

EXPONENTIAL OPERATIONAL LAWS OF PYTHAGOREAN FUZZY PROJECTION MODELS FOR DECISION MAKING

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ABSTRACT. The aim of this paper is to investigate the idea of projection models in Pythagorean fuzzy multi criteria decision making environment. Exponential operational laws are used to study the Pythagorean fuzzy ideal point, the modules of Pythagorean fuzzy numbers and the cosine of the included angle. The paper also establishes a Pythagorean projection model using exponential operational laws and the model is used to determine the degree of similarity between each alternative and the ideal point of exponential fuzzy data in Pythagorean fuzzy sets. Based on the proposed projection models, the alternatives are ranked in order to select the most desirable alternative.

1. INTRODUCTION

The concept of fuzzy set [1] was introduced by L.A. Zadeh (1965). A fuzzy Set is defined by a membership function that assigns a membership value ranging from 0 to 1. In this paper we use exponential operational laws of Pythagorean fuzzy sets [3] to create a Pythagorean projection model. Exponential operational laws is a new concept for studying the ideas of Pythagorean fuzzy ideal point and cosine of the included angle. We establish two projection models for Pythagorean fuzzy decision making, where the attribute weight is completely known.

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2. PROJECTION MODELS FOR PYTHAGOREAN FUZZY SETS

Definition 2.1. [3] Let X be a universal non empty set. A Pythagorean fuzzy set A in X is given by

$$(2.1) \quad A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \},$$

where $\mu_A(x) : X \rightarrow [0, 1]$ depicts the degree of membership and $\nu_A(x) : X \rightarrow [0, 1]$ depicts the degree of nonmembership of the element $x \in X$ to the set A , respectively, with the condition that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. The degree of indeterminacy $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

Let $\tilde{r}_j^+ = (\mu_j^+, \nu_j^+, \pi_j^+)$ ($j = 1, 2, \dots, n$), where $\mu_j^+ = \max_i \{\mu_{ij}\}$, $\nu_j^+ = \min_i \{\nu_{ij}\}$,

$$(2.2) \quad \pi_j^+ = 1 - \mu_j^+ - \nu_j^+ = 1 - \max_i \{\mu_{ij}\} - \min_i \{\nu_{ij}\}, j = 1, 2, \dots, n.$$

Definition 2.2 (Pythagorean Fuzzy Ideal Point). We form a relative pythagorean fuzzy ideal solution (RPFIS) as follows:

$$(2.3) \quad A^+ = (\tilde{r}_1^+, \tilde{r}_2^+, \dots, \tilde{r}_n^+)$$

whose module is denoted by

$$(2.4) \quad |A^+| = \sqrt{\sum_{j=1}^n |\tilde{r}_j^+|^2} = \sqrt{\sum_{j=1}^n ((\mu_j^+)^2 + (\nu_j^+)^2 + (\pi_j^+)^2)}.$$

Definition 2.3. Let $A_i = (\tilde{r}_{i1}, \tilde{r}_{i2}, \dots, \tilde{r}_{in})$ be the i^{th} alternative, and the relative pythagorean fuzzy ideal solution (RPFIS) $A^+ = (\tilde{r}_1^+, \tilde{r}_2^+, \dots, \tilde{r}_n^+)$, where $\tilde{r}_{ij} = (\mu_{ij}, \nu_{ij}, \pi_{ij})$, $i = 1, 2, \dots, m$, and $\tilde{r}_j^+ = (\mu_j^+, \nu_j^+, \pi_j^+)$, $j = 1, 2, \dots, n$. Then we call

$$(2.5) \quad \cos(A_i, A^+) = \frac{\sum_{j=1}^n (\mu_{ij}\mu_j^+ + \nu_{ij}\nu_j^+ + \pi_{ij}\pi_j^+)}{|A_i||A^+|}, i = 1, 2, \dots, m.$$

the cosine of the included angle between A_i and A^+ .

Theorem 2.1.

