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MULTI EQUILIBRIA AND STABILITY IN DISCRETE FRACTIONAL ORDER PREDATOR PREY SYSTEM WITH OVERCROWDING

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ABSTRACT. Due to the impact of crowding effects, the predator prey systems display complex behaviour. To analyze this complex behaviour on the routes to chaos which are induced by bifurcations, a discrete fractional order predator prey system with overcrowding effect is considered in this paper. The presence of fixed points which are positive and the parametric conditions essential for the local asymptotic stability at these fixed points are investigated. It is observed that the system undergoes flip bifurcation for the interior fixed point and chaos control strategy is implemented. Numerical simulations for flip bifurcation demonstrate the rich dynamics of the system including periodic-doubling cascades, periodic orbits of 2, 4, 8, 16, 5, 3, 6 and chaotic attractors. The calculation of Maximum Lyapunov Exponent establishes the existence of chaotic behavior in the system.

1. INTRODUCTION

In population dynamics, based on mathematical facts, essentially two principal modelling stratagems arise and these stratagems apply the procedures of the qualitative theory of dynamical systems, [13]. One of them uses ordinary differential equations to formulate the model which is the continuous time approach and the one closely connected to the census of a population is termed as discrete time approach.

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The study of the dynamic connection among the predator and prey which is of prime importance in ecology is one of the long-standing research themes due to its universal appeal and significance, [2, 8, 11]. In keeping with these concerns, mathematical models of population dynamics are explored to analyze the stability of diverse generalizations of the Lotka-Volterra model, the pioneer in dynamical systems to denote the predator-prey interactions as put forward by Lotka, [10] and Volterra, [14], independently.

The crowding effect is generated as a result of competition by predator within a prey for limited sources. Exactly, the severity of crowding effect surges with a proliferation of predators within a prey and shows in decreased body size and thus loss of well being. Signs of crowding effects impacts the fitness of the predator both negatively and positively. Predators are consumers with a variety of tropic plannings as seen in their life history traits. These characteristics are advantageous to forecast the effects of crowding, [4]. Research was carried out about crowding effects in prey population only, by introducing a prey growth function, [3]. Additionally, some research is carried out with under-crowding effect in both predator and prey population, [9].

In this paper, the stability of a basic Lotka-Volterra model with crowding effect, both in predator and prey is analyzed with the aid of the following equations as given in [6]:

$$\begin{aligned} \dot{x} &= x\left(a - by - \beta x\right), \\ \dot{y} &= y\left(dx - c - \mu y\right), \end{aligned} \tag{1.1}$$

with β, μ denoting the competition among individuals of prey species and predator species respectively under the influence of overcrowding.

2. Fractional Order Model and its Discrete Version

Recently, fractional calculus has garnered enormous attention from ecologist and mathematicians as it has evolved as a significant instrument to explain the dynamical behavior of various physical systems. The strong point of these fractional differential operators is their non local characteristics and also it summaries memory and transmitted properties thus making it further realistic and practical as it also efficiently exhibits the memory physiognomies of biological variables.

Hence the fractional-order continuous model of (1.1) is given as

$$\begin{aligned} D_t^{\alpha} x &= x\,(a - by - \beta x)\,, \\ D_t^{\alpha} y &= y\,(dx - c - \mu y)\,, \end{aligned} \tag{2.1}$$

where $0 < \alpha \leq 1$ is the fractional order and $D = \dfrac{d^{\alpha}}{d\alpha}$ is in the sense of Caputo derivative, [5] with the initial conditions $x(0) = x_0,\ y(0) = y_0$.

Its simulating to observe that a discretised system reveals the rich dynamical behavior in comparison to its corresponding fractional order continuous counter parts. And motivated by this fact, applying the discretization method of piecewise constant arguments, [1], to the fractional predator-prey dynamics model (2.1) yields the discrete fractional order(DFO) predator-prey dynamics model

$$\begin{aligned} x_{n+1} &= x_n + \frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[x_n\,(a - by_n - \beta x_n)\right], \\ y_{n+1} &= y_n + \frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[y_n\,(dx_n - c - \mu y_n)\right], \end{aligned} \tag{2.2}$$

with $h > 0$ representing the time interval of production.

2.1. **Existence of the fixed points and its stability analysis.** Here the existence and stability behaviour of the system (2.2) at various fixed points is explored. The system (2.2) has the following fixed points $A_0(0,0)$, $A_1(\frac{a}{\beta},0)$ and $A_2(x^*,y^*)$ with $x^*,\ y^*$ satisfying:

$$\begin{aligned} x\,(a - by - \beta x) &= 0, \\ y\,(dx - c - \mu y) &= 0. \end{aligned} \tag{2.3}$$

By simple calculation, the subsequent outcomes pertaining to the existence of multiple fixed points of system (2.2) is obtained.

Theorem 2.1. *The existence of boundary fixed points satisfies:*

(a) *The trivial point* $A_0(0,0)$ *always exists.*

(b) *If* $a > 0$*, then the persistent point* $A_1\left(\dfrac{a}{\beta},0\right)$ *exists.*

(c) *If* $a\mu + bc > 0$ *and* $ad > c\beta$ *then the interior point* $A_2(x^*,y^*)$ *exists where* $x^* = \dfrac{a\mu + bc}{bd + \beta\mu}$, $y^* = \dfrac{ad - c\beta}{bd + \beta\mu}$.

