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COMPLETE AND CYCLE GRAPH COVERS IN A ZERO DIVISOR GRAPH

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ABSTRACT. Let R be a commutative ring and let $\Gamma(Z_n)$ be the zero divisor graph of a commutative ring R, whose vertices are non-zero zero divisors of Z_n, and such that the two vertices u, v are adjacent if n divides uv. In this paper, we introduce the concept of Decomposition of Zero Divisor Graph in a commutative ring and also disscuse the some special cases of $\Gamma(Z_{2^2p^2})$, $\Gamma(Z_{3^2p^2})$, $\Gamma(Z_{5^2p^2})$, $\Gamma(Z_{7^2p^2})$ and $\Gamma(Z_{p^2q^2})$.

1. INTRODUCTION

All graphs considered here are finite and undirected, unless otherwise noted. For the standard graph-theoretic terminology the reader is referred to [4].

As usual K_n denotes the complete graph on n vertices and $K_{m,n}$ denotes the complete bipartite graph with parts of sizes m and n. Let P_k denote a path of length k and let S_k denote a star with k edges. Let C_k denotes a cycle of length K, i.e., $S_k \equiv K_{1,k}$. Let $D(K_{m,n})$ be the decomposition of complete bipartite graph. Let $L = \{H_1, H_2, ..., H_r\}$ be a family of subgraphs of G. An L-decomposition of G is an edge-disjoint decomposition of G into positive integer α_i copies of H_i where $i \in \{1, 2, 3, \ldots, r\}$. Furthermore, if each $H_i (i \in \{1, 2, 3, \ldots, r\})$ is isomorphic to a graph H, then we say that G has an H-decomposition.

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The zero divisor graph is very useful to find the algebraic structures and properties of rings. The idea of a zero divisor graph of a commutative ring was introduced by I. Beck's in [3]. Given a ring R, let G(R) denote the graph whose vertex set is R, such that distinct vertices r and s are adjacent provided that rs = 0. I.Beck's main interest was the chromatic number $\chi(G(R))$ of the graph G(R). The general terminology, notation everything based on the papers [1, 2, 6–9]. In this paper we investigate the decomposition of $\Gamma(Z_{p^2q^2})$ into cycles and stars [5, 10] and obtain the following results.

2. Preliminaries

Definition 2.1. *[1] Let R be a commutative ring (with 1) and let Z(R) be its set of zero-divisors. We associate a (simple) graph $\Gamma(R)$ to R with vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero zero-divisor of R, and for distinct $x, y \in Z(R)^*$ the vertices x and y are adjacent if and only if $xy = 0$. Thus $\Gamma(R)$ is the empty graph if and only if R is an integral domain.*

Definition 2.2. *A graph G is decomposable into $H_1, H_2, H_3, \ldots, H_k$ if G has subgraphs $H_1, H_2, H_3, \ldots, H_k$ such that*

1. *each edge of G belongs to one of the $H_i's$ for some $i = 1, 2, 3, \ldots, k$ and*
2. *If $i \neq j$, then H_i and H_j have no edges in common.*

Theorem 2.1. *[10] For any distinct prime p and q, $\Gamma(Z_{pq})$ can be decomposable into $(q-1)C_{p-1}$, where $q > p$.*

3. Decomposition of zero divisor graph $\Gamma(Z_{p^2q^2})$

In this section we investigate the problem of decomposing zero divisor graphs $\Gamma(Z_{p^2q^2})$ into complete graph K_{pq-1} and $\frac{3pq(p-1)(q-1)}{4}$ copies of C_4, for each p and q are distinct prime numbers with $q > p$.

Theorem 3.1. *If p is any prime number and $p > 2$, then $\Gamma(Z_{2^2p^2})$ is decomposition into 1-copie of star graph $K_{1,2p(p-1)}$, 1-copie of complete graph K_{2p-1} and $p(p-1)$ copies of C_4.*

Proof. Let p is any prime number with $p > 2$. Let $\Gamma(Z_{2^2p^2})$ be the non-zero zero divisor graph. The vertex set of $\Gamma(Z_{2^2p^2})$ is $V(\Gamma(Z_{2^2p^2})) = \{2, 4, 6, \ldots, 2^2p^2 - 2, p, 2p, 3p, \ldots, 2^2p^2 - p\}$.

