ &+ (-\frac{4^{(3)}}{2!}\alpha_1^3 + \frac{4^{(2)}}{1!}\alpha_1\alpha_2) f(t-4\ell) + (\frac{4^{(4)}}{4!}\alpha_1^4 - \frac{4^{(3)}}{2!}\alpha_1^2\alpha_2 + \frac{4^{(2)}}{2!}\alpha_2^2) f(t-5\ell).\end{aligned}$$

Hence, for any m^{th} *order Fibonacci delta operator, we can easily obtain the series by applying* (3.1).

4. Infinite Series of Fractional Fibonacci Sum

Throughout this section, we assume that $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, where each $\alpha_i < \ell$, $\ell \neq 0$, $\nu > 0$ and $f, g : J_\ell \to \mathbb{R}$ are two functions. Here, we derive the Generalized infinite fractional Fibonacci summation formula using Fibonacci delta operator.

Theorem 4.1. *Let $\ell \neq 0$ be a real and $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 < \ell$. If $\sum\limits_{r=0}^{\infty} \alpha^r f(t - r\ell)$ converges, $\Delta^{\nu}_{(\alpha)\ell} g(t) = f(t)$ for $\nu > 0$ and $g(-\infty) = 0$, then $g(t) = \Delta^{-\nu}_{(\alpha)\ell} f(t)$ and*

$$\Delta^{-\nu}_{(\alpha)\ell} f(t) = \sum_{r=0}^{\infty} \alpha_2^r \Delta^{-(r+1)}_{(\alpha_1)\ell} \Big(\Delta^{-(\nu-1)}_{(\alpha)\ell} f(t - (r+1)\ell) \Big). \tag{4.1}$$

Proof. For $\nu = 1$, and applying equation (1.1) on $g(t)$ for $n = 2$, we get

$$f(t) = g(t + 2\ell) - \alpha_1 g(t + \ell) - \alpha_2 g(t). \tag{4.2}$$

Replacing t by $t - 2\ell$ in (4.2), we obtain

$$g(t) = f(t - 2\ell) + \alpha_1 g(t - \ell) + \alpha_2 g(t - 2\ell). \tag{4.3}$$

Again, replacing t by $t - r\ell$ in (4.3) to get $g(t - r\ell)$ for $r = 1, 2, 3, 4, \cdots$ and then substituting all these values of $g(t - r\ell)$ again in (4.3) recursively, we arrive

$$\begin{aligned} g(t) = &\Big[f(t-2\ell) + \alpha_1 f(t-3\ell) + \alpha_1^2 f(t-4\ell) + \alpha_1^3 f(t-5\ell) + \alpha_1^4 f(t-5\ell) + \cdots \Big] \\ &+ \alpha_2 \Big[f(t-4\ell) + 2\alpha_1 f(t-5\ell) + 3\alpha_1^2 f(t-6\ell) + 4\alpha_1^3 f(t-7\ell) + 5\alpha_1^4 f(t-8\ell) \cdots \Big] \\ &+ \alpha_2^2 \Big[\frac{2^{(2)}}{2!} f(t-6\ell) + \frac{3^{(2)}}{2!} \alpha_1 f(t-7\ell) + \frac{4^{(2)}}{2!} \alpha_1^2 f(t-8\ell) + \frac{5^{(2)}}{2!} \alpha_1^3 f(t-9\ell) + \cdots \Big] \\ &+ \alpha_2^3 \Big[\frac{3^{(3)}}{3!} f(t-8\ell) + \frac{4^{(3)}}{3!} \alpha_1 f(t-9\ell) + \frac{5^{(3)}}{3!} \alpha_1^2 f(t-10\ell) + \frac{6^{(3)}}{3!} \alpha_1^3 f(t-11\ell) + \cdots \Big] \\ &+ \cdots + \cdots \end{aligned}$$