The Jacobian matrix of (2.2) with respect to fixed point (x^*, y^*) is

$$J^*(x^*, y^*) = \begin{bmatrix} 1 + S\left(a - by^* - 2\beta x^*\right) & -Sbx^* \\ Sdy^* & 1 + S\left(-c + dx^* - 2\mu y^*\right) \end{bmatrix}, \tag{2.4}$$

where $S = \dfrac{h^\alpha}{\Gamma(1+\alpha)}$.
The characteristic polynomial of (2.4) obtained at (x^*, y^*), the fixed point is

$$P(m) = m^2 - Trace(J^*)m + Determinant(J^*), \tag{2.5}$$

with $Trace = 2 + S[a - c + x^*(d - 2\beta) - y^*(2\mu + b)]$,
$Determinant = 1 + S\left[(a - c) + x^*(d - 2\beta) - y^*(2\mu + b)\right] + S^2E$, where
$E = [-ac + x^*(ad + 2\beta c) + y^*(bc - 2a\mu) + 2(b\mu y^{*2} - \beta dx^{*2} + 2\beta\mu x^* y^*) - bdx^* y^*]$.
The following lemma is essential to analyze the stability of system (2.2) at its fixed points.

Lemma 2.1. *[15], Suppose that $P(1) > 0$ in (2.5), m_1 and m_2 are the two roots of
$P(m) = 0$. Then,*

(a) $|m_1| < 1$ & $|m_2| < 1 \iff P(-1) > 0$ & $P(0) < 1$.
(b) $|m_1| < 1$ & $|m_2| > 1$ (*or* $|m_1| > 1$ & $|m_2| < 1$) $\iff P(-1) < 0$.
(c) $|m_1| > 1$ & $|m_2| > 1 \iff P(-1) > 0$ & $P(0) > 1$.
(d) $|m_1| = -1$ & $|m_2| \neq 1 \iff P(-1) = 0$ & $Tr \neq 0, -2$.
(e) m_1 *and* m_2 *are complex* & $|m_1| = |m_2| = 1 \iff T^2 - 4Det < 0$ *and* $P(0) = 1$.

Let m_1 and m_2 be the roots of equation (2.5) known as the eigenvalues at (x^*, y^*), the fixed point. The following results are obtained with the aid of lemma (4.1) from [12].

Theorem 2.2. *The trivial fixed point A_0 is always unstable.*

Proof. At A_0, the Jacobian matrix of (2.2) is

$$J^*(A_0) = J(0,0) = \begin{bmatrix} 1 + Sa & 0 \\ 0 & 1 - Sc \end{bmatrix},$$

the eigen values are $m_1 = 1 + Sa$ and $m_2 = 1 - Sc$. Since $0 < \alpha \leq 1$, $S > 0$ and $c > 0$ then $m_1 > 1$ from which it is concluded that A_0 is unstable. □

Proposition 2.1. *The trivial fixed point A_0 is*

(i) *Sink if* $a < \dfrac{c}{(1 - Sc)}$.

(ii) *Source if* $a > \frac{c}{(1-Sc)}$.

Theorem 2.3. *The fixed point* A_1 *is,*

(i) *Sink if* $\frac{ad}{\beta} < c$.

(ii) *Saddle if* $\frac{ad}{\beta} > c$.

Proof. At A_1, the Jacobian matrix of (2.2) is

$$J^*(A_1) = J\left(\frac{a}{\beta}, 0\right) = \begin{bmatrix} 1 - Sa & \frac{-Sab}{\beta} \\ 0 & 1 + S\left(\frac{ad}{\beta} - c\right) \end{bmatrix},$$

the eigen values are $m_1 = 1 - Sa$ and $m_2 = 1 + S\left(\frac{ad}{\beta} - c\right)$.

(i) If $\frac{ad}{\beta} < c$ then $m_2 < 1$. Since $m_1 < 1$ & $m_2 < 1$, A_1 is a sink, which is asymptotically stable.

(ii) If $\frac{ad}{\beta} > c$ then $m_2 > 1$. As $m_1 < 1$ & $m_2 > 1$, A_1 is a saddle, which is unstable.

□

Now the interior fixed point A_2 is considered and its stability is discussed. The Jacobian matrix (2.2) at A_2 is

$$J^*(A_2) = \begin{bmatrix} 1 + Sd_{11} & -Sd_{12} \\ Sd_{21} & 1 + Sd_{22} \end{bmatrix}, \tag{2.6}$$

where

(2.7)

$$S = \frac{h^\alpha}{\Gamma(1+\alpha)}, d_{11} = B[a(bd + \beta\mu) - b(ad - c\beta) - 2\beta(a\mu + bc)], d_{12} = Bb(a\mu + bc),$$

$$d_{21} = Bd(ad - c\beta), d_{22} = B[c(bd + \beta\mu) + d(a\mu + bc) - 2\mu(ad - c\beta)], B = \frac{1}{(bd + \beta\mu)}.$$

The characteristic equation of $J^*(A_2)$ is

$$P(m) = m^2 - (SU + 2)m + (S^2V + SU + 1), \tag{2.8}$$

where $U = d_{11} + d_{22}$ and $V = d_{11}d_{22} + d_{12}d_{21}$.

