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ON WEAK FORMS OF PYTHAGOREAN NANO OPEN SETS

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ABSTRACT. The objective is to introduce the weakest form of Pythagorean nano open sets, namely Pythagorean nano semi-open, Pythagorean nano pre-open, Pythagorean nano α-open and Pythagorean nano γ and β open sets. Various Pythagorean nano semi-open and Pythagorean nano α-open corresponding to its different characterizations have also been derived.

1. INTRODUCTION

L. A. Zadeh introduced the concept of fuzzy set having elements with degree of membership [1]. Atanassov [2] established the idea of Intuitionistic fuzzy set comprising elements with membership and non-membership degree. Chang [3] introduced the idea of fuzzy topological space along with its base properties as open, closed and continuity were defined in 1968. Following this Lowen [4] gave another definition of fuzzy topological space. The concept of intuitionistic topological space with few fundamental properties was established by Coker [5].

Pythagorean fuzzy subset which is a special fuzzy subset was instituted by Yager [6, 7] which has applications in social and natural sciences. Even when intuitionistic fuzzy subsets cannot be used can be replaced by the Pythagorean fuzzy subset. The Pythagorean fuzzy topological space was introduced following the concept of Chang by Olgun [8].

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L. Thivagar [9] instituted the perception of nano topological spaces. The nano topology and intuitionistic fuzzy topology was combined to a new form named Intuitionistic nano topology in 2017 and was coined by Ramachandran and Stephan [10]. The concept of nano topology through neutrosophic sets [11] was also coined by L. Thivagar et. al. A new structure, namely Pythagorean nano topology which is a combo of nano topology and Pythagorean fuzzy topology was coined by D. Ajay and J. J. Charisma [12] along with few properties.

2. Preliminaries

A subset of a non-void set X is said to be a fuzzy subset if it is as $A = \{\langle a, \rho_A(a)\rangle, a \in X\}$ with $\rho_A : X \to [0,1]$ (membership degree). A subset $B = \{\langle b, \sigma(b), \mu(b)\rangle, b \in X\}$ of X with $\sigma : X \to [0,1]$, $\mu : X \to [0,1]$ (membership and non-membership degree) is Intuitionistic fuzzy subset satisfying $\sigma(b) + \mu(b) \leq 1$. A non-empty subset C of X of the form $C = \{\langle c, \sigma(c), \mu(c)\rangle, c \in X\}$ with $\sigma : X \to [0,1]$, $\mu : X \to [0,1]$ as membership and non-membership degree satisfying $\sigma^2(c) + \mu^2(c) \leq 1$ is named as Pythagorean fuzzy subset.

The pair(U, R) is said *approximation space*, where U is a non-void set called as universe and R an equivalence relation. Let $K \subseteq U$. The lower approximation of K with respect to R is denoted by $L_R(x)$.$L_R(x) = \bigcup_{x \in U} R(x) : R(x) \subseteq K$, $R(x)$ is the equivalence relation determined by way of x. The upper approximation is denoted as $U_R(x)$ and described as $U_R(x) = \bigcup_{x \in U} R(x) : R(x) \cap K \neq \emptyset$. The boundary of K is $B_R(x) = U_R(x) - L_R(x)$.

Let U be an universe with equivalence relation R and $\tau_R(K) = \{\emptyset, U, L_R(x), B_R(x), U_R(x)\}$ in which $K \subseteq$ U. $\tau_R(K)$ satisfies the axioms: U,$\emptyset \in \tau_R(K)$, union of elements of $\tau_R(K)$ is in $\tau_R(K)$, intersection of finite sub-collection of $\tau_R(K)$ is in $\tau_R(K)$. (U,$\tau_R(K)$) is termed the Nano topological space.

Let A be a subset in a nano topological space $(U, \tau_R(X))$. It is named as nano semi open, nano pre-open, nano α-open if $A \subseteq Ncl(Nint(A)), A \subseteq Nint(Ncl(A))$, $A \subseteq Nint(Ncl(Nint(A)))$ respectively [13].

A non-empty set J with the collection of Pythagorean fuzzy subsets ρ is called Pythagorean fuzzy topological space if the following axioms are satisfied $1_X, 0_X \in \rho$, for any S_1, $S_2 \in \rho$, $S_1 \cap S_2 \in \rho$, for any $\{S_i\}_{i \in I} \subset \rho$ for all i, $\cup S_i \in \rho$ where I is an arbitrary index set. The pair $(\mathcal{V}, \mathfrak{R})$ is said to be a Pythagorean approximation space with D a Pythagorean set in $\mathcal{V}$ having θ_D, ω_D as membership and non-membership degree. The Pythagorean nano lower, Pythagorean nano upper and boundary of D are:

$$PNL_{\mathfrak{R}}(D) = \{\langle x, \theta_{LD}(x), \omega_{LD}(x) \mid z \in [x]_{\mathfrak{R}}, x \in \mathcal{V}\rangle\}$$

$$PNU_{\mathfrak{R}}(D) = \{\langle x, \theta_{RD}(x), \omega_{RD}(x) \mid z \in [x]_{\mathfrak{R}}, x \in \mathcal{V}\rangle\}$$

$$PNB_{\mathfrak{R}}(D) = PNU_{\mathfrak{R}}(D) - PNL_{\mathfrak{R}}(D)$$

where

$$\theta_{LD}(x) = \bigwedge_{y \in [x]_{\mathfrak{R}}} \theta_D(y)$$

$$\omega_{LD}(x) = \bigvee_{y \in [x]_{\mathfrak{R}}} \omega_{LD}(y)$$

and

$$\theta_{RD}(x) = \bigvee_{y \in [x]_{\mathfrak{R}}} \theta_D(y)$$

$$\omega_{RD}(x) = \bigwedge_{y \in [x]_{\mathfrak{R}}} \omega_D(y)$$

respectively [12].

