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FEED FORWARD AND BACKWARD DIFFERENCE EQUATIONS FOR CONTROLLING THE SIGNAL

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ABSTRACT. In this paper we consider the nonlinear time-varying discrete-time framework system, a notion of poles and zeros is developed in terms of factorizations of operator polynomials with time-varying coefficients. In the discrete-time case, it is indicated that factorizations are figured by solving a nonlinear recursive distinction condition with time-differing coefficients. The hypothesis is applied to the investigation of the input /output conditions and the response of the system and its stability are noted. Likewise we showed that, if a time-varying analogues of the difference equation is invertible, then the zero-input response itself decomposed and associated with the poles. Finally numerical examples are shown for filters stability.

1. INTRODUCTION

Difference equations and its contrast conditions are the proper portrayal of mathematical discrete process, which have exceptional significance in regions, for example, Digital Signal Processing and frameworks, channel configuration, denoise strategies (in both image and signal), etc [5, 16]. Higher order difference equations are less studied, and they have extra ordinary significance in applications where the current state depends on the previous states [4, 9].

The fundamental thought here is to consider frameworks with changes happening discretely [3, 6, 12]. We can't generally watch such systems persistently,

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so we simply screen the difference equations periodically [7, 10, 11]. This is the fundamental thought of time based analysis, which is measurable and sufficient ways are there to deal with depicting, anticipating and controlling the series over the period of time [1, 13, 15]. As an application of difference equations in signal processing, the authors, G. Britto Antony Xavier etc., [4] have defined Laplace transform with shift value ℓ using generalized difference operator.

The paper is structured as follows. In Section 2, we arrived the unique solution and the solutions are converging through poles and zeros in a discrete time interval of the higher order input/output difference equations. In Section 3, Cascade realization of the difference equation which involves the delay is modeled and extends the stability analysis for the filters. Some examples are illustrated to enrich the results. Section 4 concludes the paper.

2. CASCADED REALIZATION STRUCTURE OF THE DIFFERENCE EQUATIONS

Consider the following difference equation:

$$y(k+2) + a_1(k)y(k+1) + a_0(k)y(k) = b \tag{2.1}$$

where $y(k)$ represents the output at time k, for the input $u(k)$. $a_0(k)$ and $a_1(k)$ are elements of A which are polynomial (feed forward based) and hold a discrete values for framing the filters to reduce or enhance the noise level [2,8,14]. This holds the control of the signal. Now let z^i, which denotes the i^{th} step left shift operator on A which can be defined like,

$$\mathrm{z}^i f(k) = f_\Delta(k+i), f_\Delta \in A.$$

For any $a(k) \in A$, let $a(k)z^i$ stands that any operator which satisfied the properties of A and can be defined by

$$a(k)z^i f_\Delta(k) = a(k) f_\Delta(k+i) \ .$$

Now (2.1) can be written as:

$$(z^2 + a_1(k)z + a_0(k))y(k) = b. \tag{2.2}$$

Consider the functions $p_1(k), p_2(k) \in A$ such that

$$(z^2+a_1(k)z+a_0(k))y(k) = (z-p_1(k))[(z-p_2(k))y(k)]v(k) = (z-p_0(k))y(k) \tag{2.3}$$

$$v(k) = (z - p_0(k))y(k) \tag{2.4}$$

So that

$$y(k+1) - p_2(k)y(k) = \nu(k) \tag{2.5}$$

Inserting the expression (2.4) for $v(k)$ into (2.3) and using (2.2), we have

$$(z - p_1(k))\nu(k) = b$$

or

$$v(k+1) - p_1(k)v(k) = b \tag{2.6}$$

The cascaded realization structure of the system (2.5) and (2.6) are shown in Figure 1.

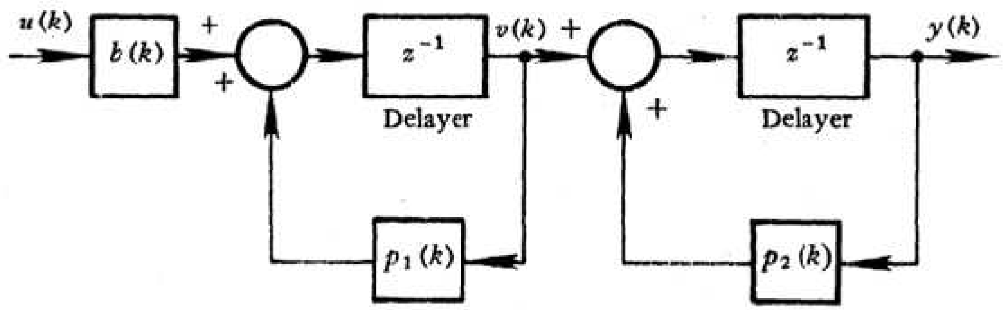

FIGURE 1. Cascaded Realization Structure of the Systems

Initially we characterized the cascaded realization structure as decomposed in terms of A, especially to reduce the noise level in the filter and its coefficients. In the above Cascade realization structured system, the delay related blocks are used to pass the previous output as it arrived.