- (1) $0 \leq \cos(A_i, A^+) \leq 1$;
- (2) $\cos(A_i, A^+) = \cos(A^+, A_i)$;
- (3) $\cos(A_i, A^+) = 1$, if $A_i = A^+$, $i = 1, 2, \dots, m$.

$\cos(A_i, A^+)$ reflects the similarity measure of A_i and A^+ . We introduce a formula of projection of A_i and A^+ as follows:

$$\begin{aligned} Proj_{A^+}(A_i) &= |A_i| \cos(A_i, A^+) \\ &= |A_i| \frac{\sum_{j=1}^n (\mu_{ij}\mu_j^+ + \nu_{ij}\nu_j^+ + \pi_{ij}\pi_j^+)}{|A_i||A^+|} = \frac{1}{|A^+|} \sum_{j=1}^n (\mu_{ij}\mu_j^+ + \nu_{ij}\nu_j^+ + \pi_{ij}\pi_j^+). \end{aligned}$$

The better alternative A_i is measured based on the greater value of $Proj_{A^+}(A_i)$ and also reflects that the closer alternative A_i to the RPFIS A^+ .

If the weighting vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ of the attributes $G = G_1, G_2, \dots, G_n$ is known, then we denote

$$(2.6) \quad |Y_i|_\omega = \sqrt{\sum_{j=1}^n (\omega_j |\tilde{r}_{ij}|)^2},$$

as the weighted module of the alternative $A_i = (\tilde{r}_{i1}, \tilde{r}_{i2}, \dots, \tilde{r}_{in})$, where $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ is the weighting vector of the attribute $G_j (j = 1, 2, \dots, n)$, where $\omega_j \in [0, 1]$, $\sum_{j=1}^n \omega_j = 1$. Furthermore, we denote weighted module of the RPFIS

$$(2.7) \quad |A^+|_\omega = \sqrt{\sum_{j=1}^n (\omega_j |\tilde{r}_j^+|)^2} = \sqrt{\sum_{j=1}^n ((\omega_j \mu_j^+)^2 + (\omega_j \nu_j^+)^2 + (\omega_j \pi_j^+)^2)}.$$

Similar to Eq.(2.7), we introduce the weighted cosine of the included angle:

$$(2.8) \quad \cos(A_i, A^+)_\omega = \frac{\sum_{j=1}^n (\omega_j)^2 (\mu_{ij}\mu_j^+ + \nu_{ij}\nu_j^+ + \pi_{ij}\pi_j^+)}{|A_i|_\omega |A^+|_\omega}, i = 1, 2, \dots, m.$$

Weighted Pythagorean Fuzzy Projection Formula: we can introduce a formula of projection of A_i on A^+ as follows:

$$\begin{aligned} (2.9) \quad Proj_{A^+}(A_i) &= |A_i|_\omega \cos(A_i, A^+)_\omega = |A_i|_\omega \frac{\sum_{j=1}^n (\omega_j)^2 (\mu_{ij}\mu_j^+ + \nu_{ij}\nu_j^+ + \pi_{ij}\pi_j^+)}{|A_i|_\omega |A^+|_\omega} \\ &= \frac{1}{|A^+|_\omega} \sum_{j=1}^n (\omega_j)^2 (\mu_{ij}\mu_j^+ + \nu_{ij}\nu_j^+ + \pi_{ij}\pi_j^+). \end{aligned}$$

2.1. Exponential Operational law of Pythagorean Fuzzy Projection Model.

Definition 2.4. Let X be a fixed set, and $\alpha = \langle \mu_A(x), \nu_A(x) \rangle$ be a PFN, then the exponential Operational of α is defined as:

$$(2.10) \quad \lambda^\alpha = \begin{cases} \left\langle (\lambda)^{\sqrt{1-\mu_A^2(x)}}, \sqrt{1 - (\lambda)^{2\nu_A(x)}} \right\rangle; & \lambda \in (0, 1) \\ \left\langle \left(\frac{1}{\lambda}\right)^{\sqrt{1-\mu_A^2(x)}}, \sqrt{1 - \left(\frac{1}{\lambda}\right)^{2\nu_A(x)}} \right\rangle; & \lambda \geq 1 \end{cases}.$$