The pair $(\mathcal{V}, \mathfrak{R})$ along with $\tau_{\mathfrak{R}}(Y) = \{\emptyset_P, \mathcal{V}_P, PNL_{\mathfrak{R}}(Y), PNU_{\mathfrak{R}}(Y), PNB_{\mathfrak{R}}(Y)\}$ is said to be Pythagorean nano topological space while satisfying these conditions: $\emptyset_P, \mathcal{V}_P \in \tau_{\mathfrak{R}}(Y)$, if $A_i \in \tau_{\mathfrak{R}}(Y)$ for $i = 1, 2, \ldots$ then $\bigcup_{i=1}^{\infty} A_i \in \tau_{\mathfrak{R}}(Y)$, if $A_i \in \tau_{\mathfrak{R}}(Y)$ for $i = 1, 2, \ldots$ then $\bigcap_{i=1}^{n} A_i \in \tau_{\mathfrak{R}}(Y)$ where $\emptyset_P = \{\langle x, 0, 1 \mid \forall\, x \in \mathcal{V}\rangle\}$, $\mathcal{V}_P = \{\langle x, 1, 0 \mid \forall\, x \in \mathcal{V}\rangle\}$ [12].

3. Pythagorean nano α open sets

Definition 3.1. *Let $(\mathcal{U}, \tau_{\mathrm{R}}(\mathrm{X}))$ be a PNT space and $\mathrm{A} \subseteq \mathcal{U}$.*

(1) *If $\mathrm{A} \subseteq \mathcal{PN}c\ell(\mathcal{PN}\imath nt(\mathrm{A}))$, then A is Pythagorean nano semi-open (PNSO)*
(2) *If $\mathrm{A} \subseteq \mathcal{PN}\imath nt(\mathcal{PN}c\ell(\mathrm{A}))$, then A is Pythagorean nano pre-open if (PNPO)*
(3) *If $\mathrm{A} \subseteq \mathcal{PN}\imath nt(\mathcal{PN}c\ell(\mathcal{PN}\imath nt(\mathrm{A})))$, then A is Pythagorean nano α-open (PNαO).*

Let PNSO$(\mathcal{U}, \mathrm{X})$, PNPO$(\mathcal{U}, \mathrm{X})$ and $\tau_{\mathrm{PR}}^{\alpha}(\mathrm{X})$ be the families of all Pythagorean nano semi-open, Pythagorean nano pre-open and Pythagorean nano α open subsets of $\mathcal{U}$ respectively. Let $\mathrm{A} \subseteq \mathcal{U}$. A is said to be a Pythagorean nano semi-closed, Pythagorean nano pre-closed, Pythagorean nano α closed if its complement is PNSO, PNPO, PN αO respectively.

Example 1. *Let* $\mathcal{U}=\{a, b, c\}$ *be the Universe and let* $\mathcal{U}/R = \{\{a, b\}, \{c\}\}$*. Let us consider*

$$A=\{\langle a, 0.7, 0.6\rangle, \langle b, 0.5, 0.4\rangle, \langle c, 0.6, 0.3\rangle\}$$

Then

$$\mathrm{PL_R}(\mathrm{A}) = \{\langle a, 0.5, 0.6\rangle, \langle b, 0.5, 0.6\rangle, \langle c, 0.6, 0.3\rangle\}$$

$$\mathrm{PU_R}(\mathrm{A}) = \{\langle a, 0.7, 0.4\rangle, \langle b, 0.7, 0.4\rangle, \langle c, 0.6, 0.3\rangle\}$$

$$\mathrm{PB_R}(\mathrm{A}) = \{\langle a, 0.5, 0.4\rangle, \langle b, 0.5, 0.4\rangle, \langle c, 0.4, 0.7\rangle\}$$

$$\tau_{\mathrm{R}}(\mathrm{A}) = \left\{ \begin{array}{c} \emptyset_{\mathrm{p}},\ \mathcal{U}_{\mathrm{p}}, \{\langle a, 0.5, 0.6\rangle, \langle b, 0.5, 0.6\rangle, \langle c, 0.6, 0.3\rangle\}, \\ \{\langle a, 0.7, 0.4\rangle, \langle b, 0.7, 0.4\rangle, \langle c, 0.6, 0.3\rangle\}, \\ \{\langle a, 0.5, 0.4\rangle, \langle b, 0.5, 0.4\rangle, \langle c, 0.4, 0.7\rangle\} \end{array} \right\}$$

The Pythagorean nano closed sets are $\emptyset_{\mathrm{p}}$, $\mathcal{U}_{\mathrm{p}}$, $\{\langle a, 0.5, 0.4\rangle, \langle b, 0.5, 0.4\rangle, \langle c, 0.4, 0.7\rangle\}$, $\{\langle a, 0.3, 0.6\rangle, \langle b, 0.3, 0.6\rangle, \langle c, 0.4, 0.7\rangle\}, \{\langle a, 0.5, 0.6\rangle, \langle b, 0.5, 0.6\rangle, \langle c, 0.6, 0.3\rangle\}$*. In this example, the set* A *is PNSO, PNPO, PNαO because it satisfies the required conditions* $\mathrm{A} \subseteq \mathcal{PN}cl(\mathcal{PN}int(\mathrm{A}))$*,* $A \subseteq \mathcal{PN}int(\mathcal{PN}cl(\mathrm{A}))$*,* $A \subseteq \mathcal{PN}int(\mathcal{PN}cl(\mathcal{PN}int(\mathrm{A})))$ *respectively.*

Theorem 3.1. *If* A *is PNO in* $(\mathcal{U}, \tau_{\mathrm{R}}(\mathrm{X}))$*, then it is a PN αO in* $\mathcal{U}$*.*