Theorem 2.4. *The fixed point A_2 of system* (2.2) *holds at least four different topological types for all permissible values of variables see, [16]:*

(i) A_2 *is asymptotically stable(sink) if one of the below criteria is satisfied:*
  (i.a) $U^2 < 4V$ & $0 < h < h_2$,
  (i.b) $U^2 \geq 4V$& $0 < h < h_1$.

(ii) A_2 *is unstable(saddle) if one of the below criteria is satisfied:*
  (i.a) $U^2 \geq 4V$& $h_1 < h < h_3$.

(iii) A_2 *is unstable(source) if one of the below criteria is satisfied:*
  (i.a) $U^2 < 4V$ & $h > h_2$,
  (i.b) $U^2 \geq 4V$& $h > h_3$.

(iv) A_2 *is non-hyperbolic if one of the below criteria is satisfied:*
  (i.a) $U^2 > 4V$ & $h = h_1$ *or* h_3,
  (i.b) $U^2 < 4V$ & $h = h_2$,

$$h_1 = \left(\Gamma(1+\alpha)\frac{-U-\sqrt{U^2-4V}}{V}\right)^{\frac{1}{\alpha}}, h_2 = \left(\frac{-\Gamma(1+\alpha)U}{V}\right)^{\frac{1}{\alpha}},$$

$$h_3 = \left(\Gamma(1+\alpha)\frac{-U+\sqrt{U^2-4V}}{V}\right)^{\frac{1}{\alpha}}.$$

Proof. We can easily obtain the stability conditions for $(i)-(iii)$ with the aid of lemma (2.6) from [16] and lemma (2.1). If $U^2 > 4V$, then there will be two real roots for equation (2.8). In addition when $P(-1) = 0$, we get

$$P(-1) = 1 + (SU+2) + (S^2V + SU + 1) = S^2V + 2SU + 4 = 0.$$

Solving the above equation we get that $h = h_1$ or h_3. Further, the eigen values $\lambda_{1,2}$ are complex roots if $(SU+2)^2 - 4(S^2V + SU + 1) < 0,$ which brings us to

$$U^2 < 4V.$$

Letting $h = h_2,$ we get

$$\lambda_{1,2} = \frac{SU+2}{2} \pm \frac{S\sqrt{4V-U^2}}{2}i,$$

which yields that equation (2.8) has two conjugate eigenvalues and the modulus of each of them equals to one. □

3. Bifurcation of Discretized Predator-Prey model of Fractional-Order

In this segment we concentrate on the occurrence of flip bifurcation about the interior fixed point A_2 of system (2.2).

3.1. **Flip bifurcation about the interior fixed point.** We follow the approach as in [16]. With the bifurcation parameter h it is observed that, the fixed point A_2 undergoes flip bifurcation under the condition that one of the eigen values of Jacobian matrix at the given fixed point is -1 and the other eigen value should neither be -1 nor 1.

Equation (2.6) is the Jacobian matrix of system (2.2) with respect to, interior fixed point A_2 and the characteristic equation of the same is given in (2.8) which is

$$P(m) = m^2 - (SU + 2)m + (S^2V + SU + 1). \tag{3.1}$$

From theorem (2.4), it is concluded that when $U^2 > 4V$ and $h^* = h_1$ or h_3 it leads to the eigenvalues about the fixed point A_2 to be $m_1 = -1$, $m_2 = SU + 3$. Further, it is essential that $|\lambda_2| \neq 1$ for flip bifurcation to occur, leading to $h^* \neq h_4$ and $h^* \neq h_5$ where $h_4 = \left(\frac{-2\Gamma(1+\alpha)}{U}\right)^{\frac{1}{\alpha}}$, $h_5 = \left(\frac{-4\Gamma(1+\alpha)}{U}\right)^{\frac{1}{\alpha}}$. From the above discussion we arrive at the following theorem essential for flip bifurcation.

Theorem 3.1. *The fixed point A_2 becomes unstable, through flip bifurcation as and when $U^2 \geq 4V$ with $h = h_1$ or $h = h_3$ and $h \neq h_4, h_5$, where*
$$h_1 = \left(\Gamma(1+\alpha)\frac{-U - \sqrt{U^2 - 4V}}{V}\right)^{\frac{1}{\alpha}}, h_3 = \left(\Gamma(1+\alpha)\frac{-U + \sqrt{U^2 - 4V}}{V}\right)^{\frac{1}{\alpha}},$$
$$h_4 = \left(\frac{-2\Gamma(1+\alpha)}{U}\right)^{\frac{1}{\alpha}}, h_5 = \left(\frac{-4\Gamma(1+\alpha)}{U}\right)^{\frac{1}{\alpha}}.$$

4. Chaos Control and Numerical Simulation

In this segment, we undertake few special cases of system (2.2) and ascertain the theoretical findings and the rich dynamical behaviour of the system. Also numerical examples support the feasibility of linear feedback control for chaos control.