Since, $\Delta^{-1}_{(\alpha_1)\ell} f(t - \ell) = \sum\limits_{r=2}^{\infty} \alpha_1^{r-2} f(t - r\ell)$ in (2.9) and $\Delta^{-1}_{(\alpha_1,\alpha_2)\ell} f(t) = g(t)$, the above expression becomes

$$\begin{aligned} \Delta^{-1}_{(\alpha_1,\alpha_2)\ell} f(t) = &\Delta^{-1}_{(\alpha_1)\ell} f(t-\ell) + \alpha_2 \Delta^{-2}_{(\alpha_1)\ell} f(t-2\ell) + \alpha_2^2 \Delta^{-3}_{(\alpha_1)\ell} f(t-3\ell) \\ &+ \alpha_2^3 \Delta^{-1}_{(\alpha_1)\ell} f(t-4\ell) + \cdots \end{aligned}$$

which is same as

$$\Delta^{-1}_{(\alpha_1,\alpha_2)\ell} f(t) = \sum_{r=0}^{\infty} \alpha_2^r \Delta^{-(r+1)}_{(\alpha_1)\ell} f(t - (r+1)\ell). \tag{4.4}$$

Again, taking $\Delta^{-1}_{(\alpha_1,\alpha_2)\ell}$ on the both sides of equation (4.4), we get

$\Delta^{-2}_{(\alpha_1,\alpha_2)\ell}f(t) = \Delta^{-1}_{(\alpha_1,\alpha_2)\ell}\{\Delta^{-1}_{(\alpha_1)\ell}f(t-\ell)+\alpha_2\Delta^{-2}_{(\alpha_1)\ell}f(t-2\ell)+\alpha_2^2\Delta^{-3}_{(\alpha_1)\ell}f(t-3\ell)+\cdots\}.$

$\Delta^{-2}_{(\alpha_1,\alpha_2)\ell}f(t) = \{\Delta^{-1}_{(\alpha_1)\ell}\Big(\Delta^{-1}_{(\alpha_1,\alpha_2)\ell}f(t-\ell)\Big) + \alpha_2\Delta^{-2}_{(\alpha_1)\ell}\Big(\Delta^{-1}_{(\alpha_1,\alpha_2)\ell}f(t-2\ell)\Big) + \cdots\},$

which can be written in the form

$$\Delta^{-2}_{(\alpha_1,\alpha_2)\ell}f(t) = \sum_{r=0}^{\infty}\alpha_2^r\Delta^{-(r+1)}_{\alpha_1(\ell)}\Big(\Delta^{-1}_{(\alpha_1,\alpha_2)\ell}f(t-(r+1)\ell)\Big). \tag{4.5}$$

Again, taking $\Delta^{-1}_{(\alpha_1,\alpha_2)\ell}$ on the both sides of equation (4.5), we get

$\Delta^{-3}_{(\alpha_1,\alpha_2)\ell}f(t) = \{\Delta^{-1}_{(\alpha_1)\ell}\Big(\Delta^{-2}_{(\alpha_1,\alpha_2)\ell}f(t-\ell)\Big) + \alpha_2\Delta^{-2}_{(\alpha_1)\ell}\Big(\Delta^{-2}_{(\alpha_1,\alpha_2)\ell}f(t-2\ell)\Big) + \cdots\},$

which yields

$$\Delta^{-3}_{(\alpha_1,\alpha_2)\ell}f(t) = \sum_{r=0}^{\infty}\alpha_2^r\Delta^{-(r+1)}_{(\alpha_1)\ell}\Big(\Delta^{-2}_{(\alpha_1,\alpha_2)\ell}f(t-(r+1)\ell)\Big). \tag{4.6}$$

In general, by taking $\Delta^{-1}_{(\alpha_1,\alpha_2)\ell}$ repeatedly on both sides of (4.3) up-to m times and continuing the above procedure, we arrive

$$\Delta^{-m}_{(\alpha_1,\alpha_2)\ell}f(t) = \sum_{r=0}^{\infty}\alpha_2^r\Delta^{-(r+1)}_{(\alpha_1)\ell}\Big(\Delta^{-(m-1)}_{(\alpha_1,\alpha_2)\ell}f(t-(r+1)\ell)\Big). \tag{4.7}$$