Proof. Since A is PNO in $\mathcal{U}$, $\mathcal{PN}int(\mathrm{A}) = \mathrm{A}$. Now:

$$\mathcal{PN}cl(\mathcal{PN}int(\mathrm{A})) = \mathcal{PN}cl(\mathrm{A}) \supseteq \mathrm{A}$$

$$\Rightarrow \mathrm{A} \subseteq \mathcal{PN}cl(\mathrm{A}) = \mathcal{PN}cl(\mathcal{PN}int(\mathrm{A}))$$

$$\Rightarrow \mathcal{PN}int(\mathrm{A}) \subseteq \mathcal{PN}int(\mathcal{PN}cl(\mathcal{PN}int(\mathrm{A})))$$

$$\Rightarrow \mathrm{A} \subseteq \mathcal{PN}int(\mathcal{PN}cl(\mathcal{PN}int(\mathrm{A})))$$

Therefore, A is PN αO in $\mathcal{U}$. □

Theorem 3.2. $\tau_{\mathrm{PR}}^{\alpha}(\mathrm{X}) \subseteq \mathrm{PNSO}(\mathcal{U}, \mathrm{X})$ *in a PNT space*$(\mathcal{U}, \tau_{\mathrm{R}}(\mathrm{X}))$.

Proof. If $\mathrm{A} \in \tau^{\alpha}_{\mathrm{PR}}(\mathrm{X})$, then $\mathrm{A} \subseteq \mathcal{PN}int(\mathcal{PN}c\ell(\mathcal{PN}int(\mathrm{A})))$. Consider any set D with interior $\mathcal{PN}int(\mathrm{D}) \subseteq \mathrm{D}$. Therefore

$$\mathcal{PN}int(\mathcal{PN}c\ell(\mathcal{PN}int(\mathrm{A}))) \subseteq \mathcal{PN}c\ell(\mathcal{PN}int(\mathrm{A}))$$

$$\Rightarrow \mathrm{A} \in \mathrm{PNSO}(\mathcal{U}, \mathrm{X})$$

Thus $\tau^{\alpha}_{\mathrm{PR}}(\mathrm{X}) \subseteq \mathrm{PNSO}(\mathcal{U}, \mathrm{X})$. □

Theorem 3.3. $\tau^{\alpha}_{\mathrm{PR}}(\mathrm{X}) \subseteq \mathrm{PNPO}(\mathcal{U}, \mathrm{X})$ *in a PNT space*$(\mathcal{U}, \tau_{\mathrm{R}}(\mathrm{X}))$.

Proof. If $\mathrm{A} \in \tau^{\alpha}_{\mathrm{PR}}(\mathrm{X})$, then $\mathrm{A} \subseteq \mathcal{PN}int(\mathcal{PN}c\ell(\mathcal{PN}int(\mathrm{A})))$. Since $\mathcal{PN}int \subseteq \mathrm{A}$, $\mathcal{PN}c\ell(\mathcal{PN}int) \subseteq \mathcal{PN}c\ell(\mathrm{A})$, $\mathcal{PN}int(\mathcal{PN}c\ell(\mathcal{PN}int)) \subseteq \mathcal{PN}int(\mathcal{PN}c\ell(\mathrm{A})) \Rightarrow$ $\mathrm{A} \subseteq \mathcal{PN}int(\mathcal{PN}c\ell(\mathrm{A}))$. Thus $\mathrm{A} \in \mathrm{PNPO}(\mathcal{U}, \mathrm{X})$. □

Theorem 3.4. $\tau^{\alpha}_{\mathrm{PR}}(\mathrm{X}) = \mathrm{PNPO}(\mathcal{U}, \mathrm{X}) \cap \mathrm{PNSO}(\mathcal{U}, \mathrm{X})$ *in a PNT space*$(\mathcal{U}, \tau_{\mathrm{R}}(\mathrm{X}))$.

Proof. Let $\mathrm{A} \in \tau^{\alpha}_{\mathrm{PR}}(\mathrm{X})$, then $\mathrm{A} \in \mathrm{PNSO}(\mathcal{U}, \mathrm{X})$ and also $\mathrm{A} \in \mathrm{PNPO}(\mathcal{U}, \mathrm{X})$ (by the previous theorems). Thus

$$\mathrm{A} \in \mathrm{PNPO}(\mathcal{U}, \mathrm{X}) \cap \mathrm{PNSO}(\mathcal{U}, \mathrm{X}) \Rightarrow \tau^{\alpha}_{\mathrm{PR}}(\mathrm{X}) \subseteq \mathrm{PNPO}(\mathcal{U}, \mathrm{X}) \cap \mathrm{PNSO}(\mathcal{U}, \mathrm{X})$$

Conversely, if $\mathrm{A} \in \mathrm{PNPO}(\mathcal{U}, \mathrm{X}) \cap \mathrm{PNSO}(\mathcal{U}, \mathrm{X})$, then:

$$\mathrm{A} \in \mathrm{PNSO}(\mathcal{U}, \mathrm{X}) \Rightarrow \mathrm{A} \subseteq \mathcal{PN}c\ell(\mathcal{PN}int(\mathrm{A})) \tag{3.1}$$

$$\mathrm{A} \in \mathrm{PNPO}(\mathcal{U}, \mathrm{X}) \Rightarrow \mathrm{A} \subseteq \mathcal{PN}int(\mathcal{PN}c\ell(\mathrm{A})) \tag{3.2}$$

From (3.1),

$$\mathcal{PN}c\ell(\mathrm{A}) \subseteq \mathcal{PN}c\ell(\mathcal{PN}c\ell(\mathcal{PN}int(\mathrm{A})))$$
$$\Rightarrow \mathrm{A} \subseteq \mathcal{PN}int(\mathcal{PN}c\ell(\mathcal{PN}int(\mathrm{A})))$$

Thus

$$\mathrm{PNSO}(\mathcal{U}, X) \subseteq \tau^{\alpha}_{PR}(\mathrm{X}) \tag{3.3}$$

