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FRACTIONAL ORDER OF ALPHA-DELTA AND ITS SUMS ON EXTORIAL FUNCTIONS

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ABSTRACT. In this paper, we obtain the anti-difference of fractional order of delta and alpha-delta difference of certain functions like extorial function arrived from exponential function by changing polynomials into polynomial factorials and trigonometric functions. Our findings are being used to obtain solutions for certain type of fractional order difference and alpha-difference equations.

1. INTRODUCTION

The forward difference and alpha delta operators are applicable in solving the problems in mathematical sciences, physical sciences, life sciences, scientific engineering. The numerical solution of m-th order difference equation is $\Delta_\ell^m g(t) = f(t)$, when $f(t) = 0$ is obtained by

$$\Delta_\ell^{-m} f(t) = \sum_{r=0}^{s-m} \frac{\Gamma(m+r)}{\Gamma(r+1)\Gamma(m)} f(t-(m+r)\ell), \tag{1.1}$$

where $\Delta_\ell f(t) = f(t+\ell) - f(t)$, and Γ is a Gamma function.

It is also possible to develop fractional order anti-difference corresponding to equation (1.1) by replacing the integer m into real number $\nu > 0$. The corresponding numerical solution for ν-th order alpha-difference equation $\Delta_{\alpha,\ell} g(t) =$

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$f(t)$ have been developed by many authors. When $\alpha = 1$, $\ell = 1$, the alpha delta operator becomes the usual forward difference operator Δ. For more details on alpha difference operator and its inverse one can refer [1–7].

2. Finite fractional order difference

In this section, first we present anti-difference Δ, Δ_ℓ and $\Delta_{\alpha(\ell)}$ for arriving general formula for numerical solution of fractional difference equation $\Delta^\nu g(t) = f(t)$, when $f(t) = 0$.

Lemma 2.1. *For any positive integer n, it holds*

$$1^{(n)} + 2^{(n)} + 3^{(n)} + \cdots + t^{(n)} = \frac{(t+1)^{(n+1)}}{n+1}, \tag{2.1}$$

where $t^{(n)} = \prod_{r=0}^{n-1} (t-r)$.

Proof. The proof follows by induction method. □

Theorem 2.1. *If $t = s\ell$, $0 < \ell < \infty$, $m < s$, $f(0) = 0$ and $m \in \mathbb{N}(1)$, then:*

$$\Delta_\ell^{-m} f(t) = \sum_{r=0}^{s-m} \frac{\Gamma(m+r)}{\Gamma(r+1)\Gamma(m)} f(t-(m+r)\ell). \tag{2.2}$$

Proof. Let ℓ be any real in $(0, \infty)$. Then, $\Delta_\ell z(t) = z(t+\ell) - z(t)$. Since $t = s\ell$, where $s \in \mathbb{N}(0)$, we take:

$$z(t) = f(t-\ell) + f(t-2\ell) + f(t-3\ell) + \cdots + f(0) \tag{2.3}$$

$$z(t+\ell) = f(t) + f(t-\ell) + f(t-2\ell) + \cdots + f(0) \tag{2.4}$$

(2.3) − (2.4) ⇒ $\Delta_\ell z(t) = f(t)$, which gives $\Delta_\ell^{-1} f(t) = z(t)$, where $z(t)$ is given in (2.3). Hence

$$\Delta_\ell^{-1} f(t) = f(t-\ell) + f(t-2\ell) + f(t-3\ell) + \cdots + f(0). \tag{2.5}$$

Taking Δ_ℓ^{-1} on both sides, we have:

$$\Delta_\ell^{-2} f(t) = \Delta_\ell^{-1} f(t-\ell) + \Delta_\ell^{-1} f(t-2\ell) + \Delta_\ell^{-1} f(t-3\ell) + \cdots + \Delta_\ell^{-1} f(0)$$

By applying (2.5) for $t-\ell$, we get:

$$\begin{aligned}\Delta_\ell^{-2}f(t) &= f(t-2\ell)+f(t-3\ell)+f(t-4\ell)+\cdots+f(0)\\ &\quad +f(t-3\ell)+f(t-4\ell)+f(t-5\ell)+\cdots+f(0)\\ &\quad +f(t-4\ell)+f(t-5\ell)+f(t-6\ell)+\cdots+f(0)\\ &\quad \vdots\\ &\quad +f(3\ell)+f(2\ell)+f(\ell)+u(0)\\ &\quad +f(2\ell)+f(\ell)+f(0)+f(\ell)+f(0)+f(0)\end{aligned}$$