The relation (4.7) is also valid for real $\nu > 0$ provided $\Delta^{-\nu}_{(\alpha_1,\alpha_2)\ell}f(t)$ should be existed and the proof follows by replacing m by ν in (4.7). □

Example 4.2. *In* (4.1)*, take* $f(t) = 2^t,\ \nu = 1.5,\ \alpha_1 = \alpha_2 = 1.5$*, we obtain*

$$\Delta^{-1.5}_{(\alpha)\ell}2^t = \sum_{r=0}^{\infty}(1.5)^r\Delta^{-(r+1)}_{1.5(\ell)}\Big(\Delta^{-(1.5-1)}_{(\alpha)\ell}2^{t-(r+1)\ell}\Big). \tag{4.8}$$

For $n = 2$ *in* (1.1)*, we get* $\Delta_{(\alpha_1,\alpha_2)\ell}2^t = 2^{t+2\ell} - \alpha_1 2^{t+\ell} - \alpha_2 2^t = 2^t(2^{2\ell} - \alpha_1 2^\ell - \alpha_2)$*, which yields* $\Delta^{-1}_{(\alpha_1,\alpha_2)\ell}2^t = \dfrac{2^t}{2^{2\ell}-\alpha_1 2^\ell - \alpha_2}$*,* $\Delta^{-2}_{(\alpha_1,\alpha_2)\ell}2^t = \dfrac{2^t}{(2^{2\ell}-\alpha_1 2^\ell - \alpha_2)^2}$*,* $\cdots$ *and*

$$\Delta^{-\nu}_{(\alpha_1,\alpha_2)\ell}2^t = \frac{2^t}{(2^{2\ell}-\alpha_1 2^\ell-\alpha_2)^\nu}. \tag{4.9}$$

Putting $\ell = 2$ *and* $t = 5$ *in* (4.9) *and then substituting in L.H.S of* (4.8)*, we arrive*

$$\Delta^{-1.5}_{(1.5,1.5)2}2^t\Big|_5 = \frac{2^t}{(2^4-(1.5)2^2-1.5)^{1.5}}\Big|_5 = \frac{32}{24.781546} = 1.294736. \tag{4.10}$$

Expanding the R.H.S of equation (4.8)*, we obtain*

$$\sum_{r=0}^{\infty}(1.5)^r\Delta^{-(r+1)}_{(\alpha_1)2}\Big(\Delta^{-0.5}_{(1.5,1.5)2}2^{t-(r+1)2}\Big)\Big|_5 = \Delta^{-1}_{(1.5)2}\Big(\Delta^{-0.5}_{(1.5,1.5)2}2^{t-\ell}\Big)\Big|_5 + \alpha_2\Delta^{-2}_{(1.5)2}\Big(\Delta^{-0.5}_{(1.5,1.5)2}2^{t-2\ell}\Big)\Big|_5 + \alpha_2^2\Delta^{-3}_{(1.5)2}\Big(\Delta^{-0.5}_{(1.5,1.5)2}2^{t-3\ell}\Big)\Big|_5 + \cdots$$

By definition 2.1 *and by the use of* (4.9)*, the right hand side of above relation becomes in the form*

$$\frac{1}{(2^4-(1.5)2^2-1.5)^{0.5}}\Big(\frac{2^{t-2}}{(2^4-1.5)} + (1.5)\frac{2^{t-4}}{(2^4-1.5)^2} + (1.5)^2\frac{2^{t-6}}{(2^4-1.5)^3} + \cdots\Big)\Big|_5,$$

which gives $(0.342997)[3.2+0.48+0.072+0.0108+0.00162+\cdots]$ *and hence*

$$\sum_{r=0}^{\infty}(1.5)^r\Delta^{-(r+1)}_{(\alpha_1)2}\Big(\Delta^{-0.5}_{(1.5,1.5)2}2^{t-(r+1)2}\Big)\Big|_5 = 1.291269+\cdots \tag{4.11}$$

Now, (4.8) *is verified by* (4.10) *and* (4.11)*.*

The following theorem is the *Generalized Infinite Fractional Fibonacci Sum*.