Again suppose that there exist $p_1(k), p_2(k) \in A$ such that (2.3) is satisfied. Now define

$$(z - p_1(k)) \circ (z - p_2(k))$$

and hence

$$[(z - p_0(k)) \circ (z - p_1(k))]y(k) = (z - p_1(k))[(z - p_2(k))y(k)] \tag{2.7}$$

We extend (2.7) and simplifying we get

$$(z - p_0(k) \circ (z - p_1(k)) = z^2 - (p_1(k) + p_2(k+1))z + p_1(k) + p_2(k)$$

$$z \circ p_0(k) = p_1(k+1)z$$

$$z^2 - (p_0(k) + p_1(k+1))z + p_1(k)p_2(k) = z^2 + a_1(k)z + a_0(k)$$

Again simplifying the above equations we get (backward based difference equation)

$$p_0(k) + p_1(k+1) = -a_1(k)$$

$$p_0(k)p_1(k) = a_0(k) - a_1(k)$$

$$p_2(k+1)p_1(k) + a_1(k)p_0(k) + a_0(k) = 0 \tag{2.8}$$

Example 1. *Consider the following coefficients of the input/output difference equation* $a_0 = 0.5$ *for all* $k \in \mathrm{Z}, a_1(k) = \begin{cases} -1, \ k \leq 0 \\ -1 + (\frac{2.5k}{200}), \ 0 < k < 200 \\ 1.5, \ k \geq 200 \end{cases}$

For $k < 0,\ z^2 + a_1(k)z + a_0\ (k)\ = z^2 - z + 0.5 = (z - 0.5 - j0.5)(z - 0.5 + j0.5)$
as $k \to \infty,\ z^2 + a_1(k)z + a_0(k) \to z^2 + 1.5z + 0.5 = (z+1)(z+0.5)$
The poles with $p_2(0) = 0.5 - j0.5$ are shown in Figure 2.

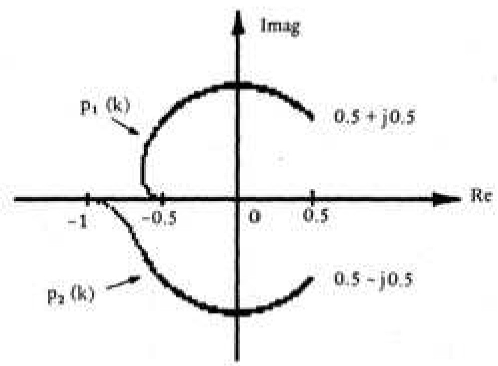

FIGURE 2. Poles with $p_2(0) = 0.5 - j0.5$

Thus we conclude that frequencies ω where $j\omega$ are near to zeros.

3. ZEROS AND POLES OF INPUT/OUTPUT DIFFERENCE EQUATIONS

Now consider the discrete-time system given by the second-order input/output difference equation (2.1). Let $\Delta_p(k)$ and $\Delta_q(k)$ denote the zeros of the polynomial

$$z^2 + a_1(k)z + a_0(k)$$

That is,

$$(z - p_1(k))(z - p_2(k)) = z^2 + a_1(k)z + a_0(k)$$

where $(z - \Delta_p(k))(z - \Delta_q)$ is the ordinary product of two polynomials. We can write this product in the form

$$(z - \Delta_p(k))(z - \Delta_q(k)) = z^2 - (\Delta_p(k) + \Delta_q(k))z + \Delta_p(k).\Delta_q(k)$$

$$= (z - \Delta_p(k) \circ (z - \Delta_q(k)) + [\Delta_q(k)(k+1) - \Delta_q(k)]z \tag{3.1}$$

