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THE CYLINDRICAL CROSSING NUMBER OF ZERO DIVISOR GRAPHS

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ABSTRACT. The concept of zero divisor was started in 1988 by Beck. He introduced this idea to coloring a commutative ring by using simple graphs and also included zero to the set vertices of zero divisors.Few years later, that is in 1999 Anderson and Livingston applied slight modification to Beck's definition by restricting the vertices set only to the nonzero zero divisors of the ring. In this paper we discuss about the cylinderical crossing number of zero divisor graphs for some graphs.

1. BASIC DEFINITIONS

Definition 1.1. *A simple graph in which each pair of distinct vertices is joined by an edge is called a* ***complete*** *graph. A complete graph on n vertices is denoted by K_n.*

Definition 1.2. *A graph G is called* ***bipartite*** *if its vertex set V can be decomposed into two disjoint subsets V_1, V_2 such that every edge in G joins a vertex in V_1 with a vertex in V_2. A* ***complete bipartite*** *graph is a bipartite graph with bipartition (V_1, V_2) such that every vertex of V_1 is joined to all the vertices of V_2. It is denoted by $K_{m,n}$, where $|V_1| = m$ and $|V_1| = n$. A star graph is a complete bipartite $K_{1,n}$.*

Definition 1.3. *A graph G is a k-**partite** graph is V(G) can be partitioned into k subsets $V_1, V_2, ..., V_k$ such that uv is an edge of G if u and v belong to different*

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partite sets. If every two vertices in different partite sets are joined by an edge, then G is a complete k-partite graph.

Definition 1.4. *A graph is **planar** if it can be embedded in the plane in which no two of its edges intersect. A **null** graph is a graph with no edges.*

Definition 1.5. *The **crossing number** $cr(G)$ of a graph G is the minimum number of edge crossings among the drawings of G in the plane[24].*

Definition 1.6. *Let G be a graph drawn in the plane with the requirement that the edges are line segments, no three vertices are col-linear, and no three edges may intersect in a point, unless the point is a vertex. Such a drawing is said to be a **rectilinear drawing** of G. The rectilinear crossing number of G, denoted $\overline{cr}(G)$, is the fewest number of edge crossings attainable over all rectilinear drawings of G. Any such a drawing is called optimal.*

Definition 1.7. *In a **k-page (book) drawing** of $G = (V, E)$ all vertices V must be drawn on astraight line (the spine of a book), and each edge in one of k half-planes incident to this line (the book pages). The **k-page crossing number** $v_k(G)$ corresponds to k-page drawings of G.*

Definition 1.8. *A nonempty set R is said to a **ring**, if in R there are defined two operations, denoted by $+$ and $\bullet$ respectively, such that for all a, b, c in R satisfies abelian group under addition, a semi group under multiplication and the both $+$ and $\bullet$ must satisfies the distributive law.*

Definition 1.9. *A ring that has finite number of elements is called **finite** ring . A ring with commutative property under multiplication is called **commutative ring**. That is, if the multiplication of Z_n is such that $a.b = b.a$ for every a, b in Z_n , then we call Z_n a commutative ring.*

Definition 1.10. *If a and b are two non-zero elements of a ring Z_n such that $a.b = 0$, then $'a'$ and $'b'$ are the **zero divisors** of commutative ring Z_n. In particular, $'a'$ is a left zero divisor and $'b'$ is a right zero divisor.*

Definition 1.11. *A commutative ring without zero divisor is called an **integral domain**. A commutative ring with zero divisor is called non-integral domain. A ring is said to be a **division ring** if it's non-zero elements form a group under multiplication.*

2. THE CYLINDERICAL CROSSING NUMBER OF ZERO DIVISOR GRAPH

Theorem 2.1. *For any distinct prime p and q, the $cc(\Gamma(Z_{pq}))$ is $(p-2)(q-2)(q-1)(p-1)/4$, where $q > p$.*