Corresponding to its membership degrees, the degree of indeterminacy function is given by

$$\pi_A(x) = \sqrt{(\lambda)^{2\nu_A(x)} - (\lambda)^{2\sqrt{1-\mu_A^2(x)}}}.$$

Based on the above mentioned equation, we develop exponential operational laws of pythagorean fuzzy projection models. We evaluate the finite collection alternatives $A_i (i = 1, 2, \dots, m)$ based on the attributes $G_j (j = 1, 2, \dots, n)$ and form exponential operational laws of pythagorean decision matrix by

$$(2.11) \quad \tilde{R}' = (\tilde{r}'_{ij})_{m \times n} = \left((\lambda)^{\sqrt{1-[\mu_{ij}(x)]^2}}, \sqrt{1 - (\lambda)^{2[\nu_{ij}(x)]}}, \sqrt{(\lambda)^{2\nu_{ij}(x)} - (\lambda)^{2\sqrt{1-\mu_{ij}^2(x)}}} \right)_{m \times n},$$

$$A_i = (\tilde{r}_{i1}, \tilde{r}_{i2}, \dots, \tilde{r}_{in}), i = 1, 2, \dots, m,$$

where r'_{ij} is the element of \tilde{R}' and introduce the module of $A_i (i = 1, 2, \dots, m)$ as:

$$(2.12) \quad |Y_i| = \sqrt{\sum_{j=1}^n |\tilde{r}_{ij}|^2},$$

where $|\tilde{r}_{ij}|$ is the module of the attribute value \tilde{r}_{ij} calculated as follows:

$$(2.13) \quad |\tilde{r}_{ij}| = \sqrt{\left((\lambda)^{\sqrt{1-[\mu_{ij}(x)]^2}} \right)^2 + \left(\sqrt{1 - (\lambda)^{2[\nu_{ij}(x)]}} \right)^2 + \left(\sqrt{(\lambda)^{2\nu_{ij}(x)} - (\lambda)^{2\sqrt{1-\mu_{ij}^2(x)}}} \right)^2}.$$

By equation (2.12), we have $0 \leq |A_i| \leq \sqrt{n}$. Let $\tilde{r}_j^+ = (\mu_j^+, \nu_j^+, \pi_j^+)(j = 1, 2, \dots, n)$, where

$$(2.14) \quad \begin{aligned} \mu_j^+ &= \max_i \left\{ (\lambda)^{\sqrt{1-[\mu_{ij}(x)]^2}} \right\}, \nu_j^+ = \min_i \left\{ \sqrt{1 - (\lambda)^{2[\nu_{ij}(x)]}} \right\}, \\ \pi_j^+ &= 1 - \max_i \left\{ (\lambda)^{\sqrt{1-[\mu_{ij}(x)]^2}} \right\} - \min_i \left\{ \sqrt{1 - (\lambda)^{2[\nu_{ij}(x)]}} \right\}. \end{aligned}$$

Definition 2.5. Let $A_i = (\tilde{r}_{i1}, \tilde{r}_{i2}, \dots, \tilde{r}_{in})$ be the i^{th} alternative, and the relative exponential operational laws of pythagorean fuzzy ideal solution $A^+ = (\tilde{r}_1^+, \tilde{r}_2^+, \dots, \tilde{r}_n^+)$,

$$\tilde{r}_{ij} = \left((\lambda)^{\sqrt{1-[\mu_{ij}(x)]^2}}, \sqrt{1 - (\lambda)^{2[\nu_{ij}(x)]}}, \sqrt{(\lambda)^{2\nu'_{ij}(x)} - (\lambda)^{2\sqrt{1-\mu_{ij}^2}(x)}} \right),$$