The delay of the systems are noted, i.e., $|\Delta_q(k+1) - \Delta_q(k)|$ is small for $k > k_0$. From (3.1), we have

$$z - \Delta_p(k)(z - \Delta_q(k)) \cong (z - \Delta_p(k) \circ (z - \Delta_q(k))$$

Now suppose that $a_0(k) \to C_0$ and $a_1(k) \to C_1$ as $k \to \infty$. Let r_1 and r_1 denote the zeros of $z^2 + c_1 z + c_0$. If $\Delta_p(k), \Delta_q(k)$ is a pole set on $k > k_0$, it turns out that $\Delta_q(k)$ will converge to r_1 or r_2 in general. This convergence ensures the control of the signal and its leads to the stability. In particular, if r_1 and r_2 are complex numbers and $\Delta_q(k)$ is a real number then by (2.8), $\Delta_q(k)$ is real for $k > k_0$ and thus $\Delta_q(k)$ cannot converge to r_1 or r_2. In a particular specified case, suppose that $a_0(k) = C_0$ and $a_1(k) = C_1$ for all $k \geq k_1$. Now consider the pole set $\Delta_p(k), \Delta_q(k)$ and let $\Delta(k) = \Delta_q(k) - r_2$, so that

$$\Delta(k+1) = \Delta_q(k)(k+1) - r_2 \tag{3.2}$$

By (2.8) we have

$$p_2(k+1) = -a_1(k) - \frac{a_0(k)}{\Delta_q(k)}$$

and thus for $k \geq k_1$, we have

$$\Delta_q(k)(k+1) = -c_1(k) - \frac{C_0}{\Delta_q(k)} \tag{3.3}$$

Substitute (3.3) into (3.2), we get

$$(k+1) = -c_1 - \frac{C_0}{p_2(k)} - r_2$$

$$\Delta(k+1) = \frac{-(C_1 + r_2)\Delta_q(k) - C_0}{\Delta_q(k)}$$

Now $\Delta_q(k) = \Delta(k) + r_2$, we have:

$$\Delta(k+1) = \frac{-(r_2 + C_1 r_2 + C_0) - (C_1 + r_2)\Delta(k)}{\Delta(k) + r_2} \tag{3.4}$$

But since r_1 and r_2 are the zeros of $z^2 + c_1 z + c_0$, now we have

$$r_1^2 + c_1 r_2 + c_0 = 0 \quad and \quad -(c_1 + c_2) = r_1 \tag{3.5}$$

Using (3.5) in (3.4), we obtain

$$\Delta(k+1) = \frac{r_1 \Delta(k)}{\Delta(k) + r_2}, \; k \geq k_1$$

$$g(k) = \frac{r_1 - \Delta(k)}{\Delta(k)}$$

$$g(k+1) = \frac{r_2}{r_1}[g(k) + 1]$$

If $r_1 \neq r_2$ and $k_1 = 0$, the result is

$$\Delta(k) = \frac{(r_1 - r_2)\Delta(0)}{\Delta(0) + [(r_1 - r_2) - \Delta(0)](\frac{r_2}{r_1})^k}, \quad k \geq 0 \tag{3.6}$$

From (3.6), we see that if $\Delta(0) \neq (r_1 - r_2)$ and $|\frac{r_2}{r_1}| > 1$, then $\Delta(k)$ is converging to zero as $k \to \infty$, this leads that $p_2(k)$ converges to r_2 as $k \to \infty$. This shows the forward and backward case of signal control and its stability. Now consider the polynomial $A[z]$ in terms of

$$\mathrm{z}^{io} z^j = \mathrm{z}^{i+j}$$

$$z^i \circ a_i(k) \; = a_i(k+i) z^i, \; a_i(k) \in A$$

Now consider

$$a_i(z,\; k) = z^n + \sum_{i=0}^{n-1} a_i(k) z^i$$

and

$$b(z,\; k) = \sum_{i=0}^{n} b_i(k) \mathrm{z}^i$$

$$a_i(z,\; k) y(k) = b(z,\; k) u_k$$

Obviously $u(k)$ represents the input value and $y(k)$ provides the output. We call $p_n(k) \in A$ which is a right pole of the cascaded realization structure if there exists $e(k,\; z) \in A[z]$ such that,

$$a_i(z,\; k) = e(z,\; k)^{o}(z - p_n(k)) \tag{3.7}$$

Now we call $q_n(k) \in A$ is a right pole of the system if there exists $b(k,\ z) \in A[z]$ such that

