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COMPARATIVE ANALYSIS OFITERATIVE METHODS FOR SOLVING NON-LINEAR EQUATIONS

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ABSTRACT. This paper present numerical study of iterative methods for solving non-linear equations. The proposed study using Classical Regula Falsi (CRF), Improved Regula Falsi(IRF) and Newton- Raphson (N-R) method for solving non linearequations and their results are compared with the help of numerical examples. It is observed that CRF requires a large number of iteration for finding a root but in most of the cases N-R is better than IRF which converge earlier. For some cases IRF converge but N-R is not converge.

1. INTRODUCTION

Numerical Analysis is the branch of mathematics which deals with the approximate solution of mathematical problems. We know that solving the non-linear equations is most important and hard problems.Algebraic equations are of the form $ax^2 + bx + c = 0$, $y^3 + y + 1 = 0$. Any equation of the form which involves trigonometric functions, exponential functions or logarithmic functions is known as transcendental equations. For example: $x + cosx = 0$, $e^x + x = 0$, $x^2 - log_e e^x = 0$.There are different methods for finding the roots and all these converge to the root at different rates. The rate of convergence of some methods is faster than others. There is the study at comparing the rate of convergence of CRF, IRF and N-R method.

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The Regula Falsi Method is a numerical technique for evaluating the roots of a polynomial $f(k)$. A value k replaces the midpoint in the Bisection Method and fills in as the new estimate of a root of $f(k)$. The goal is to make convergence faster. Assume that $f(k)$ is continuous function. It is used to find the approximate root of a non-linear equation. It is used for finding the roots in the interval $[p^1, q^1]$ given that f is continuous on the interval $[p^1, q^1]$ and $f(p^1)$ and $f(q^1)$ have opposite signs. It must be of the form $f(p^1)f(q^1) < 0$.

In this paper, we have considered the problem of finding roots of equations.

$$f(k) = 0, \tag{1.1}$$

where f be a continuous function. Suppose that $f(k)$ has at least one zero say in $[p^1, q^1]$. There exist many methods such as B.M, CRF method, N-R method etc. [1]. The CRF method for solving (1.1) consists of successive application cite6, [8].

We here study the CRF,IRF and N-R method.There are different methods for finding the root and all these converge to the root at different rates. The rate of convergence of some methods is faster than other methods.N-R method is very speedy and successful as compared to other methods [10]. As we realize that order of convergence of CRF is one, IRF is of order 3 and N-R method is of order 2. Out of these methods N-R gives better efficient result and CRF requires a large number of iteration for finding a root but in most of the cases N-R is better than IRF which converge earlier. For few cases IRF converge but N-R is not converge.

2. Main Results

Order of Convergence

The order of convergence is one of the essential approaches to evaluate the actual rate of convergence, the speed at which the mistakes go to zero.

2.1. **Order of Convergence for Classical Regula Falsi Method.** Let us assume that ω is a root of equation $f(k) = 0$. The general iterative formula for RegulaFalsistrategy is

$$k_{m+1} = k_m - \frac{k_m - k_{m-1}}{f(k_m) - f(k_{m-1}}.f(k_m). \tag{2.1}$$

If h_{m-1},h_m and h_{m+1} are the errors in the approximations k_{m-1},k_m and k_{m+1} respectively then

$$h_{m-1} = k_{m-1} - \omega, h_m = k_m - \omega \quad \text{and} \quad h_{m+1} = k_{m+1} - \omega$$

i.e.

$$k_{m-1} = \omega + h_{m-1}, k_m = \omega + h_m \quad \text{and} \quad k_{m+1} = \omega + h_{m+1}.$$

Substituting these values in formula (2.1), we have

$$h_{m+1} = h_m - \frac{(h_m - h_{m-1}) f(\omega + h_m)}{f(\omega + h_m) - f(\omega + h_{m-1})}. \tag{2.2}$$

Expanding $f(\omega + h_m)$ and $f(\omega + h_{m-1})$ using Taylor's series about the point ω and substituting $f(\omega) = 0$ in (2.2), we get

$$h_{m+1} = h_m - \frac{h_m + 1/2 h_m^2 \frac{f''(\omega)}{f'(\omega)} + \dots}{1 + 1/2(h_m + h_{m-1}) \frac{f''(\omega)}{f'(\omega)} + \dots}.$$