Proof. The vertex set of $\Gamma(Z_{pq})$ is $\{p, 2p, 3p, ..(p-1)p, q, 2q, ..(p-1)q\}$. That is $|V(\Gamma(Z_{pq}))| = p + q - 2$. Using the above theorem (crossing number), $\Gamma(Z_{pq})$ can be decomposed into $K_{p-1,q-1}$ for $p < q$. The construction is based on the drawing of $\Gamma(Z_{pq})$ on a cylinder, placing $(q-1)$ vertices evenly around the rim on one lid and $(p-1)$ vertices on the other. All vertices on both lids are joined pairwise by means of straight line from one lid to another.

Considering one vertex say p from the upper lid which is adjacent to all the $(p-1)$ vertices on the other. Here, crossing is zero. Now considering another vertex say $2p$ from the upper lid which is adjacent to all the $(p-1)$ vertices on the other. Then, crossing is $(p-2)$. Considering the next vertex say $3p$ from the upper lid which is adjacent to all the $(p-1)$ vertices on the other. Then, crossing is $2(p-2)$. Proceeding in this manner there will be $(q-2)$ times of $(p-2)$. The cylindrical crossing is summing up all the crossings between the two rim. That is $cy(\Gamma(Z_{pq}))$ is $(p-2)(q-2)(q-1)(p-1)/4$, where $q > p$. □

Theorem 2.2. *For any prime $p > 4$, $cc(\Gamma(Z_{4p}))$ is $3(p-2)(p-1)/2$.*

Proof. The vertex set of $\Gamma(Z_{4p}) = \{2, 4, 6...2(2p-1), p, 2p, 3p\}$. That is $|V(\Gamma(Z_{4p}))| = 2p + 1$. Let the vertex set be partitioned into three parts, namely $V_1 = \{p, 2p, 3p\}, V_2 = \{4, 8, 16, ..., 4(p-1)\}$ and $V_3 = \{2, 6, 14, ..2(2p-1)\}$. It is easy to see that any vertices of V_1 is adjacent to all the vertices of V_2 which forms a complete bipartite graph $K_{3,p-1}$. Similarly, vertex $2p$ of V_1 is adjacent to all the vertices of V_3 which forms a complete bipartite graph $K_{1,p-1}$. Hence $\Gamma(Z_{4p})$ can be decomposed into $K_{1,p-1}$ and $K_{3,p-1}$.

The construction is based on the drawing of $\Gamma(Z_{4p})$ on a cylinder by placing V_1 vertices evenly around the rim on one lid and V_2 and V_3 vertices on the other. All vertices on both lids are joined pairwise by means of straight line from one lid to another. By the above theorem, the crossing number for $K_{3,p-1}$ is $3(p-2)(p-1)/2$. Similarly, the crossing number for $K_{1,(p-1)}$ is zero since it is a star graph. The cylindrical crossing is summing up all the crossings between the two rim. That is $cy(\Gamma(Z_{4p}))$ is $3(p-2)(p-1)/2$. □

Theorem 2.3. *For any graph $\Gamma(Z_{2p^2})$,where p is any prime $p > 3$, then*

$$cc(\Gamma(Z_{2p^2})) = (p-1)C_4 + p(p-1)^2(p-2)/4.$$

Proof. The vertex set of $\Gamma(Z_{2p^2})$, is $\{2, 4, 6, 8, ..., 2(p^2-1), p, 2p, 3p, ..., p(2p-1)\}$. Hence, $|V(\Gamma(Z_{2p^2}))| = p^2 + p - 1$. Let the vertex set be partitioned into three parts, namely $V_1 = \{p, 3p, 5p, .., p(2p-1)\}, V_2 = \{2p, 4p, ..., 2p(p-1)\}$ and $V_3 = V(\Gamma(Z_{2p^2})) - \{V_1 + V_2\}$. It is easy to see that any vertices of V_1 are adjacent to all the vertices of V_2 which forms a complete bipartite graph $K_{p,p-1}$. Similarly, vertex p^2 of V_1 is adjacent to all the vertices of V_3 which forms a complete bipatite graph $K_{1,p(p-1)}$. Moreover the vertices of V_2 are adjacent among itself and hence form a complete graph $K_{2p(p-1)}$. Hence $\Gamma(Z_{2p^2})$ can be decomposed into $K_{1,p(p-1)}, K_{p,p-1}$ and $K_{2p(p-1)}$.