From (3.2),

$$\mathrm{A} \subseteq \mathcal{PN}int(\mathcal{PN}c\ell(\mathrm{A})) \Rightarrow \mathrm{A} \subseteq \mathcal{PN}int(\mathcal{PN}c\ell(\mathcal{PN}int(\mathrm{A})))$$

Thus

$$\mathrm{PNPO}(\mathcal{U}, \mathrm{X}) \subseteq \tau^{\alpha}_{\mathrm{PR}}(\mathrm{X}) \tag{3.4}$$

Equations (3.3) and (3.4) imply that $\mathrm{PNSO}(\mathcal{U}, \mathrm{X}) \cap \mathrm{PNPO}(\mathcal{U}, \mathrm{X}) \subseteq \tau^{\alpha}_{\mathrm{PR}}(\mathrm{X})$. Therefore $\tau^{\alpha}_{\mathrm{PR}}(\mathrm{X}) = \mathrm{PNPO}(\mathcal{U}, \mathrm{X}) \cap \mathrm{PNSO}(\mathcal{U}, \mathrm{X})$. □

Theorem 3.5. *If in a PNT space* $(\mathcal{U}, \tau_{\mathrm{R}}(\mathrm{X}))$*,* $\mathrm{PL_R}(\mathrm{X}) = \mathrm{PU_R}(\mathrm{X}) = \mathrm{X}$*, then* $\mathcal{U}_{\mathrm{p}}, \emptyset_{\mathrm{p}}, \mathrm{PNL_R}(\mathrm{X})\ (= \mathrm{PNU_R}(\mathrm{X}))$ *and any set A such that* $\mathrm{PNL_R}(\mathrm{X}) \subseteq \mathrm{A}$ *is the only PN* α*O set in* $\mathcal{U}$*.*

Proof. Since $\mathrm{PNL_R}(\mathrm{X}) = \mathrm{PNU_R}(\mathrm{X}) = \mathrm{X}$, the PNT is $\tau_{\mathrm{R}}(\mathrm{X}) = \{\mathcal{U}_{\mathrm{p}}, \emptyset_{\mathrm{p}}, \mathrm{PNL_R}(\mathrm{X})\}$. From Theorem 3.3, any PN open set is PN αO set. Thus $\mathcal{U}_{\mathrm{p}}, \emptyset_{\mathrm{p}}, \mathrm{PNL_R}(\mathrm{X})$ are all PN αO. Let us assume $\mathrm{A} \subset \mathrm{PNL_R}(\mathrm{X})$, then $\mathcal{PN}int(\mathrm{A}) = \emptyset_{\mathrm{p}}$, because the only open set contained in A is $\emptyset_{\mathrm{p}}$. Then $\mathcal{PN}c\ell(\mathcal{PN}int(\mathrm{A})) = \mathcal{PN}c\ell(\emptyset_{\mathrm{p}}) = \emptyset_{\mathrm{p}}$. Thus A is not a PN αO set. If $\mathrm{PNL_R}(\mathrm{X}) \subset \mathrm{A}$, then $\mathrm{PNL_R}(\mathrm{X})$ is the largest PN open subset of A (i.e. $\mathcal{PN}int(\mathrm{A}) = \mathrm{PNL_R}(\mathrm{X})$).

$$\mathcal{PN}int\ (\mathcal{PN}c\ell(\mathcal{PN}int(\mathrm{A}))) = \mathcal{PN}int(\mathcal{PN}c\ell(\mathrm{PL_R}(\mathrm{X}))) \Rightarrow \mathcal{PN}int(\mathcal{U}_{\mathrm{p}}) = \mathcal{U}_{\mathrm{p}}$$

Thus $\mathrm{A} \subseteq \mathcal{PN}int(\mathcal{PN}c\ell(\mathcal{PN}int(\mathrm{A}))) \Rightarrow \mathrm{A}$ is a PN αO. $\square$

Theorem 3.6. $\mathcal{U}_{\mathrm{p}}, \emptyset_{\mathrm{p}}, \mathrm{PNU_R}(\mathrm{X})$ *and any set that contains* $\mathrm{PNU_R}(\mathrm{X})$ *are the only PN* α*O sets in a PNT space if* $\mathrm{PNL_R}(\mathrm{X}) = \emptyset_{\mathrm{p}}$.

Proof. Since $\mathrm{PNL_R}(\mathrm{X}) = \emptyset_{\mathrm{p}}$, $\mathrm{PNB_R}(\mathrm{X}) = \mathrm{PNU_R}(\mathrm{X})$. Thus the PNT is $\{\mathcal{U}_{\mathrm{p}}, \emptyset_{\mathrm{p}}, \mathrm{PU_R}(\mathrm{X})\}$ and by Theorem 3.3 the members of PNT are all PN αO sets. Let $\mathrm{A} \subset \mathrm{PNU_R}(\mathrm{X})$. If $\mathrm{A} \subset \mathrm{PU_R}(\mathrm{X})$, then A is not a PN αO set. If $\mathrm{PNU_R}(\mathrm{X}) \subset \mathrm{A}$, then $\mathrm{PNU_R}(\mathrm{X})$ is the largest PNO subset of A

$$\Rightarrow \mathcal{PN}int(\mathcal{PN}c\ell(\mathcal{PN}int(\mathrm{A}))) = \mathcal{U}_{\mathrm{p}}$$

Thus A is a PN αO in $\mathcal{U}$. Hence $\mathcal{U}_{\mathrm{p}}, \emptyset_{\mathrm{p}}, \mathrm{PNU_R}(\mathrm{X})$ and any superset of $\mathrm{PNU_R}(\mathrm{X})$ is the only PN αO set in $\mathcal{U}$, if $\mathrm{PNL_R}(\mathrm{X}) = \emptyset_{\mathrm{p}}$. $\square$