Example 1. *We consider* $a = 1.38$*,* $b = 0.18$*,* $\beta = 1.1$*,* $c = 0.01$*,* $d = 0.02$*,* $\mu = 0.9$*,* $\alpha = 0.5$ *and* $1 \leq h \leq 4$ *in system* (2.2) *and we elaborate on flip bifurcation. The unique fixed point of* (2.2) *is* $A_2(x^*, y^*) = (1.2518, 0.0167)$*. Further, verifying the conditions of theorem (3.1) we obtain* $U = -1.3720$*,* $V = -0.0068$*,* $U^2 - 4V = 1.9095 > 0$*,* $h_1 = 1.6570$*,* $h_3 = 0.000013035$*,* $h_4 = 1.6689$ *and* $h_5 = 6.6755$*. The characteristic equation with* $h_1 = 1.6570$ *and calculated at the fixed point* A_2 *is*

$$P(\lambda) = \lambda^2 - 0.0072\lambda - 1.0072 = 0. \tag{4.1}$$

$\lambda_1 = -1.0000$*,* $\lambda_2 = 1.0071$ *are the roots of* (4.1)*. Thus is the criteria for flip bifurcation occurance close to the fixed point* A_2 *with the bifurcation parameter* $h_1 = 1.6570$*.*

Also the flip bifurcation diagram for prey population and predator population are provided in figures 1(a), 1(c) and local amplification for prey population for $2.75 \leq h \leq 3.5$ in figure 1(b). Maximum Lyapunov exponent(MLE) is presented in figure 1(d) wherein we see that MLE is negative for $h < h_1$, indicating the stablity of system (2.2) in this region. For $h = h_1$, the MLE is zero, indicating that system (2.2) is unstable at fixed point A_2. And for $h > h_1$, the MLE is greater than zero, clearly indicating chaos being present. In particular the positive MLE indicates the existence of chaos.

It is clear from figure 1(a) and 1(c) and figure (2), that fixed point A_2 is stable for $h < h_1$, at $h = h_1$ it is unstable leading to periodic $2, 4, 8, 16$ orbits followed by uncertain periodic windows. For example, consider $h = 3.11,\ h = 3.33,\ h = 3.35$ there appears periodic $5, 3, 6$ orbits respectively. And in figure 3(j), 3(k) and 3(l) for $h = 3.5, h = 3.6, h = 3.65$ the presence of chaotic attractors.

Chaos control is vital in biological systems as they generally adopt to chaotic regimes in reacting to some new situation also, they have broad range of possible behaviours in such situations. Basically, chaos control depends on the richness of responses of chaotic behaviour. A chaotic attractor has a dense set of unstable periodic orbits (UPOs) and the system tends to visit the neighborhood of each of them. Also, chaotic response is sensitive dependent on initial conditions, resulting in systems evolution which may be changed by small perturbations. Thus, chaos control can be comprehended as the usage of small perturbations to stabilize an UPO embedded in a chaotic attractor, making this kind of behavior to be sought after in a variety of applications. We apply the linear feedback control method, [7] to system (2.2). Towards this,

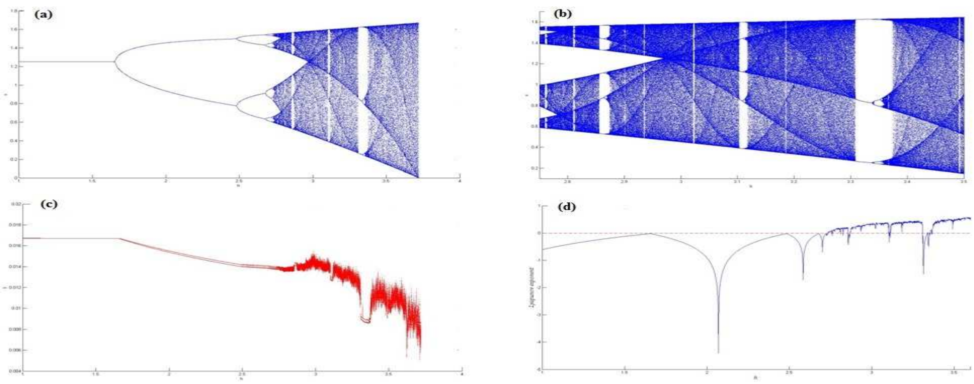

FIGURE 1. (a) Bifurcation diagram in(h, x) plane; (b) Local Amplification in (h, x) plane for $2.75 \leq h \leq 3.5$; (c) Bifurcation diagram in(h, y) plane; (d) Maximal Lyapunov Exponent of System (2.2).

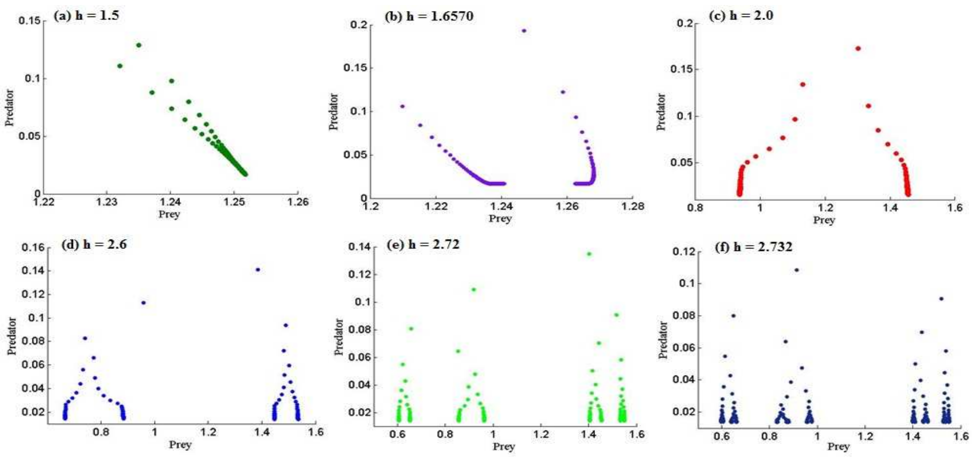

FIGURE 2. Phase portrait diagrams of period-1, period-2, period-4, period-8, period-16 with $a = 1.38$, $b = 0.18$, $\beta = 1.1$, $c = 0.01$, $d = 0.02$, $\mu = 0.9$, $\alpha = 0.5$ and initial values $(0.7, 0.4)$.