Case (i): Let us consider vertex subsets are $V_1, V_2, V_3 \in V$ where $V_1 = \{2p^2\}$, $V_2 = \{4, 8, 12, \ldots, 4p(p-1)\}$, $V_3 = V_3 \setminus V_2 = \{2, 6, 10, \ldots, 2(2p^2-1)\}$. That is $|V_1| = 1$, $|V_2| = p(p-1)$, $|V_3| = p(p-1)$.

The vertex $V_1 = \{2p^2\}$ be the middle vertex this vertex $v_1 \in V_1$ is adjacent to all the vertex sets V_1 and V_2. Then clearly there exits two star graphs namely $K_{1,p(p-1)}$ and $K_{1,p(p-1)}$. Hence $K_{1,2(p-1)}$ with 2(p-1) edges.

Case (ii): Let assume the vertex subset in $V_4 \in \Gamma(Z_{2^2q^2})$ where $V_4 = \{2p, 4p, 6p, \ldots, 2p(2p-1)\}$. If any two vertices in $u, v \in V_4$ and every vertex u is an adjacent to v then there exits an edge between u and v. Clearly the vertex set V_4 is complete graph K_{2p-1} with $2p-1$ vertices.

Case (iii): Let the zero-divisor graph $\Gamma(Z_{2^2q^2})$ is decomposite three type of complete bipartite graphs are $K_{2(p-1),(p-1)}$, $K_{2,(p-1)}$ and $K_{2,p(p-1)}$. By theorem[2.3] clearly shows this three complete bipatite graph covers in $p(p-1)$ copies of C_4. Hence the above three cases clearly shows the given graph $\Gamma(Z_{2^2p^2})$ is covers 1-copie of star graph, 1-copie of complete graph and $p(p-1)$ copies of C_4. □

Theorem 3.2. *If p is any prime number and $p > 3$, then $\Gamma(Z_{3^2p^2})$ is decomposition into 1-copie of complete graph K_{3p-1} with $3p-1$ vertices and $\frac{9p(p-1)}{2}$ copies of C_4.*

Proof. Let p is any prime number with $p > 3$ and let $\Gamma(Z_{3^2p^2})$ be the non-zero zero divisor graph. The vertex set of $\Gamma(Z_{3^2p^2})$ is $V = \{3, 6, 9, \ldots, 3(3p^2-1), p, 2p, 3p, \ldots, p(9p-1)\}$.

Case (i) Let the vertex subset $V_1 \in V$ where $V_1 = \{3p, 6p, 9p, \ldots, 3p(3p-1)\}$. The cordinality of V_1 is $3p-1$. If any two vertices $v_1, v_2 \in V_1$ are adjacent then clearly the vertex set V_1 is complete graph K_{3p-1} with $3p-1$ vertices.

Case (ii) Consider the vertex subsets are $V_2, V_3, V_4, V_5, V_6, V_7 \in V(\Gamma(Z_{3^2p^2}))$ where $V_2 = V_2 \setminus V_1 = \{p, 2p, 3p, \ldots, 8p\}$, $V_3 = \{9p, 18p, 27, \ldots, 9p(p-1)\}$, $V_4 = \{p^2, 2p^2, 3p^2, \ldots, 8p^2\}$., $V_5 = \{9, 18, 27, \ldots, 9(p^2-1)\}$, $V_6 = V_6 \setminus V_5 = \{3, 6, 9, \ldots, 3(3p^2-1)\}$ and $V_7 = \{3p^2, 6p^2\}$. The cardinality of above vertex sets are $|V_2| = 6(p-1)$, $|V_3| = p-1$, $|V_4| = 6$, $|V_5| = p(p-1)$, $|V_6| = 2p(p-1)$ and $|V_7| = 2$. If the pairs of vertex sets (V_2, V_3), (V_3, V_4), (V_4, V_5), (V_5, V_7) and (V_7, V_6) are adjcent then clearly there exits $K_{6(p-1),(p-1)}$, $K_{(p-1),6}$, $K_{6,p(p-1)}$, $K_{p(p-1),2}$, $K_{2,2p(p-1)}$ complete bipartite graphs. By the theorem[2.3] shows complete bipartite graph coves some copies of C_4. Then follows sum of all $K_{6(p-1),(p-1)} + K_{(p-1),6} + K_{6,p(p-1)} + K_{p(p-1),2} + K_{2,2p(p-1)} = \frac{6(p-1)(p-1)}{4} + \frac{6(p-1)}{4} + \frac{6p(p-1)}{4} + \frac{2p(p-1)}{4} + \frac{4p(p-1)}{4} = \frac{9p(p-1)}{2}$. Clearly above cases shows that the graph of $\Gamma(Z_{3^2p^2})$ is decomposition into 1 - copie of complete graph K_{3p-1} with $3p-1$ vertices and $\frac{9p(p-1)}{2}$ copies of C_4. □