Theorem 3.7. *If* $\mathrm{PNU_R}(\mathrm{X}) = \mathcal{U}_{\mathrm{p}}$ *and* $\mathrm{PNL_R}(\mathrm{X})$ *is non-empty in a PNT space* $(\mathcal{U}, \tau_{\mathrm{R}}(\mathrm{X}))$*, then* $\mathcal{U}_{\mathrm{p}}, \emptyset_{\mathrm{p}}, \mathrm{PNL_R}(\mathrm{X})\ \&\mathrm{PNU_R}(\mathrm{X})$ *are the only PN* α*O sets in* $\mathcal{U}$*.*

Proof. Since $\mathrm{PNU_R}(\mathrm{X}) = \mathcal{U}_{\mathrm{p}}$ and $\mathrm{PNU_R}(\mathrm{X}) = \emptyset_{\mathrm{p}}$, the PNO sets in $\mathcal{U}$ are $\mathcal{U}_{\mathrm{p}}$, $\emptyset_{\mathrm{p}}$, $\mathrm{PNL_R}(\mathrm{X})$ and $\mathrm{PNB_R}(\mathrm{X})$ and hence they are also PN αO. If $\mathrm{A} = \emptyset$, then obviously it is PN αO.

Now, let A be non-empty, and let $\mathrm{A} \subset \mathrm{PNL_R}(\mathrm{X})$. It implies $\mathcal{PN}int(\mathrm{A}) = \emptyset_{\mathrm{p}}$, because $\emptyset_{\mathrm{p}}$ is the largest PNO in A and hence $\mathrm{A} \not\subset \mathcal{PN}int(\mathcal{PN}c\ell(\mathcal{PN}int(\mathrm{A})))$. This implies A is not PN αO set. When $\mathrm{PNL_R}(\mathrm{X}) \subset \mathrm{A}$, $\mathcal{PN}int(\mathrm{A}) = \mathrm{PNL_R}(\mathrm{X})$ and therefore

$$\mathcal{PN}int(\mathcal{PN}c\ell(\mathcal{PN}int(\mathrm{A}))) = \mathcal{PN}int(\mathcal{PN}c\ell(\mathrm{PNL_R}(\mathrm{X}))) = \mathrm{PL_R}(\mathrm{X}) \subset \mathrm{A}$$

$$\Rightarrow \mathrm{A} \not\subset \mathcal{PN}\imath nt\left(\mathcal{PN}c\ell\left(\mathcal{PN}\imath nt\left(\mathrm{A}\right)\right)\right)$$

Therefore A is not PN αO set.

Similarly it can be proved that A is not PN αO for $\mathrm{PNB_R}\left(\mathrm{X}\right) \subset \mathrm{A}$ and $\mathrm{A} \subset \mathrm{PNB_R}\left(\mathrm{X}\right)$. If A has one element in $\mathrm{PNB_R}\left(\mathrm{X}\right)$ or $\mathrm{PNL_R}\left(\mathrm{X}\right)$, then A is not PN αO set. $\mathcal{U}_\mathrm{p}, \emptyset_\mathrm{p}, \mathrm{PNL_R}\left(\mathrm{X}\right) \& \mathrm{PNU_R}\left(\mathrm{X}\right)$ are the only PN αO sets in $\mathcal{U}$ when $\mathrm{PNU_R}\left(\mathrm{X}\right) = \mathcal{U}_\mathrm{p}$ and $\mathrm{PNL_R}\left(\mathrm{X}\right) = \emptyset_\mathrm{p}$. □

Corollary 3.1. $\tau^{\alpha}_{\mathrm{PR}}\left(\mathrm{X}\right) = \tau_\mathrm{R}\left(\mathrm{X}\right)$*, if* $\mathrm{PNU_R}\left(\mathrm{X}\right) = \mathcal{U}_\mathrm{p}$.

Theorem 3.8. *Let* $\mathrm{PNL_R}\left(\mathrm{X}\right) \neq \mathrm{PNU_R}\left(\mathrm{X}\right)$ *where* $\mathrm{PNL_R}\left(\mathrm{X}\right) \neq \emptyset_\mathrm{p}$ *and* $\mathrm{PNU_R}\left(\mathrm{X}\right) \neq \mathcal{U}_\mathrm{p}$ *in a PNT space* $(\mathcal{U}, \tau_\mathrm{R}\left(\mathrm{X}\right))$*. Then* $\mathcal{U}_\mathrm{p}, \emptyset_\mathrm{p}, \mathrm{PNL_R}\left(\mathrm{X}\right), \mathrm{PNU_R}\left(\mathrm{X}\right), \mathrm{PNB_R}\left(\mathrm{X}\right)$ *and any set A such that it contains* $\mathrm{PNU_R}\left(\mathrm{X}\right)$ *are the only PN* α*O sets in* $\mathcal{U}$*.*

Proof. The PNT on $\mathcal{U}$ is given by $\tau_\mathrm{R}\left(\mathrm{X}\right) = \{\mathcal{U}_\mathrm{p}, \emptyset_\mathrm{p}, \mathrm{PNL_R}\left(\mathrm{X}\right), \mathrm{PNU_R}\left(\mathrm{X}\right), \mathrm{PNB_R}\left(\mathrm{X}\right)\}$ and hence $\mathcal{U}_\mathrm{p}, \emptyset_\mathrm{p}, \mathrm{PNL_R}\left(\mathrm{X}\right), \mathrm{PNU_R}\left(\mathrm{X}\right), \mathrm{PNB_R}\left(\mathrm{X}\right)$ are PN αO sets in $\mathcal{U}$. Let A be any PN set in $\mathcal{U}$ such that $\mathrm{PNU_R}\left(\mathrm{X}\right) \subset \mathrm{A}$, then $\mathcal{PN}\imath nt\left(\mathrm{A}\right) = \mathrm{PNU_R}(\mathrm{X})$. Therefore $\mathcal{PN}\imath nt\left(\mathcal{PN}c\ell\left(\mathcal{PN}\imath nt\left(\mathrm{A}\right)\right)\right) = \mathcal{PN}\imath nt\left(\mathcal{PN}c\ell\left(\mathrm{PNU_R}\left(\mathrm{X}\right)\right)\right) = \mathcal{PN}\imath nt\left(\mathcal{U}_\mathrm{p}\right)$.