Case (i) Let us assume that controller of (2.2) is given by

$$
\begin{aligned}
x_{n+1} &= x_n + \frac{h^\alpha}{\Gamma(1+\alpha)} \left[x_n \left(a - by_n - \beta x_n\right)\right] + S_n, \\
y_{n+1} &= y_n + \frac{h^\alpha}{\Gamma(1+\alpha)} \left[y_n \left(dx_n - c - \mu y_n\right)\right],
\end{aligned} \tag{4.2}
$$

where $S_n = -p_1(x_n - x^*) - p_2(y_n - y^*)$ is feedback controlling force, $p_{1,2}$ stands for the feed back gains while (x^*, y^*) is the unique positive fixed point of system

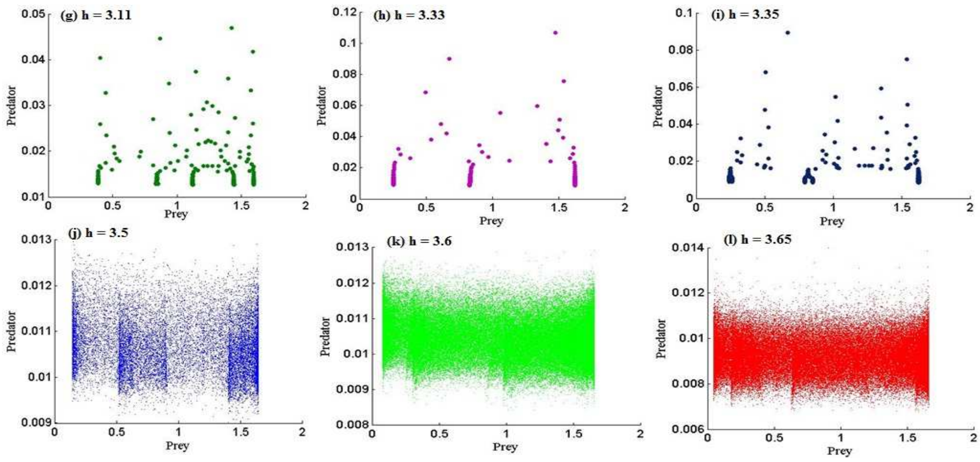

FIGURE 3. Phase portrait diagrams of period-5, period-3, period-6 and chaotic attractors.

(2.2).
The Jacobian matrix of (4.2) taken at (x^*, y^*), the unique positive fixed point is

$$J_1^*(x^*, y^*) = \begin{bmatrix} 1 + Sd_{11} - p_1 & -Sd_{12} - p_2 \\ Sd_{21} & 1 + Sd_{22} \end{bmatrix}, \tag{4.3}$$

where $S, d_{11}, d_{12}, d_{21}, d_{22}$ are as in (2.7). The characteristic equation corresponding to (4.3) is

$$\lambda^2 - \lambda(SU + 2 - p_1) + (S^2V + SU + 1 + Sd_{21}p_2 - Sd_{22}p_1 - p_1) = 0, \tag{4.4}$$

where $U = d_{11} + d_{22}$ and $V = d_{11}d_{22} + d_{12}d_{21}$. Let the eigenvalues with respect to (4.4), the characteristic equation be λ_1, λ_2. From this we arrive at

$$\lambda_1\lambda_2 = (1 + Sd_{22})(1 + Sd_{11} - p_1) + Sd_{21}(Sd_{12} + p_2). \tag{4.5}$$

The lines of marginal stability are obtained by solving $\lambda_1 = \pm 1$ and $\lambda_1\lambda_2 = 1$ which assures that λ_1 and λ_2 have absolute value less than 1. If $\lambda_1\lambda_2 = 1$ then from (4.5) we arrive at

$$l_1 : (Sd_{22} + 1)p_1 - Sd_{21}p_2 = S^2V + SU. \tag{4.6}$$

If $\lambda_1 = 1$ or -1 then from (4.4) we get

$$l_2 : d_{22}p_1 - d_{21}p_2 = SV. \tag{4.7}$$

$$l_3 : (Sd_{22} + 2)p_1 - Sd_{21}p_2 = S^2V + 2SU + 4. \tag{4.8}$$

The stable eigenvalues lie within the triangular region bounded by the lines l_1, l_2 and l_3.

