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CREEPING FLOW OF NON-NEWTONIAN FLUID PAST A FLUID SPHERE WITH NON-ZERO SPIN BOUNDARY CONDITION

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ABSTRACT. The paper concerns the creeping flow of non-newtonian fluid past a fluid sphere, assuming uniform stream far away from the body along its axis of symmetry. For outside and inside the fluid sphere we consider micropolar fluid. The stream function is determined by matching the solution of micropolar field equation for the flow outside the fluid sphere with that of the Stokes equation for the flow inside the fluid sphere. Two known boundary conditions are considered. No spin and spin boundary condition. The drag force experienced by the fluid sphere is determined. The variation of drag for different values of the permeability parameter (η), the coupling number N and the micropolar parameter (m) is studied. Some well-known result then deduced as a limiting case from present analysis.

1. INTRODUCTION

Micropolar fluids are fluids with micro structure. Micropolar fluids are also called as polar fluids. When micropolar fluids are suspended in a viscous medium they exhibit rigid, randomly oriented or spherical particles with their own spin and microrotation, where the deformation of a particle is ignored. Eringen [2] models of microfluid deals with a class of fluids which shows certain microscopic effects which raises from the local structure and micromotion of the fluid elements [4]. Some of the physical example of micropolar fluids are ferrofluids, blood flows, bubbly liquids, liquid crystals and so on, they all containing intrinsic polarities.

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Eringen [2] introduced the theory of simple microfluids. According to this theory simple microfluids is a fluent medium whose properties and behaviour are affected by the local motion of the fluid elements. Micropolar fluids can support body moments and stress moments and are also influenced by spin inertia. Micropolar fluids has applicants in wider range including lubrication problem, stokes flow about a sphere, stagnation flow, boundary layer flow over a plate and Taylor-Benard instability. Micropolar theory also modeled the problems on body fluids and biological flow.

Satya Deo and Pankaj Shukla [6] studied the microrotation on the boundary of the fluid sphere which is assumed to be proportional to the rotation rate of the velocity on the boundary. Further they have also evaluated the drag force experienced by the fluid sphere and also studied its variation with respect to the material parameter. The problem of an incompressible micropolar fluid flow through a porous sphere is studied by K. Ramalakshmi and Pankaj Shukla [4] and also they analysed the drag coefficient dependence by numerically and graphically with different values of micropolar parameter.

The paper concerns the creeping flow of non-newtonian fluid past a fluid sphere, assuming uniform stream far away from the body along its axis of symmetry. For outside and inside the fluid sphere we consider micropolar fluid. The stream function is determined by matching the solution of micropolar field equation for the flow outside the fluid sphere with that of the Stokes equation for the flow inside the fluid sphere. Two known boundary conditions are considered. No spin and spin boundary condition. The drag force experienced by the fluid sphere is determined. The variation of drag for different values of the permeability parameter **(**η**)**, the coupling number N and the micropolar parameter **(**m**)** is studied. Some well-known result then deduced as a limiting case from present analysis.

2. Micropolar Field Equation

For micropolar fluid, the field equation (Eringen [2]) is given by

$$\frac{\partial \rho}{\partial t} + div\rho\mathbf{v} = 0 \tag{2.1}$$

$$\rho\frac{d\mathbf{v}}{dt} = \rho\mathbf{f} - \nabla p + \kappa\nabla\times\omega - (\mu+\kappa)\nabla\times\nabla\times\mathbf{v} + (\lambda+2\mu+\kappa)\nabla(div\mathbf{v}) \tag{2.2}$$

$$\rho J\frac{d\omega}{dt} = \rho\mathbf{I} - 2\kappa\omega + \kappa\nabla\times\mathbf{v} - \gamma\nabla\times\nabla\times\omega + (\alpha+\beta+\gamma)\nabla(div\omega), \tag{2.3}$$

where ρ the density, **v** the velocity field, ω the micro-rotation field, J the gyration paramter, **f** the body force per unit mass, **I** the micro-rotation driving force per unit mass, p the hydrostatic pressure, μ the classical viscosity coefficient, κ,λ the vortex viscosity coefficient and α,β,γ are the gyroviscosity coefficients which satisfies the following inequalities,

$$\begin{aligned} 3\alpha+\beta+\gamma\geq 0, \quad 2\mu+\kappa\geq 0, \quad 3\lambda+2\mu+\kappa\geq 0,\\ \gamma\geq|\beta|, \quad \kappa\geq 0, \quad \gamma\geq 0. \end{aligned} \tag{2.4}$$