The construction is based on the drawing of $\Gamma(Z_{2p^2})$ on a cylinder by placing V_1 vertices evenly around the rim on one lid and V_2 and V_3 vertices on the other. All vertices on both lids are joined pairwise by means of straight line from one lid to another. By the above theorem, the crossing number for $K_{p,p-1}$ = $(p)(p-1)(p-2)(p-1)/4 = p(p-1)^2(p-2)/4$ and the crossing number for $K_{1,p(p-1)}$ is zero since it is a star graph. Since, the vertices of V_2 form a complete graph $K_{2p(p-1)}$ the crossing number for $K_{2p(p-1)}$ is $(p-1)C_4$. The cylindrical crossing is summing up all the crossings between the two rim and also on the bottom of the rim. That is $cy(\Gamma(Z_{2p^2})) = (p-1)C_4 + p(p-1)^2(p-2)/4$. □

Theorem 2.4. *For any graph $\Gamma(Z_{3p^2})$, where p is any prime $p > 3$, then*

$$cc(\Gamma(Z_{3p^2})) = (p+1)C_4 + \frac{(p-1)}{2}\{p^2(3p-10) + 6(2p-1)\}.$$

Proof. The vertex set of $\Gamma(Z_{3p^2})$, is $\{3, 6, 9, ...3(p^2-1), p, 2p, 3p, ...p(3p-1)\}$. Hence, $|V(\Gamma(Z_{2p^2}))| = p^2 + 2p - 1$.Let the vertices of $\Gamma(Z_{3p^2})$ be partitioned into three parts. Namely, $V_1 = \{p^2, 2.P^2\}$, $V_2 = \{1.3p, 2.3p, 3.3p, ..(p-1).3p\}$, $V_3 = \{3, 6, 9, ...3(p^2-1)\}$ and $V_4 = V(\Gamma(Z_{3p^2})) - \{V_1 + V_2 + V_3\}$. It is easy to see that any vertices of V_1 are adjacent to all the vertices of V_3 which forms a complete bipartite graph $K_{2,p(p-1)}$. Similarly,any vertices of V_2 are adjacent to all the vertices of V_4 which forms a complete bipartite graph $K_{p-1,2(p-1)}$. Moreover the vertices of V_2 are adjacent among itself and hence form a complete graph $K_{(p-1)}$. Also the vertices of V_1 is adjacent to all the vertices of V_2 but vertices of V_1 are non adjacent among itself. Therefore it forms a complete graph $K_{(p+1)}$

minus an edge. Hence $\Gamma(Z_{3p^2})$ can be decomposed into $K_{2,p(p-1)}, K_{p-1,2(p-1)}$and $K_{(p+1)}$ minus an edge.

The construction is based on the drawing of $\Gamma(Z_{3p^2})$ on a cylinder by placing V_1 and V_2 vertices evenly around the rim on one lid and V_3 and V_4 vertices on the other. All vertices on both lids are joined pairwise by means of straight line from one lid to another. By the above theorem, the crossing number for $K_{2,p(p-1)}$ is $\frac{(p-1)^2(p-2)(2p-3)}{2}$. Similarly, the crossing number for $K_{p-1,2(p-1)}$ is $\frac{(p(p-1)-1)(p(p-1))}{2}$. Since, the vertices of V_1 and V_2 form a complete graph K_{p+1} minus an edge cross on the lid with the crossing number $(p+1)C_4$. The cylindrical crossing is summing up all the crossings between the two rim and also on the top of the rim. That is $cy(\Gamma(Z_{3p^2})) = (p+1)C_4 + \frac{(p-1)^2(p-2)(2p-3)}{2} + \frac{(p(p-1)-1)(p(p-1))}{2} = (p+1)C_4 + \frac{(p-1)}{2}\{p^2(3p-10)+6(2p-1)\}$. □

Theorem 2.5. *The cylindrical crossing number of* $\Gamma(Z_{3^n})$ *where* $n \geq 4$*, is* $\frac{1}{2}\{140 + \frac{9(3^(n-3)-1)(9(3^(n-3)-1)+1)}{2}\}$.