Thus $\mathrm{A} \subseteq \mathcal{PN}\imath nt\left(\mathcal{PN}c\ell\left(\mathcal{PN}\imath nt\left(\mathrm{A}\right)\right)\right) \Longrightarrow$ A is PN αO in $\mathcal{U}$ when $\mathrm{PNU_R}\left(\mathrm{X}\right) \subset \mathrm{A}$. When $\mathrm{A} \subset \mathrm{PL_R}\left(\mathrm{X}\right)$, $\mathcal{PN}\imath nt\left(\mathrm{A}\right) = \emptyset_\mathrm{p} \Longrightarrow \mathcal{PN}\imath nt\left(\mathcal{PN}c\ell\left(\mathcal{PN}\imath nt\left(\mathrm{A}\right)\right)\right) = \emptyset_\mathrm{p}$. Thus A is not a PN αO set. When $\mathrm{A} \subset \mathrm{PNU_R}\left(\mathrm{X}\right)$ but is neither a subset of $\mathrm{PL_R}\left(\mathrm{X}\right)$ nor of $\mathrm{PNB_R}\left(\mathrm{X}\right)$, then $\mathcal{PN}\imath nt\left(\mathrm{A}\right) = \emptyset_\mathrm{p} \Longrightarrow$ A is not a PN αO set. Thus $\mathcal{U}_\mathrm{p}, \emptyset_\mathrm{p}, \mathrm{PNL_R}\left(\mathrm{X}\right), \mathrm{PNU_R}\left(\mathrm{X}\right), \mathrm{PNB_R}\left(\mathrm{X}\right)$ and any subset $\mathrm{PNU_R}\left(\mathrm{X}\right) \subset \mathrm{A}$ are the only PNαO sets. □

4. FORMS OF PYTHAGOREAN NANO SEMI OPEN AND PYTHAGOREAN NANO REGULAR OPEN SETS

Remark 4.1. $\mathcal{U}_\mathrm{p}$*,* $\emptyset_\mathrm{p}$ *are always Pythagorean Nano Semi Open(PNSO) since* $\mathcal{PN}c\ell\left(\mathcal{PN}\imath nt\left(\mathcal{U}_\mathrm{p}\right)\right) = \mathcal{U}_\mathrm{p}$ *and* $\mathcal{PN}c\ell\left(\mathcal{PN}\imath nt\left(\emptyset_\mathrm{p}\right)\right) = \emptyset_\mathrm{p}$.

Theorem 4.1. *Let* $(\mathcal{U}, \tau_\mathrm{R}\left(\mathrm{X}\right))$ *be a PNT space with* $\mathrm{PNU_R}\left(\mathrm{X}\right) = \mathrm{PNL_R}\left(\mathrm{X}\right)$*, then* $\emptyset_\mathrm{p}$ *and sets A such that* $\mathrm{PNL_R}\left(\mathrm{X}\right) \subseteq \mathrm{A}$ *are the only PNSO subsets of* $\mathcal{U}$*.*

Proof. Since $\mathrm{PNU_R}\left(\mathrm{X}\right) = \mathrm{PNL_R}\left(\mathrm{X}\right)$, $\tau_\mathrm{R}\left(\mathrm{X}\right) = \{\mathcal{U}_\mathrm{p}, \emptyset_\mathrm{p}, \mathrm{PNL_R}\left(\mathrm{X}\right)\}$. $\mathcal{PN}\imath nt\left(\emptyset_\mathrm{p}\right) = \emptyset_\mathrm{p}$, $\mathcal{PN}c\ell\left(\mathcal{PN}\imath nt\left(\emptyset_\mathrm{p}\right)\right) = \emptyset_\mathrm{p}$ implying $\emptyset_\mathrm{p}$ is PNSO. Let $\mathrm{A} \neq \emptyset_\mathrm{p}$ be a subset of $\mathcal{U}$ and $\mathrm{A} \subset \mathrm{PNL_R}\left(\mathrm{X}\right)$, then $\mathcal{PN}c\ell\left(\mathcal{PN}\imath nt\left(\mathrm{A}\right)\right) = \mathcal{PN}c\ell\left(\emptyset_\mathrm{p}\right) = \emptyset_\mathrm{p}$. If $\mathrm{A} \subset \mathrm{PNL_R}\left(\mathrm{X}\right)$ then A is not a PNSO.

Consider $\mathrm{PNL_R}(\mathrm{X}) \subseteq \mathrm{A}$, thus $\mathcal{PN}cl(\mathcal{PN}int(\mathrm{A})) = \mathcal{PN}cl(\mathrm{PNL_R}(\mathrm{X})) = \mathcal{U}_\mathrm{p}$ (since $\mathrm{PNL_R}(\mathrm{X}) = \mathrm{PNU_R}(\mathrm{X})$). Thus $\mathrm{A} \subseteq \mathcal{PN}cl(\mathcal{PN}int(\mathrm{A}))$ and A is PNSO. Therefore $\emptyset_\mathrm{p}$ and any set which contains $\mathrm{PNL_R}(\mathrm{X})$ are the only PNSO sets in $\mathcal{U}$ whenever $\mathrm{PNU_R}(\mathrm{X}) = \mathrm{PNL_R}(\mathrm{X})$. □

Theorem 4.2. *If* $\mathrm{PNL_R}(\mathrm{X}) = \emptyset_\mathrm{p}$ *and* $\mathrm{PNU_R}(\mathrm{X}) \neq \mathcal{U}_\mathrm{p}$, *then only those sets containing* $\mathrm{PNU_R}(\mathrm{X})$ *are the PNSO sets in* $\mathcal{U}$.