For the stress tensor T_{ij} and couple stress tensor m_{ij}, the constitutive equations are given by

$$T_{ij} = (-p + div\nu)\delta_{ij} + (2\mu+\kappa)d_{ij} + \kappa\epsilon_{ijm}(\xi_m - \omega_m) \tag{2.5}$$

and

$$m_{ij} = (\alpha div\nu)\delta_{ij} + \beta\xi_{i,j} + \gamma\xi_{j,i}, \tag{2.6}$$

d_{ij} the rate of strain components, ω_m the components of microrotation vector, $2\epsilon_m$ the components of vorticity vector and δ_{ij} the kronecker delta.

Assuming a uniform, axi-symmetric slow viscous flow of an unbounded incompressible micropolar fluid past a Non-Newtonian fluid sphere. The governing differential equation for creeping flow around and through the fluid sphere written for two regions seperated by the interface. The flow of fluid for both outside and inside region are considered to be Stokesian, that is it is assumed that the inertial terms in the momentum equation and bilinear terms in balance of first stress moments can be ignored. Also, for both region, let us consider that the body force and body couple terms are not present. Hence, the governing equations for outside and inside flow are given by

$$div\mathbf{v}^{(i)} = 0 \tag{2.7}$$

$$-\nabla p^{(i)} + \kappa\nabla\times\omega^{(i)} - (\mu+\kappa)\nabla\times\nabla\times\mathbf{v}^{(i)} = 0 \tag{2.8}$$

$$-2\kappa\omega^{(i)} + \kappa\nabla\times\mathbf{v}^{(i)} - \gamma\nabla\times\nabla\times\omega^{(i)} + (\alpha+\beta+\gamma)\nabla(\nabla.\omega^{(i)}) = 0 \tag{2.9}$$

3. Stream Function Formulation

Considering the velocity and microrotation in spherical polar coordinates (r,θ,ϕ) as

$$\mathbf{v}^{(i)} = v_r^{(i)}(r,\theta)^{)}e_r + v_\theta^{(i)}(r,\theta)^{)}e_\theta \tag{3.1}$$

and

$$\omega^{(i)} = \nu_\phi^{(i)}(r,\theta)^{)}e_\phi. \tag{3.2}$$

The velocity components $v_r^{(i)}$ and $v_\theta^{(i)}$ mentioned below should satisfy the equation of continuity

$$v_r^{(i)} = -\frac{1}{r^2 sin\theta}\frac{\partial\psi^{(i)}}{\partial\theta}; \quad v_\theta^{(i)} = \frac{1}{rsin\theta}\frac{\partial\psi^{(i)}}{\partial r}, \quad i = 1, 2. \tag{3.3}$$

Now eliminating pressure from (2.8) and using in (3.3), we get

$$E^4\psi^{(i)} - NE^2(rsin\theta\nu_\phi^{(i)}) = 0. \tag{3.4}$$

Now using (3.4) in (2.9), we find that

$$\nu_\phi^{(i)} = \frac{1}{2rsin\theta}[E^2\psi^{(i)} + \frac{2-N}{Nm^2}E^4\psi^{(i)}]. \tag{3.5}$$

As of (3.4) and (3.5) by removing $\nu_\phi^{(i)}$, we can find the stream function formulation meant for both outside and inside flow

$$E^4(E^2 - m^2)\psi^{(i)} = 0, \tag{3.6}$$

where

$$\begin{aligned} &E^2 = \frac{\partial^2}{\partial r^2} + \frac{(1-\zeta^2)}{r^2}\frac{\partial^2}{\partial\zeta^2}, \zeta = cos\theta, m^2 = \frac{\kappa(2\mu+\kappa)}{\gamma(\mu+\kappa)}a^2 \quad and \\ &N = \frac{\kappa}{\mu+\kappa} \quad being \quad the \quad coupling \quad number \quad (0 \le N \le 1). \end{aligned} \tag{3.7}$$