$$b(z,\ k) = e(z,\ k)^{\mathrm{o}}(z - q_n(k))$$

This implies that (3.7) holds arbitrarily. Then we have

$$e(z,\ k) = \mathrm{z}^{n-1} + \sum_{i=0}^{n-2} e_i(k)\mathrm{z}^i$$

Now consider the equation which is equivalent to (3.7):

$$e_{n-2} = p_0(k+n-1) + a_{n-1}\ (k) \tag{3.8}$$

$$e_{i-1}(k) = e_i(k)p_0(n+i) + a_i(k)\ ,\ i = n-2, n-3, ..., 1$$

$$0 = e_0(k)p_0(k) + a_0(k) \tag{3.9}$$

According to the initial values $p_0(k_0 - n + 1), p_0(k_0 - n + 2)\ , \ .\ .\ .\ ,\ p_n(k_0)$ and from (3.8) to (3.9) we compute $p_n(k)$ and $e_i(k)$ for $k > k_0$. Now combine (3.8) to (3.9) and we get

$$e_{n-3}(k+1) = p_n(k+n-1)p_n(k+n-2) + a_{n-1}(k)p_n(k+n-2) + a_{n-2}(k)$$

Simplifying the above equations, we get

$$\begin{aligned} e_{n-4}(k)\ &= p_n(k+n-p)p_n(k+n-2)p_n(k+n-p) \\ &+a_{n-1}(k)p_n(k+n-2)p_n(k+n-p) \\ &+a_{n-p+1}(k)p_n(k+n-p) + a_{n-p}(k) \end{aligned}$$

Proceeding like this we obtain

$$0 = p_0(k+n-1)p_1(k+n-2)\ p_n(k)$$

$$0 = \sum_{i=1}^{n-1} a_i(k)p_0(k+i-1)p_1(k+i-2)p_2(k) + a_0(k) \tag{3.10}$$

If $p_n(k) \neq 0$ for $k \geq k_0 - n + p$, then we can write (3.10) in the following form:

$$p_0(k+n-1) = -a_{n-1}(k) - \sum_{i=1}^{n-2} \frac{a_i(k)}{p_0(k+n-2)p_1(k+n-3)p_n(k+i)}$$

$$-\frac{a_0(k)}{p_0(k+n-2)p_1(k+n-3)p_n(k)} \tag{3.11}$$

Equation (3.11) is solved recurrently to find the pole value of $p_n(k)$. Let $q(k)$ is denoted as zero of the cascaded realization structured system defined in the earlier case as:

$$b(z-k+p)=b(z,\ k+p)^{\text{o}}(z-q(k+p)).$$

Define $$\varphi_q(k,\ k_0)=\begin{cases} -1,\ k=0 \\ q(k-1+p)q(k-2+p)\ q(k_0+p),\ k>k_0 \\ 0,\ k<k_0 \end{cases}$$

The function $\varphi_q(k,\ k_0)$ is called the method connected with the zero $q(k)$. If $q(k)$ is constant, then $q(k)=q$ for $k>k_0$, then

$$\varphi_q(k,\ k_0)=q^{k-k_0}$$

Initially we have that

$$b(z,\ k+p)u(k)=b(z,\ k+p)[(z-q(k+p))u(k+p)]$$

$$=b(z,\ k+p)[\varphi_q(k+1,\ k_0),\ q(k)\varphi_q(k+p,\ k_0)]$$

By definition of $\varphi_q(k+p,\ k_0)$,

$$\varphi_q(k+1,\ k_0)=q(k+p)\varphi_q(k+p,\ k_0),\ k\geq k_0$$

and hence we have

$$b(z,\ k+p)u(k+p)=0 \quad for \quad k\geq k_0.$$

Restructuring the input/output difference equation, we get

$$y(k+n+p)=-\sum_{i=0}^{n-1}a_i(k+p)y(k+p+i),\ k\geq k_0 \tag{3.12}$$

If $y(k_0+i)=0$ for $i=0,1,2,\cdots,n-1$, it follows from (3.12) that $y(k)=0$ for all $k\geq k_0$ as desired. Now it can be chosen that $y(k_0+i)=0$ for $i=0,1,2,\cdots,n-1$.

4. Conclusion

This paper demonstrates an advanced level approach to the higher order (both feed forward and backward) difference equation's convergence through poles and zeros. In addition, stability principles are focused on the concepts of difference equation in the field of signal control and its stability. Finally, few examples are given to demonstrate the efficiency of the projected structure.

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