Theorem 4.3. *Let* $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$*,* $\alpha_1,\alpha_2,\cdots,\alpha_n<\ell\neq 0$*. If* $\sum\limits_{r=0}^{\infty}\alpha^r f(t-r\ell)$ *converges,* $\Delta^{\nu}_{(\alpha)\ell}g(t)=f(t)$ *for* $\nu>0$ *and* $g(-\infty)=0$*, then* $g(t)=\Delta^{-\nu}_{(\alpha)\ell}f(t)$ *and*

$$\Delta^{-\nu}_{(\alpha)\ell}f(t)=\sum_{r=\tau}^{\infty}\Big[\sum_{\tau=0}^{n-2}\alpha^r_{\tau+2}\Delta^{-(r+1)}_{(\alpha_1)\ell}\Big(\Delta^{-(\nu-1)}_{(\alpha)\ell}f(t-(r+\tau+n-1)\ell)\Big)\Big]. \tag{4.12}$$

Proof. Since $\Delta^{-\nu}_{(\alpha_1,\alpha_2)\ell}f(t)$ is already proved in Theorem 4.1. By applying $n=3$ in (1.1) on $g(t)$, we get $f(t)=g(t+3\ell)-\alpha_1g(t+2\ell)-\alpha_2g(t+\ell)-\alpha_3g(t)$, and then replacing t by $t-3\ell$, we arrive

$$g(t)=f(t-3\ell)+\alpha_1g(t-\ell)+\alpha_2g(t-2\ell)+\alpha_3g(t-3\ell). \tag{4.13}$$

Now, follow the steps just like the proof given in Theorem 4.1, we find

$$\Delta^{-\nu}_{(\alpha_1,\alpha_2,\alpha_3)\ell}f(t)=\sum_{r=\tau}^{\infty}\Big[\sum_{\tau=0}^{1}\alpha^r_{\tau+2}\Delta^{-(r+1)}_{(\alpha_1)\ell}\Big(\Delta^{-(\nu-1)}_{(\alpha_1,\alpha_2,\alpha_3)\ell}f(t-(r+\tau+2)\ell)\Big)\Big]. \tag{4.14}$$

Next, by applying $\Delta_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\ell}$ on $g(t)$, and then replacing t by $t-4\ell$, we get

$$g(t)=f(t-4\ell)+\alpha_1g(t-\ell)+\alpha_2g(t-2\ell)+\alpha_3g(t-3\ell)+\alpha_4g(t-4\ell). \tag{4.15}$$

From equation (4.15) and by the proof of Theorem 4.1, we obtain

$$\Delta^{-\nu}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\ell}f(t)=\sum_{r=\tau}^{\infty}\Big[\sum_{\tau=0}^{2}\alpha^r_{\tau+2}\Delta^{-(r+1)}_{(\alpha_1)\ell}\Big(\Delta^{-(\nu-1)}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\ell}f(t-(r+\tau+3)\ell)\Big)\Big]. \tag{4.16}$$

Similarly, proceeding like this up-to $(\alpha_1,\alpha_2,\alpha_3,\cdots,\alpha_n)$ times, we get (4.12). $\square$

Example 4.4. *In* (4.12)*, take* $f(t)=2^t,\ \nu=3$*, and* $n=4$*, we get*

$$\Delta^{-3}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\ell}2^t=\sum_{r=\tau}^{\infty}\Big[\sum_{\tau=0}^{2}\alpha^r_{\tau+2}\Delta^{-(r+1)}_{(\alpha_1)\ell}\Big(\Delta^{-2}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\ell}2^{t-(r+\tau+3)\ell}\Big)\Big]. \tag{4.17}$$

For $n=4$ *in* (1.1)*, we get* $\Delta_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\ell}2^t=2^t(2^{4\ell}-\alpha_1 2^{3\ell}-\alpha_2 2^{2\ell}-\alpha_3 2^{\ell}-\alpha_4)$ *and* $\Delta^{-1}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\ell}2^t=\dfrac{2^t}{2^{4\ell}-\alpha_1 2^{3\ell}-\alpha_2 2^{2\ell}-\alpha_3 2^{\ell}-\alpha_4}$*. So, it is easy to obtain*

$$\Delta^{-\nu}_{(\alpha_1,\cdots,\alpha_4)\ell}2^t=\frac{2^t}{(2^{4\ell}-\alpha_1 2^{3\ell}-\alpha_2 2^{2\ell}-\alpha_3 2^{\ell}-\alpha_4)^{\nu}}. \tag{4.18}$$