By applying the seperation of variables, the general regular solution of (3.6) can be expressed as

$$\begin{aligned} \psi^{(i)}(r,\zeta) = \sum_{n=2}^{\infty} [&A_n r^n + B_n r^{-n+1} + C_n r^{-n+2} + D_n r^{-n+3} + \\ &E_n r^{\frac{1}{2}} K_{n-\frac{1}{2}}(mr) + F_n r^{\frac{1}{2}} I_{n-\frac{1}{2}}(mr)]G_n(\zeta), \end{aligned} \tag{3.8}$$

$I_v(mr)$ and $K_v(mr)$ are the modified Bessel functions of first and second kind of non-integer index v=$n - \frac{1}{2}$ respectively and $G_n(\zeta)$ the Gegenbauer function of first kind as defined in Abramowitz and Stegun [1]. The solution for the region outside the sphere contain only the terms of the order $n = 2$ in the general solution (3.8), we get

$$\psi^{(1)}(r,\zeta) = [r^2 + A_2 r^{-1} + B_2 r + C_2\sqrt{r}K_{3/2}(mr)]G_2(\zeta). \tag{3.9}$$

For the region inside the fluid sphere we have

$$\psi^{(2)}(r,\zeta) = [A_2^* r^2 + C_2^* r^4 + F_2^*\sqrt{r}I_{3/2}(mr)]G_2(\zeta). \tag{3.10}$$

The microrotation components are

$$\nu_\phi^{(1)}(r,\zeta) = \frac{1}{rsin\theta}[-B_2 r^{-1} + \frac{m^2(\mu_1+\kappa)}{\kappa}C_2\sqrt{r}K_{3/2}(mr)]G_2(\zeta) \tag{3.11}$$

and

$$\nu_\phi^{(2)}(r,\zeta) = \frac{1}{rsin\theta}[5A_2^* r + \frac{m^2(\mu_2+\kappa)}{\kappa\sqrt{r}}F_2^* I_{3/2}(mr)]G_2(\zeta). \tag{3.12}$$

4. Boundary Condition

To determine the unknowns in equations (3.9) and (3.10), the following boundary conditions are used,

$$\psi^{(1)} = \psi^{(2)} \quad on \quad r = 1. \tag{4.1}$$

Let us consider the continuity of tangential velocity,

$$\frac{\partial\psi^{(1)}}{\partial r} = \frac{\partial\psi^{(2)}}{\partial r} \quad on \quad r = 1. \tag{4.2}$$

Assuming the continuity of tangential stress,

$$\tau_{r\theta}^{(1)} = \tau_{r\theta}^{(2)} \quad on \quad r = 1. \tag{4.3}$$

Considering the continuity of normal stresses,

$$p^{(1)} = p^{(2)} \quad on \quad r = 1. \tag{4.4}$$

No-spin condition on the boundary, i.e,

$$\nu_\phi^{(1)} = 0 \quad on \quad r = 1, \tag{4.5}$$

$$\nu_\phi^{(2)} = 0 \quad on \quad r = 1. \tag{4.6}$$

Applying these above boundary conditions (4.1) - (4.6), we get the following equations respectively as

$$A_2 + B_2 + C_2 K_{3/2}(m) - A_2^* - C_2^* - F_2^* I_{3/2}(m) = -1 \tag{4.7}$$

$$\begin{aligned} 2A_2^* + 4C_2^* + F_2^*[mI_{1/2}(m) - I_{3/2}(m)] + A_2 - B_2 + \\ C_2[mK_{1/2}(m) + K_{3/2}(m)] = 2 \end{aligned} \tag{4.8}$$

$$\begin{aligned} 6\eta(A_2 - C_2^*) + C_2\eta[m^2 K_{3/2}(m) + 6K_{3/2}(m) + 2mK_{1/2}(m)] + \\ F_2^*[-m^2 I_{3/2}(m) - 6I_{3/2}(m) + 2mI_{1/2}(m)] = 0 \end{aligned} \tag{4.9}$$

$$(2\mu_1 + \kappa)\beta = 10C_2^*(\mu_2 + \kappa) \tag{4.10}$$

$$-B_2 + m^2 \frac{(\mu_1 + \kappa)}{\kappa} C_2 K_{3/2}(m) = 0 \tag{4.11}$$

$$5A_2^* + m^2 \frac{(\mu_2 + \kappa)}{\kappa} F_2^* I_{3/2}(m) = 0. \tag{4.12}$$