Grouping the terms, we find

$$\Delta_\ell^{-2}f(t)=\frac{1^{(1)}}{1!}f(t-2\ell)+\frac{2^{(1)}}{1!}f(t-3\ell)+\cdots+\frac{(s-1)^{(2)}}{1!}f(0)$$

where $t-s\ell=0$. Again taking Δ_ℓ^{-1} on both sides and by using (2.1), we get

$$\Delta_\ell^{-3}f(t)=\frac{2^{(2)}}{1!}f(t-3\ell)+\frac{3^{(2)}}{1!}f(t-4\ell)+\cdots+\frac{(s-1)^{(2)}}{1!}f(0)$$

where $t-s\ell=0$. Proceeding like this, we arrive

$$\begin{aligned}\Delta_\ell^{-m}f(t) &= \frac{(m-1)^{(m-1)}}{(m-1)!}f(t-m\ell)+\frac{m^{(m-1)}}{(m-1)!}f(t-(m+1)\ell)\\ &\quad +\cdots+\frac{(s-1)^{(m-1)}}{(m-1)!}f(0)\end{aligned}$$

where $t-s\ell=0$.

$$\begin{aligned}\Delta_\ell^{-m}f(t) &= \frac{\Gamma(m)}{\Gamma(m-(m-1))\Gamma(m)}f(t-m\ell)+\frac{\Gamma(m+1)}{\Gamma(m+1-(m-1))\Gamma(m)}\\ &\quad \times f(t-(m+1)\ell)+\frac{\Gamma(m+(s-m))}{\Gamma(m+(s-m)-(m-1))\Gamma(m)}f(0)\end{aligned}$$

which gives (2.2). □

Corollary 2.1. *Assume that $f(0)=0$. The m-th inverse finite difference of $f(t)$ is*

$$\Delta^{-m}f(t)=\sum_{r=0}^{t-m}\frac{(m-1+r)^{(m-1)}}{(m-1)!}f(t-(m+r)),n\in\mathbb{N}(m).$$

Proof. The proof follows by taking $\ell=1$ in Theorem 2.1. □

Corollary 2.2. *If $t > m\ell$, $0 < \ell < \infty$ and $f(t - m\ell) = 0$, then:*

$$\Delta_\ell^{-\nu} f(t)\Big|_{t-m\ell}^{t} = \sum_{r=0}^{m} \frac{(\nu + r)^{(\nu)}}{\ell^\nu} f(t - \ell - r\ell)$$

where $n^{(\nu)} = \dfrac{\Gamma(n+1)}{\Gamma(n+1-\nu)}$.

Example 1. *Consider the extorial function,* $f(t) = e((mt)^{(1)}_{(m\ell)})$, *where*

$$e((mt)^{(1)}_{(m\ell)}) = 1 + \frac{(mt)^{(1)}_{(m\ell)}}{1!} + \frac{(mt)^{(2)}_{(m\ell)}}{2!} + \frac{(mt)^{(3)}_{(m\ell)}}{3!} + \cdots$$

From the property of extorial function [8], we have:

$$\Delta_\ell^s e((mt)^{(1)}_{(m\ell)}) = (m\ell)^s e((mt)_{(m\ell)}) \quad \text{and} \quad \Delta_\ell^{-\nu} e((mt)^{(1)}_{(m\ell)}) = \frac{e((mt)^{(1)}_{(m\ell)})}{(m\ell)^\nu}$$

Replacing m by s, we get:

$$\Delta_\ell^{-\nu} e(st^{(1)}_{(s\ell)}) = \frac{e(st^{(1)}_{(s\ell)})}{(s\ell)^\nu}$$

$$\Delta_\ell^{-\nu} e(st^{(1)}_{(s\ell)})\Big|_{t-m\ell}^{t} = \frac{e(st^{(1)}_{(s\ell)}) - e(s(t-m\ell)^{(1)}_{(s\ell)})}{(s\ell)^\nu}$$

$$\sum_{r=0}^{m} \frac{(\nu + r)^{(\nu)}}{\ell^\nu} e\big((s(t - \ell - r\ell))^{(1)}_{(s\ell)}\big) = \frac{e(st^{(1)}_{(s\ell)})}{(s\ell)^{(\nu)}}\Big|_{t=t-m\ell}^{t}$$

$$\sum_{r=0}^{m} \frac{\Gamma(\nu + r)}{\Gamma(r+1)\Gamma(\nu+1)} e((s(t - \ell - r\ell))^{(1)}_{(s\ell)}) = \frac{e((st)^{(1)}_{(s\ell)}) - e((s(t - \ell - r\ell))^{(1)}_{(s\ell)})}{(s\ell)^\nu}$$