Taking $t=3,\ \ell=2,\ \alpha_1=\cdots=\alpha_4=1.5$ *in* (4.18)*, and then substituting in* (4.17)*, we arrive*

$$\Delta^{-3}_{(1.5,\cdots,1.5)2}2^t\Big|_3=\frac{2^t}{(2^8-(1.5)2^6-(1.5)2^4-(1.5)2^2-1.5)^3}\Big|_3=3.770341\times10^{-6}. \tag{4.19}$$

Now, the expanded form of right hand side of equation (4.17) *is*

$$\sum_{r=0}^{\infty}(1.5)^r\Delta^{-(r+1)}_{(1.5)2}\Big(\Delta^{-2}_{(1.5,\cdots,1.5)2}2^{t-(r+3)2}\Big)\Big|_3+\sum_{r=1}^{\infty}(1.5)^r\Delta^{-(r+1)}_{(1.5)2}\Big(\Delta^{-2}_{(1.5,\cdots,1.)2}2^{t-(r+4)2}\Big)\Big|_3$$
$$+\sum_{r=2}^{\infty}(1.5)^r\Delta^{-(r+1)}_{(1.5)2}\Big(\Delta^{-2}_{(1.5,\cdots,1.5)2}2^{t-(r+5)2}\Big)\Big|_3 \tag{4.20}$$

Consider the term $\sum_{r=0}^{\infty}(1.5)^r\Delta^{-(r+1)}_{(1.5)2}\Big(\Delta^{-2}_{(1.5,\cdots,1.5)2}2^{t-(r+3)2}\Big)\Big|_3$*, which can be written in the form* $(1.5)^0\Delta^{-1}_{(1.5)2}\Big(\Delta^{-2}_{(1.5,\cdots,1.5)2}2^{t-6}\Big)\Big|_3+(1.5)\Delta^{-2}_{(1.5)2}\Big(\Delta^{-2}_{(1.5,\cdots,1.5)2}2^{t-8}\Big)\Big|_3+\cdots$

While solving the above series and then substituting the values, we get $\dfrac{1}{(2^8-(1.5)2^6-(1.5)2^4-(1.5)2^2-1.5)^2}\Big[\dfrac{2^{-3}}{2^2-1.5}+(1.5)\dfrac{2^{-5}}{(2^2-1.5)^2}+\cdots\cdots\Big]$ *and*

$$\sum_{r=0}^{\infty}(1.5)^r\Delta^{-(r+1)}_{1.5(2)}\Big(\Delta^{-2}_{(1.5,\cdots,1.5)2}2^{t-(r+3)2}\Big)\Big|_3=3.562386\times10^{-6}+\cdots \tag{4.21}$$

Now, consider the second and third term of (4.20) *and proceeding the similar steps given above, we obtain*

$$\sum_{r=1}^{\infty}(1.5)^r\Delta_{(1.5)2}^{-(r+1)}\left(\Delta_{(1.5,\cdots,1.5)2}^{-2}2^{t-(r+4)2}\right)\Big|_3 = 1.335079\times 10^{-7}+\cdots \tag{4.22}$$

and

$$\sum_{r=2}^{\infty}(1.5)^r\Delta_{(1.5)2}^{-(r+1)}\left(\Delta_{(1.5,\cdots,1.5)2}^{-2}2^{t-(r+5)2}\right)\Big|_3 = 5.007114\times 10^{-9}+\cdots \tag{4.23}$$

Hence, equation (4.17) *is verified by* (4.19)*,* (4.21)*,* (4.22) *and* (4.23).

5. Conclusion

We have developed the certain formulas for finding the value of infinite series by using numerical and closed form solutions of fractional order forward Fibonacci difference equation. In our example, we have applied fractional order Fibonacci forward difference operators with several variables on geometric functions.

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