Solving these equations mathematically we get the following unknowns,

(4.13)
$$\begin{aligned} A_2 = 2\mu_1(\kappa(2 - m^2(-2 + \eta) - 2m(-1 + \eta) - 2\eta + m^3\eta)9m^2(1 + m)\mu_2) + \\ \kappa(\kappa(6 - m^2(-2 + \eta) - 6m(-1 + \eta) - 6\eta + m^3\eta) - 3m(m^2(-4 + \eta) + \\ 3m^3(-2 + \eta) + 2(-1 + \eta) + 2m(-1 + \eta)\mu_2)/\Delta \end{aligned}$$

$$B_2 = -\frac{3m^2(1 + m)(\kappa(-2 + \eta) - 2\mu - 1)(\kappa + 2\mu_1 - \mu_2)}{\Delta} \tag{4.14}$$

$$C_2 = -\frac{e^m m^{\frac{3}{2}} \sqrt{\frac{2}{\pi}} \kappa(-1 + \eta)(\kappa + 2\mu_1 - \mu_2)}{\Delta/2} \tag{4.15}$$

$$\begin{aligned} A_2^* = \kappa^2(6(-1 + \eta) + 6m(-1 + \eta) + m^2(-7 + 2\eta) + m^2(-9 + 4\eta)) + \\ \kappa(8(-1 + \eta) + 8m(-1 + \eta) + m^2(-23 + 4\eta) + \\ m^2(-27 + 8\eta))\mu_1 - 18m^2(1 + m)\mu_1^2/\Delta \end{aligned} \tag{4.16}$$

$$C_2^* = \frac{\beta(\kappa + 2\mu_1}{10(\kappa + \mu_2)} \tag{4.17}$$

$$F_2^* = \kappa^2(6 + m^3(9 - 4\eta) + m^2(7 - 2\eta) - 6m(-1 + \eta) - 6\eta) + \kappa(8 + m^3(27 - 8\eta) + m^2(23 - 4\eta) - 8m(-1 + \eta) - 8\eta)\mu_1 + 18m^2(1 + m)\mu_1^2/\Delta \tag{4.18}$$

where

$$\Delta = 2(-6m^2(1 + m)\mu_1^2 + \mu_1(\kappa(8(-1 + \eta)8m(-1 + \eta) + m^2(-9 + 2\eta) + m^2(-11 + 4\eta)) - 6m^2(1 + m)\mu_2) + \kappa(\kappa(m^3(-3 + \eta) + 2m^2(-2 + \eta) + 6(-1 + \eta) + 6m(-1 + \eta)) + 3m^2(-1 + m(-2 + \eta))\mu_2)) \tag{4.19}$$

5. Evaluation of Drag Force

The drag on the fluid sphere is given by Ramkissoon and Mazumdar [5], the formula is

$$D = 4\pi(2\mu_e + k) \lim_{r\to\infty} \frac{r(\psi^{(i)} - \psi_\infty)}{\bar{\omega}}, \tag{5.1}$$

where ψ_∞ the stream function corresponding to the fluid motion at infinity and $\bar{\omega}$ the cylindrical radius coordinate.

The values of ψ_∞ and $\bar{\omega}$ are given by

$$\psi_\infty = \frac{1}{2}Ur^2 sin^2\theta, \quad \bar{\omega} = r sin\theta. \tag{5.2}$$

Thus for the present case, the drag force is given by

$$D = 2\pi(2\mu_e + k)UaA_2. \tag{5.3}$$

6. Special Cases

Case I:

Drag on a fluid sphere embedded in a another fluid. If $k \to 0$ i.e. $m \to 0$, the micropolar fluid turns out to be a Newtonian fluid. Hence the drag reduces to

$$F = -6\pi\mu Ua\frac{(1 + \frac{2}{3}\eta)}{1 + \eta}, \tag{6.1}$$

which agree with "the result reported earlier by Happel and Brenner [3] for the drag force experienced by fluid sphere in a clear fluid".

Case II:

If $\eta \to 0$ the fluid sphere acts like a solid sphere, thus the drag force turns to be

$$F = -6\pi\mu Ua, \tag{6.2}$$

which is "well-known Stokes result for flow past a rigid sphere in unbounded medium".

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