Proof. The vertex set of $\Gamma(Z_{3^n})$ is $\{3, 6, 9...3(3^{n-1}-1)\}$. Hence, $|V(\Gamma(Z_{3^n}))| = 3^{n-1}-1$. Let the vertices of $\Gamma(Z_{3^n})$ be partitioned into three parts. Namely, $V_1 = \{3^{n-1}, 2.3^{n-1}\}, V_2 = \{1.3^{n-2}, 2.3^{n-2}, ..., 6.3^{n-2}\}$ and $V_3 = V(\Gamma(Z_{3^n})) - \{V_1 + V_2\}$. It is easy to see that any vertices of V_1 are adjacent to all the vertices of V_2 moreover the vertices of V_2 are adjacent among itself and hence form a complete graph K_6. Also the vertices of V_1 are adjacent among itself which together form a complete graph K_8. Similarly,any vertices of V_1 are adjacent to all the vertices of V_3 which forms a complete bipartite graph $K_{2,9(3^(n-3)-1)}$. Hence $\Gamma(Z_{3^n})$ can be decomposed into $K_8$and $K_{2,9(3^(n-3)-1)}$

The construction is based on the drawing of $\Gamma(Z_{3^n})$ on a cylinder by placing V_1 vertices evenly around the rim on one lid and V_2 and V_3 vertices on the other. All vertices on both lids are joined pairwise by means of straight line from one lid to another. By the above theorem, the crossing number for $K_{2,9(3^(n-3)-1)}$ is $\frac{9(3^(n-3)-1)(9(3^(n-3)-1)+1)}{2}$ and the complete graph cross on the lid with the crossing number $(8)C_4 = 70$. The cylindrical crossing is summing up all the crossings between the two rim and also on the top of the rim. That is $cy(\Gamma(Z_{3^n})) = 70 + \frac{9(3^(n-3)-1)(9(3^(n-3)-1)+1)}{2} = \frac{1}{2}\{140 + \frac{9(3^(n-3)-1)(9(3^(n-3)-1)+1)}{2}\}$. □

Theorem 2.6. *The cylindrical crossing number of* $\Gamma(Z_{8p})$*,where* p *is any prime* $p > 3$ *is* $(p-1)(20p-3)$.

Proof. The vertex set of $\Gamma(Z_{8p})$ is $\{2, 4, 6, ..., 2(4p-1), p, 2p, 3p, ..., 7p\}$. Where $|V(\Gamma(Z_{8p}))| = 4p + 3$. Let the vertices of $\Gamma(Z_{8p})$ be partitioned into five parts. Namely, $V_1 = \{2p, 4p, 6p\}, V_2 = \{p, 3p, 5p, 7p\}$, V_3 is multiples of 8, V_4 is multiples of 4 other than multiples of 8 and $V_5 = V(\Gamma(Z_{8p})) - \{V_1 + V_2 + V_3 + V_4\}$. It is easy to see that any vertices of V_2 along with V_1 are adjacent to all the vertices of V_3 which forms a complete bipartite graph $K_{7,p-1}$.Also, the vertex $4p$ of V_1 are adjacent to $\{2p, 6p\}$Similarly, any vertices of V_1 are adjacent to all the vertices of V_4 which forms a complete bipartite graph $K_{3,p-1}$. . Also the vertex $4p$ in V_1 are adjacent to all the vertices of V_5 which forms a complete bipartite graph $K_{1,2(p-1)}$. Hence $\Gamma(Z_{8p})$ can be decomposed into $K_{5,p-1}, K_{3,p-1}, K_{1,2}$ and $K_{1,2(p-1)}$.