Theorem 3.3. *If p is any prime number and $p > 5$, then $\Gamma(Z_{5^2p^2})$ is decomposition into 1-copie of complete graph K_{5p-1} with $5p-1$ vertices and $15p(p-1)$ copies of C_4.*

Proof. Let p is any prime number with $p > 5$ and let $\Gamma(Z_{5^2p^2})$ be the non-zero zero divisor graph. The vertex set of $\Gamma(Z_{5^2p^2})$ is $V = \{5, 10, 15, \ldots, 5(5p^2 - 1), p, 2p, 3p, \ldots, p(25p-1)\}$.

Case (i) Let the vertex subset $V_1 \in V$ where $V_1 = \{5p, 10p, 15p, \ldots, 5p(5p-1)\}$. The cordinality of V_1 is $5p-1$. If any two vertices $v_1, v_2 \in V_1$ are adjacent then clearly the vertex set V_1 is complete graph K_{5p-1} with $5p-1$ vertices.

Case (ii) Consider the vertex subsets are $V_2, V_3, V_4, V_5, V_6, V_7 \in V(\Gamma(Z_{5^2p^2}))$ where $V_2 = V_2 \setminus V_1 = \{p, 2p, 3p, \ldots, 24p\}$, $V_3 = \{25p, 50p, 75p, \ldots, 25p(p-1)\}$, $V_4 = \{p^2, 2p^2, 3p^2 \ldots, 24p^2\}$., $V_5 = \{25, 50, 75, \ldots, 25(p^2-1)\}$, $V_6 = V_6 \setminus V_5 = \{5, 10, 15, \ldots, 5(5p^2-1)\}$ and $V_7 = \{5p^2, 10p^2, 15p^2, 20p^2\}$. The cardinality of above vertex sets are $|V_2| = 20(p-1)$, $|V_3| = p-1$, $|V_4| = 20$, $|V_5| = p(p-1)$, $|V_6| = 4p(p-1)$ and $|V_7| = 4$. If the pairs of vertex sets (V_2, V_3), (V_3, V_4), (V_4, V_5), (V_5, V_7) and (V_7, V_6) are adjcent then clearly there exits $K_{20(p-1),(p-1)}$, $K_{(p-1),20}$, $K_{20,p(p-1)}$, $K_{p(p-1),4}$, $K_{4,4p(p-1)}$ complete bipartite graphs. By the theorem[2.3] shows complete bipartite graph coves some copies of C_4. Then follows sum of all $K_{20(p-1),(p-1)} + K_{(p-1),20} + K_{20,p(p-1)} + K_{p(p-1),4} + K_{4,4p(p-1)} = \frac{20(p-1)(p-1)}{4} + \frac{20(p-1)}{4} + \frac{20p(p-1)}{4} + \frac{4p(p-1)}{4} + \frac{16p(p-1)}{4} = 15p(p-1)$. Clearly above cases shows that the graph of $\Gamma(Z_{5^2p^2})$ is decomposition into 1 - copie of complete graph K_{5p-1} with $5p-1$ vertices and $15p(p-1)$ copies of C_4. □

Theorem 3.4. *If p is any prime number and $p > 7$, then $\Gamma(Z_{7^2p^2})$ is decomposition of 1-copie of complete graph K_{7p-1} with $7p-1$ vertices and $\frac{63p(p-1)}{2}$ copies of C_4.*

Proof. Let p is any prime number with $p > 7$ and let $\Gamma(Z_{7^2p^2})$ be the non-zero zero divisor graph. The vertex set of $\Gamma(Z_{7^2p^2})$ is $V = \{7, 14, 21, \ldots, 7(7p^2 - 1), p, 2p, 3p, \ldots, p(49p-1)\}$.