Example 2. *We now, consider $a = 1.38$, $b = 0.18$, $\beta = 1.1$, $c = 0.01$, $d = 0.02$, $\mu = 0.9$, $\alpha = 0.5$ and $h = 1.6570$. Here the unique fixed point $A_2(x^*, y^*) = (1.2518, 0.0167)$ of system* (2.2) *is unstable. The stable region for system* (4.2) *is given by the region bounded by the lines $l_1, 1_2$ and l_3 as given in* (4.6), (4.7) *and* (4.8). *To make the fixed point A_2 stable we introduce the linear feedback control strategy. In system* (4.2) *the feed control term is $S_n = -p_1(x_n - 1.2518) - p_2(y_n - 0.0167)$. The bifurcation diagram for feedback gain $p_1 = -0.18$ and $p_2 = 0.5$ is in figure 4(a) for system* (2.2) *and controlled system* (4.2) *and the corresponding Maximal Lyapunov Exponent is in figure 5(a). Also for $p_1 = -0.25$ and $p_2 = 0.5$ in 4(b) and 5(b), for $p_1 = -0.35$ and $p_2 = 0.5$ in 4(c) and 5(c) and for $p_1 = -0.45$ and $p_2 = 0.5$ in 4(d) and 5(d). These diagrams show the rich dynamics with the introduction of feedback gains as in system* (4.2). *The plot of x_n for system* (4.2) *are in figure 6(a)-6(d).*

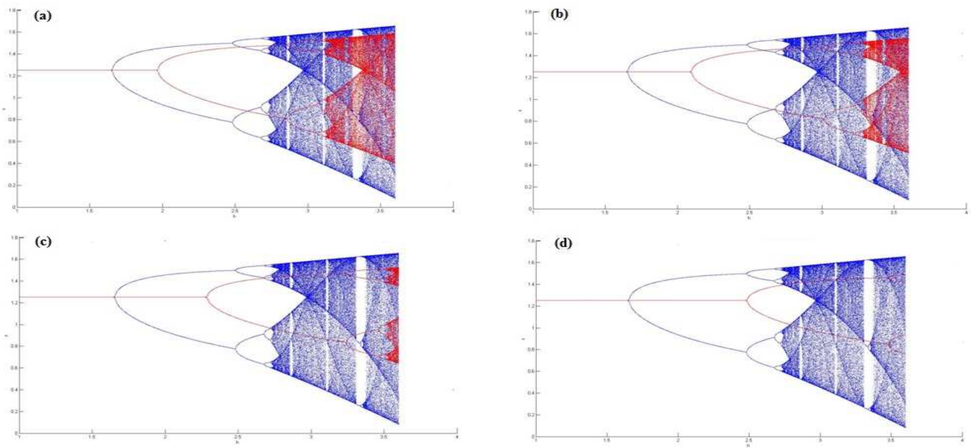

FIGURE 4. Bifurcation diagrams for system (2.2) and controlled system (4.2) with (a) $p_1 = -0.18$ and $p_2 = 0.5$ (b) $p_1 = -0.25$ and $p_2 = 0.5$ (c) $p_1 = -0.35$ and $p_2 = 0.5$ (d) $p_1 = -0.45$ and $p_2 = 0.5$

Case (ii) Let us assume that controller of (2.2) is given by

$$x_{n+1} = x_n + \frac{h^\alpha}{\Gamma(1+\alpha)} \left[x_n \left(a - b y_n - \beta x_n\right)\right] + S_n,$$
$$y_{n+1} = y_n + \frac{h^\alpha}{\Gamma(1+\alpha)} \left[y_n \left(d x_n - c - \mu y_n\right)\right] + S_n, \tag{4.9}$$

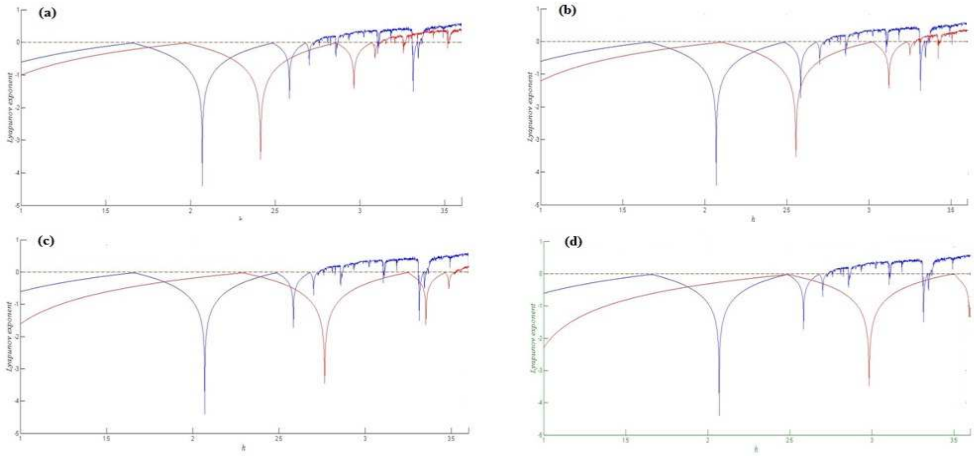

FIGURE 5. Maximun Lyapunov Exponent for system (2.2) and controlled system (4.2) with (a) $p_1 = -0.18$ and $p_2 = 0.5$ (b) $p_1 = -0.25$ and $p_2 = 0.5$ (c) $p_1 = -0.35$ and $p_2 = 0.5$, (d) $p_1 = -0.45$ and $p_2 = 0.5$.