As $m \to \infty$, $e(s(t - m\ell)^{(1)}_{s\ell}) \to 0$, and hence the above equation becomes:

$$\sum_{r=0}^{m} \frac{\Gamma(\nu + r)}{\Gamma(r+1)\Gamma(\nu+1)} e((s(t - \ell - r\ell))^{(1)}_{(s\ell)}) = \frac{e((st)^{(1)}_{(s\ell)})}{(s\ell)^\nu}$$

3. Fractional order of delta operator

Theorem 3.1. *For any real value ν and $\ell \neq 0$, we have*

$$\Delta_\ell^\nu \sin t = 2^\nu \sin^\nu(\frac{\ell}{2}) \sin(\frac{\nu\pi}{2} + \frac{\nu\ell}{2} + t)$$

Proof. By the linear operator Δ_ℓ on $\sin t$, we have $\Delta_\ell \sin t = \sin(t+\ell) - \sin t$ which is similar to $\Delta_\ell \sin t = 2\cos(\frac{t+\ell+t}{2})\sin(\frac{t+\ell-t}{2}) = 2\cos(t+\frac{\ell}{2})\sin(\frac{\ell}{2})$. Thus,

$$\Delta_\ell \sin t = 2\sin(\frac{\ell}{2}) \sin(\frac{\pi}{2} + \frac{\ell}{2} + t) \tag{3.1}$$

Taking Δ_ℓ again on both sides of (3.1), we get

$$\Delta_\ell^2 \sin t = 2\sin(\frac{\ell}{2})\Delta_\ell \sin(\frac{\pi}{2} + \frac{\ell}{2} + t)$$

While solving the above relation, we obtain

$$\Delta_\ell^2 \sin t = 2^2 \sin^2(\frac{\ell}{2}) \sin(2\frac{\pi}{2} + 2\frac{\ell}{2} + t) \tag{3.2}$$

Again taking Δ_ℓ on both sides of (3.2), we get

$$\Delta_\ell^3 \sin t = 2^3 \sin^3(\frac{\ell}{2}) \sin(\frac{3\pi}{2} + \frac{3\ell}{2} + t)$$

Proceeding the steps up-to m times, we find

$$Delta_\ell^m \sin t = 2^m \sin^m(\frac{\ell}{2}) \sin(\frac{m\pi}{2} + \frac{m\ell}{2} + t)$$

Hence, the proof follows by extending the integer m to any real value ν. □

Theorem 3.2. *For any real value ν and $\ell \neq 0$, we have:*

$$\Delta_\ell^\nu \cos t = 2^\nu \sin^\nu(\frac{\ell}{2}) \cos(\frac{\nu\pi}{2} + \frac{\nu\ell}{2} + t).$$

Proof. The proof is similar to Theorem 3.1. □

Example 2. *Take $\nu = \frac{1}{2}$ in (3.1) and (3.2), then:*

$$\Delta_\ell^{\frac{1}{2}}(\sin t) = 2^{\frac{1}{2}} \sin^{\frac{1}{2}}(\frac{\ell}{2}) \sin(\frac{\frac{1}{2}\pi}{2} + \frac{\frac{1}{2}\ell}{2} + t)$$

$$\Delta_\ell^{\frac{1}{2}}(\cos t) = 2^{\frac{1}{2}} \sin^{\frac{1}{2}}(\frac{\ell}{2}) \cos(\frac{\frac{1}{2}\pi}{2} + \frac{\frac{1}{2}\ell}{2} + t)$$

4. Fractional order of alpha delta operator and its inverse

In this section, we develop the theory of fractional order of alpha delta operator and its sums on trigonometric functions.

Definition 4.1. *Let $\ell > 0$ and f, g be a functions. Then the alpha delta operator $\Delta_{\alpha,\ell}$ on $f(t)$ is defined by*

$$\Delta_{\alpha,\pm\ell} f(t) = f(t \pm \ell) - \alpha f(t)$$

If $\Delta_{\alpha,\pm\ell} g(t) = f(t)$, then the inverse is $\Delta^{-1}_{\alpha,\pm\ell} f(t) = g(t) + c$, c is constant and

$$\Delta^{-1}_{\alpha,\pm\ell} f(t)||_a^d = f(a) - \alpha f(d)$$

Theorem 4.1. *The alpha fractional difference of the sine function is*

$$\Delta^{\gamma}_{\alpha,\ell} \sin t = \sin^{\gamma} \ell \sin\left(\frac{\gamma\pi}{2} + t\right), \alpha = \cos \ell \tag{4.1}$$