$i = 1, 2, \dots, m$ and $\tilde{r}_j^+ = (\mu_j^+, \nu_j^+, \pi_j^+)$, ($j = 1, 2, \dots, n$). Then we defined the cosine of the included angle between A_i and A^+

$$(2.15) \quad \cos(A_i, A^+) = \frac{\sum_{j=1}^n \left((\lambda)^{\sqrt{1-[\mu_{ij}]^2}} \mu_j^+ + \sqrt{1 - (\lambda)^{2[\nu_{ij}]}} \nu_j^+ + \sqrt{(\lambda)^{2\nu_{ij}} - (\lambda)^{2\sqrt{1-\mu_{ij}^2}}} \pi_j^+ \right)}{|A_i||A^+|}.$$

We introduce a formula of exponential operational laws of pythagorean projection of A_i and A^+ as follows:

$$(2.16) \quad Proj_{A^+}(A_i) = |A_i| \cos(A_i, A^+) = \frac{1}{|A^+|} \sum_{j=1}^n \left((\lambda)^{\sqrt{1-[\mu_{ij}(x)]^2}} \mu_j^+ + \sqrt{1 - (\lambda)^{2[\nu_{ij}(x)]}} \nu_j^+ + \sqrt{(\lambda)^{2\nu_{ij}(x)} - (\lambda)^{2\sqrt{1-\mu_{ij}^2}(x)}} \pi_j^+ \right).$$

Pythagorean weighted exponential operational laws of pythagorean fuzzy projection formula: If the weighting vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ of the attributes $G = G_1, G_2, \dots, G_n$ is known, then the weighted projection formula as follows:

$$(2.17) \quad Proj_{A^+}(A_i) = |A_i|_\omega \cos(A_i, A^+)_\omega = \frac{1}{|A^+|_\omega} \sum_{j=1}^n (\omega_j)^2 \left((\lambda)^{\sqrt{1-[\mu_{ij}(x)]^2}} \mu_j^+ + \sqrt{1 - (\lambda)^{2[\nu_{ij}(x)]}} \nu_j^+ + \sqrt{\lambda^{2\nu_{ij}} - \lambda^{2\sqrt{1-\mu_{ij}^2}}} \pi_j^+ \right).$$

3. ILLUSTRATION OF THE PROPOSED WEIGHTED PROJECTION MODELS

In this section we present an illustrative example to explain the weighted projection method for potential evaluation of emerging technology commercialization with pythagorean fuzzy information. Let us consider a panel have to select emerging technology enterprises among possible five enterprises A_i ($i = 1, 2, 3, 4, 5$). The experts are identifying six attributes for evaluating the five possible emerging technology companies: x_1 is the technical advancement, x_2

TABLE 1. Pythagorean Fuzzy Decision Matrix

	A_1	A_2	A_3	A_4	A_5
x_1	(0.53,0.09)	(0.73,0.08)	(0.91,0.02)	(0.85,0.05)	(0.90,0.02)
x_2	(0.89,0.03)	(0.13,0.21)	(0.07,0.05)	(0.74,0.10)	(0.68,0.21)
x_3	(0.42,0.18)	(0.03,0.13)	(0.04,0.10)	(0.02,0.05)	(0.05,0.06)
x_4	(0.08,0.02)	(0.73,0.08)	(0.68,0.06)	(0.08,0.06)	(0.13,0.09)
x_5	(0.33,0.12)	(0.52,0.16)	(0.15,0.07)	(0.16,0.05)	(0.15,0.08)
x_6	(0.17,0.13)	(0.51,0.21)	(0.31,0.25)	(1.00,0.00)	(0.91,0.05)

is the potential market and marketrisk, x_3 is the industrialization infrastructure, x_4 is the development of science and technology, x_5 is the financial conditions, x_6 is the employment creation. To avoid influencing each other, decision-makers are expected to evaluate the five potential emerging technology companies $A_i (i = 1, 2, 3, \dots, 5)$ under the six attributes mentioned above and decision maker provides the decision matrix $\tilde{R} = (\tilde{r}_{ij})_{5 \times 6}$ in Table.1, where $\tilde{r}_{ij} (i = 1, 2, 3, 4, 5, j = 1, 2, 3, 4, 5, 6)$ are in the form of pythagrean fuzzy numbers. The Weight vector of $x_i (i = 1, 2, \dots, 6)$ is $\omega = (0.12, 0.25, 0.09, 0.16, 0.2, 0.18)^T$. The following steps are involved in weighted pythagorean fuzzy projection formula for getting the most desirable emerging technology companies:

Step 1: Table 1. represents the five emerging technology enterprises A_i .