Proof. Let $\tau_\mathrm{R}(\mathrm{X}) = \{\mathcal{U}_\mathrm{p}, \emptyset_\mathrm{p}, \mathrm{PNU_R}(\mathrm{X})\}$ and A be a non-empty subset of $\mathcal{U}$. If $\mathrm{A} \subset \mathrm{PNU_R}(\mathrm{X})$, then $\mathcal{PN}int(\mathrm{A}) = \emptyset_\mathrm{p} \implies \mathcal{PN}cl(\mathcal{PN}int(\mathrm{A})) = \emptyset_\mathrm{p}$, thus A is not PNSO.

Consider $\mathrm{PNU_R}(\mathrm{X}) \subseteq \mathrm{A}$, then $\mathcal{PN}int(\mathrm{A}) = \mathrm{PNU_R}(\mathrm{X}) \implies \mathcal{PN}cl(\mathcal{PN}int(\mathrm{A})) = \mathcal{U}_\mathrm{p}$. This implies that A is a PNSO set. Thus the sets A, a superset of $\mathrm{PNU_R}(\mathrm{X})$ are the only PNSO sets in $\mathcal{U}$ whenever $\mathrm{PNL_R}(\mathrm{X}) = \emptyset_\mathrm{p}$ and $\mathrm{PNU_R}(\mathrm{X}) \neq \mathcal{U}_\mathrm{p}$. □

Theorem 4.3. *If* $\mathrm{PNU_R}(\mathrm{X}) = \mathcal{U}_\mathrm{p}$ *is a PNT space, then* $\mathcal{U}_\mathrm{p}, \emptyset_\mathrm{p}, \mathrm{PNB_R}(\mathrm{X})$ *and* $\mathrm{PNL_R}(\mathrm{X})$ *are the only PNSO sets in* $\mathcal{U}$.

Proof. Let $\tau_\mathrm{R}(\mathrm{X}) = \{\mathcal{U}_\mathrm{p}, \emptyset_\mathrm{p}, \mathrm{PNL_R}(\mathrm{X}), \mathrm{PNB_R}(\mathrm{X})\}$ and A be a non-empty subset of $\mathcal{U}$. Obviously A is not PNSO set when $\mathrm{A} \subset \mathrm{PNL_R}(\mathrm{X})$. If $\mathrm{A} = \mathrm{PNL_R}(\mathrm{X})$ then $\mathcal{PN}cl(\mathcal{PN}int(\mathrm{A})) = \mathrm{PNL_R}(\mathrm{X})$ and hence $\mathrm{A} \subset \mathcal{PN}cl(\mathcal{PN}int(\mathrm{A}))$ which implies that A is a PNSO set. But A is not a PNSO set when $\mathrm{PNL_R}(\mathrm{X}) \subset \mathrm{A}$ since $\mathcal{PN}cl(\mathcal{PN}int(\mathrm{A})) = \mathrm{PNL_R}(\mathrm{X})$ but $\mathrm{A} \not\subset \mathrm{PNL_R}(\mathrm{X})$.

Similarly if $\mathrm{A} \subset \mathrm{PNB_R}(\mathrm{X})$ and $\mathrm{PNL_R}(\mathrm{X}) \subset \mathrm{A}$, then $\mathcal{PN}cl(\mathcal{PN}int(\mathrm{A})) = \emptyset_\mathrm{p}$ and $\mathcal{PN}cl(\mathcal{PN}int(\mathrm{A})) = \mathrm{PNB_R}(\mathrm{X})$ respectively. Thus $\mathrm{A} \not\subset \mathcal{PN}cl(\mathcal{PN}int(\mathrm{A}))$. Therefore A is not a PNSO set. If A has at least one element in $\mathrm{PNL_R}(\mathrm{X})$ and at least one element in $\mathrm{PNB_R}(\mathrm{X})$, then A is not a PNSO. Thus $\mathcal{U}_\mathrm{p}, \emptyset_\mathrm{p}, \mathrm{PNL_R}(\mathrm{X})$, $\mathrm{PNB_R}(\mathrm{X})$ are the only PNSO sets in $\mathcal{U}$ when $\mathrm{PNU_R}(\mathrm{X}) = \mathcal{U}_\mathrm{p}$ and $\mathrm{PNL_R}(\mathrm{X}) \neq \emptyset_\mathrm{p}$. □

Corollary 4.1. *If* $\mathrm{PL_R}(\mathrm{X}) = \emptyset_\mathrm{p}$ *in Theorem 4.3, then* $\emptyset_\mathrm{p}$ *and* $\mathcal{U}_\mathrm{p}$ *are the only PNSO sets in* $\mathcal{U}$.

Theorem 4.4. *If A and B are PNSO in* $\mathcal{U}$, *then the union is also a PNSO in* $\mathcal{U}$.

Proof. Since A and B are PNSO, $\mathrm{A} \subseteq \mathcal{PN}cl(\mathcal{PN}int(\mathrm{A}))$ and $\mathrm{B} \subseteq \mathcal{PN}cl(\mathcal{PN}int(\mathrm{B}))$.