Proof. From the definition of $\Delta_{\alpha,\ell}$, we have:

$$\Delta_{\alpha,\ell} \sin t = \sin(t + \ell) - \alpha \sin t$$

and taking $\alpha = \cos \ell$

$$\begin{aligned}\Delta_{\alpha,\ell} \sin t &= \sin t \cos \ell + \cos t \sin \ell - \cos \ell \sin t \\ &= \cos t \sin \ell \\ &= \sin \ell \sin\left(\frac{\pi}{2} + t\right)\end{aligned}$$

Similarly,

$$\Delta^2_{\alpha,\ell} \sin t = \sin^2_{\ell} \sin(\frac{2\pi}{2} + n)$$

In general,

$$\Delta^{\gamma}_{\alpha,\ell} \sin t = \sin^{\gamma} \ell \sin\left(\frac{\gamma\pi}{2} + t\right)$$

which is (4.1). □

Theorem 4.2. *The alpha fractional derivative of the cosine function is*

$$\Delta^{\gamma}_{\alpha,\ell} \cos t = \sin^{\gamma} \ell \cos\left(\frac{\gamma\pi}{2} + t\right), \alpha = \cos \ell \tag{4.2}$$

Proof. Since $\Delta_{\alpha,\ell}\cos t = \cos(t+\ell) - \alpha\cos t$ and taking $\alpha = \cos\ell$, we have:

$$\begin{aligned}\Delta_{\alpha,\ell}\cos t &= \cos t\cos\ell - \sin t\sin\ell - \cos\ell\cos t\\ &= -\sin t\sin\ell\\ &= \sin\ell\cos\left(\frac{\pi}{2}+t\right)\\ \Delta^2_{\alpha,\ell}\cos t &= \sin^2\ell\cos\left(\frac{2\pi}{2}+t\right)\end{aligned}$$

In general,

$$\Delta^\gamma_{\alpha,\ell}\cos t = \sin^\gamma\ell\cos\left(\frac{\gamma\pi}{2}+t\right)$$

which is (4.2). □

Theorem 4.3. *The fractional alpha difference of* $\sin at$ *is*

$$\Delta^\gamma_{\alpha,\ell}\sin at = \sin^\gamma a\ell\sin a\left(\frac{\gamma\pi}{2}+t\right), \alpha = \cos\ell \tag{4.3}$$

Proof. Since $\Delta_{\alpha,\ell}\sin at = \sin a(t+\ell) - \alpha\sin at$, and taking $\alpha = \cos a\ell$ we have:

$$\begin{aligned}\Delta_{\alpha,\ell}\sin at &= \sin at\cos a\ell + \cos at\sin a\ell - \alpha\sin at\\ &= \cos at\sin a\ell\\ &= \sin a\ell\sin a\left(\frac{\pi}{2}+t\right)\\ \Delta^2_{\alpha,\ell}\sin at &= \sin^2 a\ell\sin a(\frac{2\pi}{2}+n)\end{aligned}$$

In general,

$$\Delta^\gamma_{\alpha,\ell}\sin at = \sin^\gamma a\ell\sin a\left(\frac{\gamma\pi}{2}+t\right)$$

which is (4.3). □

Theorem 4.4. *The fractional alpha difference of* $\cos at$ *is*

$$\Delta^\gamma_{\alpha,\ell}\cos at = \sin^\gamma a\ell\cos a\left(\frac{\gamma\pi}{2}+t\right), \alpha = \cos\ell \tag{4.4}$$

Proof. From $\Delta_{\alpha,\ell}\cos at = \cos a(t+\ell) - \alpha\cos at$ and $\alpha = \cos a\ell$, we have:

$$\begin{aligned}\Delta_{\alpha,\ell}\cos at &= \cos at\cos a\ell - \sin at\sin a\ell - \cos a\ell\cos at\\ &= -\sin at\sin a\ell\\ &= \sin a\ell\cos a\left(\frac{\pi}{2}+t\right)\\ \Delta^2_{\alpha,\ell}\cos t &= \sin^2 a\ell\cos a\left(\frac{2\pi}{2}+t\right)\end{aligned}$$