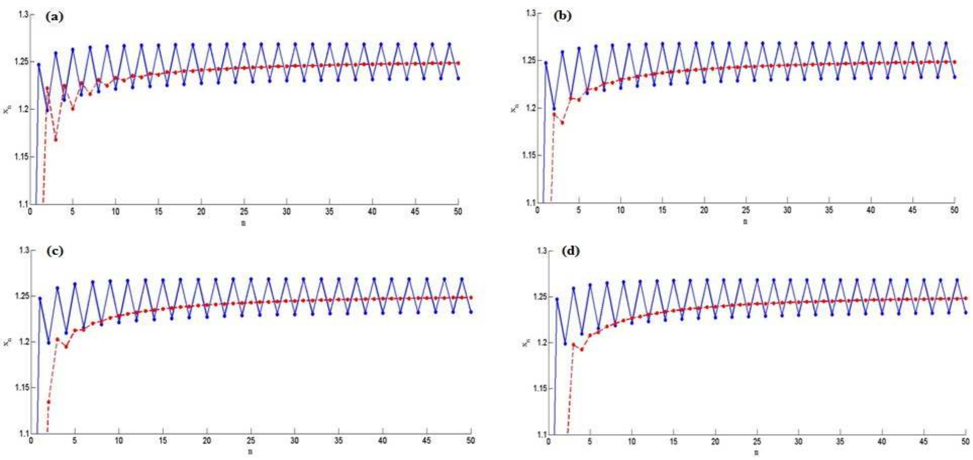

FIGURE 6. Plots for system (2.2) and controlled system (4.2) with $a = 1.38$, $b = 0.18$, $\beta = 1.1$, $c = 0.01$, $d = 0.02$, $\mu = 0.9$, $\alpha = 0.5$ and initial values $(0.7, 0.4)$.

where $S_n = -p_1(x_n - x^*) - p_2(y_n - y^*)$ is feedback controlling force, $p_{1,2}$ stands for the feed back gains and (x^*, y^*) be unique positive fixed point of system (2.2).

The Jacobian matrix of (4.9) taken at the unique positive fixed point (x^*, y^*) is

$$J_1^*(x^*, y^*) = \begin{bmatrix} 1 + Sd_{11} - p_1 & -Sd_{12} - p_2 \\ Sd_{21} - p_1 & 1 + Sd_{22} - p_2 \end{bmatrix}, \tag{4.10}$$

where $S, d_{11}, d_{12}, d_{21}, d_{22}$ are as in (2.7). The characteristic equation corresponding to (4.10) is

$$\begin{aligned} \lambda^2 - \lambda\left[2 + SU - (p_1 + p_2)\right] + 1 + SU + S^2V - S(d_{12} + d_{22})p_1 \\ + S(d_{21} - d_{11})p_2 - (p_1 + p_2) = 0, \end{aligned} \tag{4.11}$$

where $U = d_{11} + d_{22}$ and $V = d_{11}d_{22} + d_{12}d_{21}$.
Let the eigenvalues with respect to (4.11), the characteristic equation be λ_1, λ_2. Following as in case (i) we arrive at

$$\lambda_1\lambda_2 = (1 + Sd_{22})(1 + Sd_{11} - p_1) - p_2(Sd_{11} - 1) + Sd_{12}(Sd_{21} - p_1). \tag{4.12}$$

If $\lambda_1\lambda_2 = 1$ then from (4.12) we obtain

$$l_1 : S(d_{12} + d_{22})p_1 - S(d_{21} - d_{11})p_2 + (p_1 + p_2) = S^2V + SU. \tag{4.13}$$

If $\lambda_1 = 1$ or -1 then from (4.11) we get

$$l_2 : (d_{12} + d_{22})p_1 - (d_{21} - d_{11})p_2 = SV. \tag{4.14}$$

$$l_3 : S(d_{12} + d_{22})p_1 - S(d_{21} - d_{11})p_2 + (p_1 + p_2) = S^2V + 2SU + 4. \tag{4.15}$$

The stable eigenvalues lie within the triangular region bounded by the lines l_1, l_2 and l_3.

5. Conclusion

A discrete fractional order predator prey system with overcrowding effect is considered in this paper. The local asymptotic stability at the fixed points is investigated. It is observed that the system undergoes flip bifurcation for interior fixed point and chaos control strategy is implemented. Numerical simulations for flip bifurcation demonstrate the rich dynamics of the system. The calculation of Maximum Lyapunov Exponent establishes the existence of chaotic behaviors in the system.