Step 2: Based on the Table.1 and Eqs.(2.2),(2.3), we can get the PFFIS A^+ :

$$A^+ = \{(0.91, 0.02, 0.07), (0.89, 0.03, 0.08), (0.42, 0.05, 0.53), \\ (0.73, 0.02, 0.25), (0.52, 0.05, 0.43), (1.00, 0.00, 0.00)\}$$

Step 3: The projection of $A_i (i = 1, 2, 3, 4, 5)$ on the PFFIS A^+ by using Eq.(2.9):

$$Prj_A^+(A_1) = 0.248, Prj_A^+(A_2) = 0.200, \\ Prj_A^+(A_3) = 0.180, Prj_A^+(A_4) = 0.314, Prj_A^+(A_5) = 0.298.$$

Rank the emerging technology enterprises $A_i (i = 1, 2, 3, 4, 5)$ in accordance with the values $Prj_{A^+}(A_i) (i = 1, 2, 3, 4, 5)$: $A_4 \succ A_5 \succ A_1 \succ A_2 \succ A_3$.

Next, we have taken exponential operational laws of weighted projection model to validate the same illustration problem mentioned above. Then the

following steps are involved as follows:

Step 1. Based on the table 1, we denote the five emerging technology A_i ($i = 1, 2, 3, 4, 5$) by $\tilde{r}_{ij} = (\mu_{ij}, \nu_{ij}, \pi_{ij})$. Here the values mentioned only for A_1 ;

$$A_1 = \{(0.55556, 0.34249, 0.757668), (0.90326, 0.11532, 0.41331), \\ (0.435371, 0.53007, 0.72765), (0.050482, 0.33605, 0.94049), \\ (0.320932, 0.50095, 0.8037724), (0.103408, 0.67116, 0.734062)\}$$

Step 2. Based on the Table and Eqs.(2.14), we can get the EOLPFFIS A^+ :

$$A^+ = \{(0.912245, 0.094266, -0.006511), (0.90326, 0.115323, -0.018583), \\ (0.435371, 0.530074, 0.0345542), (0.783669, 0.220543, -0.004212), \\ (0.553185, 0.415695, 0.0311197), (1, 0, 0)\}$$

Step 3. Calculate the projection of A_i ($i = 1, 2, 3, 4, 5$) on the EOLPFFIS A^+ by using Eq.(2.17):

$$Prj_A^+(A_1) = 0.071121, Prj_A^+(A_2) = 0.065985, Prj_A^+(A_3) = 0.049501, \\ Prj_A^+(A_4) = 0.091301, Prj_A^+(A_5) = 0.090124$$

Rank the emerging technology enterprises A_i ($i = 1, 2, 3, 4, 5$) in accordance with the values $Prj_{A^+}(A_i)$ ($i = 1, 2, 3, 4, 5$): $A_4 \succ A_5 \succ A_1 \succ A_2 \succ A_3$.

The Proposed projection models is applied to multi criteria decision making problem. We listed the rank of the alternatives in table3. We clearly see that the optimistic results of the problem is same in weighted pythagorean fuzzy projection models.

TABLE 2. Rank of the pythagorean fuzzy projection models

Proposed Projection Models	Ranking order
Pythagorean Fuzzy	$A_4 \succ A_5 \succ A_1 \succ A_2 \succ A_3$
Exponential of Pythagorean Fuzzy	$A_4 \succ A_5 \succ A_1 \succ A_2 \succ A_3$

4. CONCLUSION

We investigated the idea of projection models in pythagorean fuzzy multi criteria decision making environment. We established a new fuzzy projection model based on exponential operational laws of Pythagorean fuzzy sets. Then, the proposed model is demonstrated with real life MCDM problems. Finally the alternatives are ranked in order to select the most desirable alternative. We found that the ranking order of Pythagorean and exponential operational laws of projection models are same, while comparing the results.

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