In general,

$$\Delta^{\gamma}_{\alpha,\ell} \cos at = \sin^{\gamma} a\ell \cos a\Big(\frac{\gamma\pi}{2} + t\Big)$$

which is (4.4). □

Theorem 4.5. *The fractional inverse of alpha-difference of the sine function is*

$$\Delta^{-\gamma}_{\alpha,\ell} \sin t = \sin^{-\gamma} \ell \sin\left(t - \frac{\gamma\pi}{2}\right), \alpha = \cos\ell. \tag{4.5}$$

Proof. From $\Delta^{\gamma}_{\alpha,\ell} \sin t = \sin^{\gamma} \ell \sin(\frac{\gamma\pi}{2} + t)$ and $\gamma = -\gamma$, we have:

$$\Delta^{-\gamma}_{\alpha,\ell} \sin t = \frac{\sin\left(t - \frac{\gamma\pi}{2}\right)}{\sin^{\gamma} \ell}$$

$$\Delta^{\gamma}_{\alpha,\ell} \sin\left(t - \frac{\gamma\pi}{2}\right) = \sin t \sin^{\gamma} \ell$$

Similarly,

$$\Delta^{-\gamma}_{\alpha,\ell} \sin t = \sin^{-\gamma} \ell \sin\left(t - \frac{\gamma\pi}{2}\right)$$

which is (4.5). □

Theorem 4.6. *For any real function $f(t)$, and $m > 0$ it holds:*

$$\Delta^{-1}_{\alpha,\ell} f(t) - \alpha^m \Delta^{-1}_{\alpha,\ell} f(t - m\ell) = \sum_{r=1}^{m} \alpha^{r-1} f(t - r\ell). \tag{4.6}$$

Proof. From the definition of inverse of alpha difference, $\Delta^{-1}_{\alpha,\ell} f(t) = g(t)$

$$g(t + \ell) = f(t) + \alpha g(t) \tag{4.7}$$

Replacing t by $t - \ell$, $t - 2\ell$, $t - 3\ell, \cdots$, $t - m\ell$ in (4.7) we obtain:

$$\begin{aligned} g(t) = {} & f(t-\ell) + \alpha f(t-2\ell) + \alpha^2 f(t-3\ell) + \alpha^3 f(t-4\ell) + \cdots \\ & + \alpha^{m-1} f(t-m\ell) + \alpha^m g(t-m\ell) \end{aligned}$$

$$\begin{aligned} g(t) - \alpha^m g(t-m\ell) = {} & f(t-\ell) + \alpha f(t-2\ell) + \alpha^2 f(t-3\ell) + \alpha^3 f(t-4\ell) \\ & + \cdots + \alpha^{m-1} f(t-m\ell) \end{aligned}$$

$$\begin{aligned} \Delta^{-1}_{\alpha,\ell} f(t) - \alpha^m \Delta^{-1}_{\alpha,\ell} f(t-m\ell) = {} & f(t-\ell) + \alpha f(t-2\ell) + \alpha^2 f(t-3\ell) \\ & + \alpha^3 f(t-4\ell) + \cdots + \alpha^{m-1} f(t-m\ell) \end{aligned}$$

which gives (4.6). □

Example 3. *If $f(t) = \sin t$, then the equation (4.6) becomes:*

$$\Delta_{\alpha,\ell}^{-1} \sin t - \alpha^m \Delta_{\alpha,\ell}^{-1} \sin(t - m\ell) = \sum_{r=1}^{m} \alpha^{r-1} \sin(t - r\ell)$$

Taking $t = 5$, $\alpha = \cos \ell$, $\ell = 1$, $m = 2$, we obtain:

$$\begin{aligned} &\Delta_{\alpha,\ell}^{-1} \sin t - (\cos \ell)^m \Delta_{\alpha,\ell}^{-1} \sin(t - m\ell) \\ &\qquad = \sin^{-1}(1) \sin(5 - \frac{\pi}{2}) - \sin^{-1}(1)(\cos(1))^2 \sin(3 - \frac{\pi}{2}) \\ &\qquad = \sin^{-1}(1) \Big[\sin(5 - \frac{\pi}{2}) - (\cos(1))^2 \sin(3 - \frac{\pi}{2}) \Big] \end{aligned}$$

$$\sin(4) + \cos(1)\sin(3) = \frac{\sin(-85) - \cos^2(1)\sin(-87)}{\sin(1)} \Rightarrow 0.1221 = 0.1221$$

5. Conclusion

Through this research, we have obtained methods for finding fractional order of difference, alpha delta operator difference and finding the values of sum of extorial function and trigonometric function with Gamma coefficients. By our results it is possible to find solutions of fractional order difference equation as well as fractional differential equation when ℓ tends to zero.

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