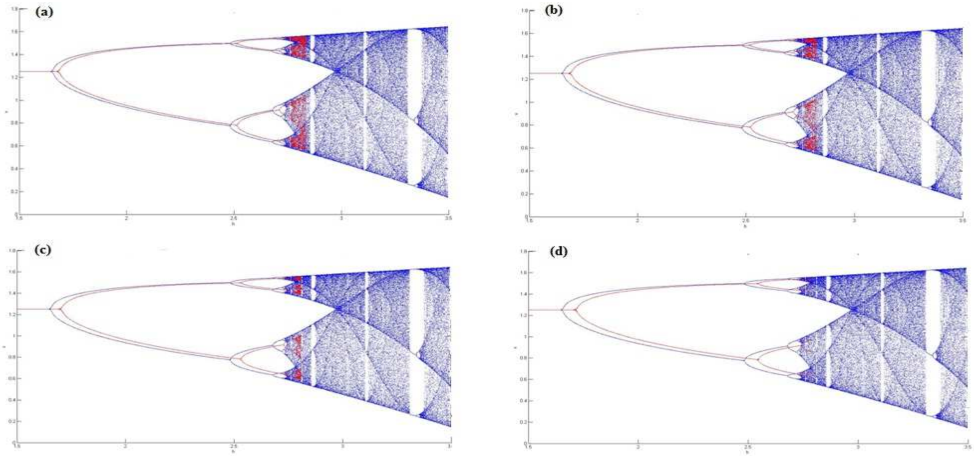

FIGURE 7. Bifurcation diagrams of system (2.2) and controlled system (4.9) with feedback gains (a) $p_1 = -0.009$ and $p_2 = 0.9$ (b) $p_1 = -0.0115$ and $p_2 = 0.9$ (c) $p_1 = -0.014$ and $p_2 = 0.9$ (d) $p_1 = -0.018$ and $p_2 = 0.9$

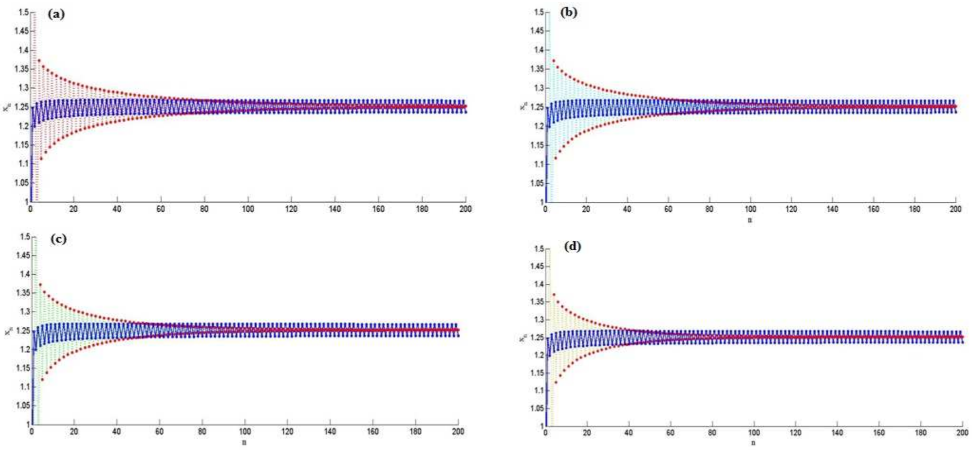

FIGURE 8. Plots for system (2.2) and controlled system (4.9) with $a = 1.38$, $b = 0.18$, $\beta = 1.1$, $c = 0.01$, $d = 0.02$, $\mu = 0.9$, $\alpha = 0.5$ and initial values $(0.7, 0.4)$.

REFERENCES

[1] R. P. AGARWAL, A. M. EL-SAYED, S. M. SALMAN: *Fractional-order Chua's system: discretization, bifurcation and chaos*, Advanced Difference Equation, **2013**(320) (2013).

[2] A. A. BERRYMAN: *The origins and evolutions of predator-prey theory*, Ecology, **73**(1992), 1530–1543.

[3] G. BUFFONI, M. GROPPI, C. SORESINA: *Effects of prey over under crowding in predator-prey systems with prey-dependent tropic functions*, Nonlinear Analysis Real World Applications, **12**(5) (2011), 2871–2887.

[4] A. O. BUSH, J. M. LOTZ: *The ecology of crowding*, Journal of Parasitology, **86**(2) (2000), 212–213.

[5] M. CAPUTO: *Linear models of dissipation whose Q is almost frequency independent-II*, Geophys J R Astron Society, **13**(1967), 529–539.

[6] C. L. BENNETT: *Stability Analysis of Lotka-Volterra Equations for Predator-Prey Systems*, National Science Foundation, Virginia 22314.

[7] D. QAMAR: *Complexity and chaos control in a discrete-time prey-predator model*, Communications in Nonlinear Science and Numerical Simulation, **49**(2017), 113–134.

[8] Y. KUANG, E. BERETTA: *Global qualitative analysis of a ratio-dependent predator-prey system*, Journal of Mathematical Biology, **36**(4) (1998), 389–406.

[9] E. KUNO: *Principles of predator-prey interaction in theoretical, experimental and natural population system*, Advances in Ecological Research, **16**(1987), 249–337.

[10] A. J. LOTKA: *Elements of Physical Biology*, Baltimore Wiliams & Wilkins, 1925.

[11] R. M. MAY: *Limit cycles in predator-prey communities*, Science, **177**(4052) (1972), 900–902.

[12] G. A. K. OZLEM, A. G. M. SELVAM, D. A. VIANNY: *Bifurcation and stability analysis of a discrete time SIR epidemic model with vaccination*, International Journal of Analysis and Applications, **17**(5) (2019), 809–820.

[13] D. RUI: *Mathematical models in population dynamics and Ecology*, Biomathematics, (2006), 399–449.

[14] V. VOLTERRA: *Fluctuations in the abundance of species considered mathematically*, Nature, **118**(1926), 558–560.

[15] S. WIGGINS: *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer, New York, 1990.

[16] Y. SHI, Q. MA, X. DING: *Dynamical behaviors in a discrete fractional-order predator-prey system*, Filomat, **32**(2018), 5857–5874.

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