The construction is based on the drawing of $\Gamma(Z_{8p})$ on a cylinder by placing V_1, V_2 vertices evenly around the rim on one lid and V_3, V_4 and V_5 vertices on the other. All vertices on both lids are joined pairwise by means of straight line from one lid to another. By the above theorem, the crossing number for $K_{5,p-1}$ is $\frac{20(p-1)p}{4} = 5(p-1)p$, the crossing number for $K_{3,p-1}$ is $\frac{6(p-1)p}{4} = \frac{3(p-1)p}{2}$. Moreover, the vertex $\{2p, 6p\}$ of V_1 which are adjacent to V_3 cross the complete bipartite graph $K_{4,p-1}$ which is $(p-1)^2$. The cylindrical crossing is summing up all the crossings between the two rim and also on the top of the rim. That is $cy(\Gamma(Z_{8p})) = 5(p-1)p + \frac{3(p-1)p}{2} + (p-1)^2 = \frac{(p-1)(14p-1)}{2}$. □

Theorem 2.7. *For any prime $p > 6$, the cylindrical crossing number of $\Gamma(Z_{6p})$ is $\frac{(p-1)(13p-1)}{2}$.*

Proof. The $V(\Gamma(Z_{6p})) = \{2, 4, 6...2(3p-1), 3, 6, \ldots, 3(2p-1), p, 2p, 3p, 4p, 5p\}$. That is $|V(\Gamma(Z_{6p}))| = 4p + 1$. Let the vertices of $\Gamma(Z_{6p})$ be partitioned into five parts. Namely, $V_1 = \{2p, 3p, 4p\}, V_2 = \{p, 5p\}$, V_3 is multiples of 6, V_4 is multiples of 3 other than multiples of 6 and $V_5 = V(\Gamma(Z_{6p})) - \{V_1 + V_2 + V_3 + V_4\}$. It is easy to see that any vertices of V_2 along with V_1 are adjacent to all the vertices of V_3 which forms a complete bipartite graph $K_{5,p-1}$. Also the vertex $3p$ is adjacent to the vertices $\{2p, 4p\}$ Similarly, the vertices $\{2p, 4p\}$ of V_1 are adjacent to all the vertices of V_4 which forms a complete bipartite graph $K_{2,p-1}$. Also the vertex $3p$ in V_1 are adjacent to all the vertices of V_5 which forms a complete bipartite graph $K_{1,2(p-1)}$. Hence $\Gamma(Z_{8p})$ can be decomposed into $K_{5(p-1}, K_{2,p-1}, K_{1,2}$ and $K_{1,2(p-1)}$.

The construction is based on the drawing of $\Gamma(Z_{6p})$ on a cylinder by placing V_1 and V_2 vertices evenly around the rim on one lid and V_3, V_4 and V_5 vertices on the other. All vertices on both lids are joined pairwise by means of straight line from one lid to another. By the above theorem, the crossing number for $K_{5,p-1}$ is $\frac{20(p-1)p}{4} = 5(p-1)p$, the crossing number for $K_{2,p-1}$ is $\frac{6(p-1)p}{4} = \frac{3(p-1)}{2}p$. Moreover, the vertex $\{2p, 6p\}$ of V_1 which are adjacent to V_3 cross the complete bipartite graph $K_{2,p-1}$ which is $(p-1)^2$. The cylindrical crossing is summing up all the crossings between the two rim and also on the top of the rim. That is $cy(\Gamma(Z_{6p})) = 5(p-1)p + \frac{(p-1)p}{2} + (p-1)^2 = \frac{(p-1)(13p-1)}{2}$. □

Theorem 2.8. *For any prime $p > 2$, $cy(\Gamma(Z_{2p})) = 0$.*

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