Consider $\mathrm{A} \cup \mathrm{B}$.

$$\mathrm{A} \cup \mathrm{B} \subseteq \mathcal{PN}cl(\mathcal{PN}int(\mathrm{A})) \cup \mathcal{PN}cl(\mathcal{PN}int(\mathrm{B}))$$

$$\subseteq \mathcal{PN}cl\,(\mathcal{PN}int\,(\mathrm{A}) \cup \mathcal{PN}int\,(\mathrm{B}))$$
$$\subseteq \mathcal{PN}cl\,(\mathcal{PN}int\,(\mathrm{A} \cup \mathrm{B}))$$

since $\mathcal{PN}int\,(\mathrm{A}) \cup \mathcal{PN}int\,(\mathrm{B}) = \mathcal{PN}int\,(\mathrm{A} \cup \mathrm{B})$. Therefore the union of two PNSO sets is a PNSO set, but intersection of two PNSO sets need not be PNSO. □

Definition 4.1. *A subset A of a PNT space $(\mathcal{U}, \tau_{\mathrm{R}}\,(\mathrm{X}))$ is Pythagorean Nano Regular Open(PNRO) in $\mathcal{U}$, if $\mathcal{PN}int\,(\mathcal{PN}cl\,(\mathrm{A})) = \mathrm{A}$.*

Theorem 4.5. *Any PNRO set is PNO.*

Proof. If A is PNRO in $\mathcal{U}$, then $\mathcal{PN}int\,(\mathcal{PN}cl\,(\mathrm{A})) = \mathrm{A}$. Since

$$\mathcal{PN}int\,(\mathcal{PN}cl\,(\mathrm{A})) = \mathrm{A}$$

$$\mathcal{PN}int\,(\mathrm{A}) = \mathcal{PN}int\,(\mathcal{PN}int\,(\mathcal{PN}cl\,(\mathrm{A}))) \implies \mathcal{PN}int\,(\mathrm{A}) = \mathcal{PN}int\,(\mathcal{PN}cl\,(\mathrm{A})) = \mathrm{A}$$

Thus $\mathcal{PN}int\,(\mathrm{A}) = \mathrm{A}$. Therefore A is a PNO set in $\mathcal{U}$. But the converse of the theorem need not be true. □

Theorem 4.6. *In a PNT space $(\mathcal{U}, \tau_{\mathrm{R}}\,(\mathrm{X}))$, if $\mathrm{PNL_R}\,(\mathrm{X}) = \mathrm{PNU_R}\,(\mathrm{X})$, then the only PNRO sets are $\mathcal{U}_{\mathrm{p}}$ and $\emptyset_{\mathrm{p}}$.*

Proof. The PNO sets in $\mathcal{U}$ are $\mathcal{U}_{\mathrm{p}}$, $\emptyset_{\mathrm{p}}$ and

$$\mathcal{PN}int\,(\mathcal{PN}cl\,(\mathrm{PNL_R}\,(\mathrm{X}))) = \mathcal{U}_{\mathrm{p}} \neq \mathrm{PNL_R}\,(\mathrm{X})$$

We have $\mathrm{PNL_R}\,(\mathrm{X}) \subseteq \mathcal{U}_{\mathrm{p}}$ but for PNRO $\mathcal{PN}int\,(\mathcal{PN}cl\,(\mathrm{A})) = \mathrm{A}$. Thus $\mathrm{PNL_R}\,(\mathrm{X})$ is not PNRO. Therefore the only PNRO sets are $\mathcal{U}_{\mathrm{p}}$ and $\emptyset_{\mathrm{p}}$. □

Definition 4.2. *Let $(\mathcal{U}, \tau_{\mathrm{R}}\,(\mathrm{X}))$ be a PNT space and $\mathrm{A} \subseteq \mathcal{U}$. Then A is Pythagorean nano γ open $(PN\gamma O)$ and Pythagorean nano β open $(PN\beta O)$ if*

$$\mathrm{A} \subseteq \mathcal{PN}cl\,(\mathcal{PN}int\,(\mathrm{A})) \cup \mathrm{A} \subseteq \mathcal{PN}int\,(\mathcal{PN}cl\,(\mathrm{A}))$$

and

$$A \subseteq \mathcal{PN}cl\,(\mathcal{PN}int\,(\mathcal{PN}cl\,(\mathrm{A})))$$

respectively.

Definition 4.3. *Let $(\mathcal{U}, \tau_{\mathrm{R}}\,(\mathrm{X}))$ and $(\mathcal{V}, \tau_{\mathrm{R}}\,(\mathrm{Y}))$ be PNT spaces. The mapping $f : (\mathcal{U}, \tau_{\mathrm{R}}\,(\mathrm{X})) \to (\mathcal{V}, \tau_{\mathrm{R}}\,(\mathrm{Y}))$ is said to be PNβcontinuous if $f^{-1}\,(A)$ is $PN\beta O$ in U for every PNO A in V.*

Proposition 4.1. *For every PNT space $(\mathcal{U}, \tau_{\mathrm{R}}(\mathrm{X}))$, we have that: PNSO$(\mathcal{U}, \mathrm{X}) \cup$ PNPO$(\mathcal{U}, \mathrm{X}) \subseteq (PN\gamma O) \subseteq (PN\beta O)$ holds but none of them are vice versa.*

Proposition 4.2. *Let $(\mathcal{U}, \tau_{\mathrm{R}}(\mathrm{X}))$ be a PNT space then:*

(1) *If $\mathrm{A} \subseteq \mathcal{U}$ is PNO and $B \subseteq \mathcal{U}$ is PNSO (resp. PNPO, PNβO, PNγO) then $A \cap B$ is PNSO (resp. PNPO, PNβO, PNγO).*

(2) *For every subset $A \subseteq \mathcal{U}$, $A \cap PNint\,(PNcl\,(A))$ is PNPO.*

(3) *$A \subseteq U$ is $PN\gamma O$, if and only if A is union of PNSO and PNPO.*

5. CONCLUSION

Herein, weak Pythagorean nano open sets have been introduced. Along with it, various forms of Pythagorean nano semi-open, Pythagorean nano pre-open and Pythagorean nano α-open sets were derived. Furthermore, strongest forms of Pythagorean nano open sets have beeb studied.

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