Case (i) Let the vertex subset $V_1 \in V$ where $V_1 = \{7p, 14p, 21p, \dots, 7p(7p-1)\}$. The cordinality of V_1 is $7p-1$. If any two vertices $v_1, v_2 \in V_1$ are adjacent then clearly the vertex set V_1 is complete graph K_{7p-1} with $7p-1$ vertices.

Case (ii) Consider the vertex subsets are $V_2, V_3, V_4, V_5, V_6, V_7 \in V(\Gamma(Z_{7^2p^2}))$ where $V_2 = V_2 \setminus V_1 = \{p, 2p, 3p, \dots, 48p\}$, $V_3 = \{49p, 98p, 147p, \dots, 49p(p-1)\}$, $V_4 = \{p^2, 2p^2, 3p^2 \dots, 48p^2\}$., $V_5 = \{49, 98, 147, \dots, 49(p^2-1)\}$, $V_6 = V_6 \setminus V_5 = \{7, 14, 21,$ $\dots,$
$7(7p^2-1)\}$ and $V_7 = \{7p^2, 14p^2, 21p^2, 28p^2, 35p^2, 42p^2\}$. The cordinality of above vertex sets are $|V_2| = 42(p-1)$, $|V_3| = p-1$, $|V_4| = 42$, $|V_5| = p(p-1)$, $|V_6| = 6p(p-1)$ and $|V_7| = 6$. If the pairs of vertex sets (V_2, V_3), (V_3, V_4), (V_4, V_5), (V_5, V_7) and (V_7, V_6) are adjcent then clearly there exits $K_{42(p-1),(p-1)}$, $K_{(p-1),42}$, $K_{42,p(p-1)}$, $K_{p(p-1),6}$, $K_{6,6p(p-1)}$ complete bipartite graphs. By the theorem[2.3] shows complete bipartite graph coves some copies of C_4. Then follows sum of all $K_{42(p-1),(p-1)} + K_{(p-1),42} + K_{42,p(p-1)} + K_{p(p-1),2} + K_{6,6p(p-1)} = \frac{42(p-1)(p-1)}{4} + \frac{42(p-1)}{4} + \frac{42p(p-1)}{4} + \frac{6p(p-1)}{4} + \frac{36p(p-1)}{4} = \frac{63p(p-1)}{2}$. Clearly above cases shows that the graph of $\Gamma(Z_{7^2p^2})$ is decomposition into 1 - copie of complete graph K_{7p-1} with $7p-1$ vertices and $\frac{63p(p-1)}{2}$ copies of C_4. □

Theorem 3.5. *If p and q are distint prime numbers with $p < q$, then $\Gamma(Z_{p^2q^2})$ is decomposition of 1-copie of complete graph K_{pq-1} with $pq-1$ vertices and $\frac{3pq(p-1)(q-1)}{4}$ copies of C_4.*

Proof. Let p and q are distinct prime numbers with $p < q$ and let $\Gamma(Z_{p^2q^2})$ be the non-zero zero divisor graph. The vertex set of $\Gamma(Z_{p^2q^2})$ is $V = \{p, 2p, 3p, \dots, p(pq^2-1), q, 2q, 3q, \dots, q(p^2q-1)\}$. Using the above theorem $\Gamma(Z_{p^2q^2})$ is 1-complete graph and cycle of length 4. Therefore decomposition of zero divisor graph of $\Gamma(Z_{p^2q^2})$ into 1-copie of complete graph K_{pq-1} with $pq-1$ vertices and $\frac{3pq(p-1)(q-1)}{4}$ copies of C_4. □

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