By dividing numerator and denominator of second term in right-hand side by $f'(\omega) \neq 0)$ we receive

$$h_{m+1} = h_m - \left[h_m + 1/2 h_m^2 \frac{f''(\omega)}{f'(\omega)} + \dots \right] \left[1 - 1/2(h_m + h_{m-1}) \frac{f''(\omega)}{f'(\omega)} - \dots \right].$$

Using binomial theorem for any index:

$$h_{m+1} = h_m - [h_m - 1/2 h_m (h_m + h_{m-1}) \frac{f''(\omega))}{(f'(\omega)} + 1/2 h_m^2 \frac{f''(\omega)}{f'(\omega)} + \cdots.$$

And neglecting the terms having higher orders of errors:

$$h_{m+1} = 1/2 \frac{f''(\omega)}{f'(\omega)} (h_m h_{m-1}), \tag{2.3}$$

$$h_{m+1} = A h_m.$$

where A is the asymptotic error constant.

It is clear that the CRF method is of linear order convergent [9].

2.2. **Algorithm of Classical Regula falsi.**

(1) Let p^1 and q^1 be such that $f(p^1)^* f(q^1) < 0$.
(2) Let $r^1 = \frac{(f(q^1)^* p^1 - f(p^1)^* q^1)}{(f(q^1) - f(p^1))}$.
(3) If $f(p^1) * f(r^1) < 0$ then $q^1 = r^1$, if $f(p^1)^* f(q^1) < 0$ set else $p^1 = r^1$.
(4) If more accuracy is required go to step 2.
(5) If $|p^1 - q^1| < h$, h being precision.

3. Improved Exponential Regula Falsi method

In this section, we will initially give some fundamental outcomes so as to propose the strategy

$$f(k) = 0.$$

As we know that basic condition of Regula Falsi for lying a root is $f(p^1)f(q^1) < 0$, there is no compulsion that $f(p^1) < 0$ and $f(q^1) > 0$ it should also be of the form $f(p^1) > 0$ and $f(q^1) < 0$.

The iteration formulae with a restriction under study of [5], [6] is given by

$$K_{m+1} = k_m exp\left\{\frac{-f^2(h_m)}{k_m(s(k_m)f^2(k_m) + f(k_m) - f(k_m - f(k_m)))}\right\}, \tag{3.1}$$

$m = 0, 1,, s \in R, |s| < \infty$.

(3.2)

$$s(k_m) = -\frac{f(k_m - f(k_m))[f(k_m - f(k_m)) + f(k_m + f(k_m)) - 2f(k_m)]}{2[f(k_m) - f(k_m - f(k_m))]f^2(k_m)} - \frac{1}{2k_m}.$$

Following statement shows that this method is cubically convergent.

Give f be a continuously differentiable and $f(\omega) = 0$ be its root. Let $U^1(\omega)$ be an adequately little part of ω to such an extent that $f^{'}(\omega) \neq 0$, $f^{'}(\omega)$ and $f^{''}(\omega)$ occur in $U^1(\omega)$. Then the sequence of iteration develop by iteration equation (3.1) with (3.2) is cubically convergent.

Ibrahim [6] gives result:

$$\lim_{m\to\infty} \frac{h_{m+1}h_m^3}{=} \left[\frac{f^{'''}(\omega)}{6f^{'}(\omega)}(f^{'^2}(\omega) - 1) + \frac{f^{''^2}(\omega)}{4f^{'^2}(\omega)}(1 - f^{'}(\omega)) + \frac{1}{12\omega^2}\right].$$

This shows that iteration (5) with (6) is cubically convergent.

3.1. **Improved method for non linear equations.**

In [2], Wu and Fu present that iteration given by

$$K_{m+1} = k_m - \frac{f^2(k_m)}{sf^2(k_m) + f(k_m) - f(k_m - f(k_m))}, \ldots \tag{3.3}$$

$m = 0, 1, 2,\ |s| < \infty,\ s \in R$ are quadratically convergent. Result is corrected in [3] and given as

$$\lim_{m\to\infty} h_{m+1}/h_m^2 = s + \frac{1}{2}\frac{f^{''}(\omega^*)}{f^{'}(\omega^*)} - \frac{f^{''}(\omega^*)}{2}. \tag{3.4}$$

This shows that the iteration is least cubically convergent under a few suppositions. This motivates the new iteration as follows:

(3.5) $$k_{m+1} = k_m - \frac{f^2(k_m)}{s(k_m)f^2(k_m)+f(k_m)-f(k_m-f(k_m))},$$

(3.6) $$s\left(k_m\right) = -\frac{f(k_m-f(k_m))[f(k_m-f(k_m))+f(k_m+f(k_m))-2f(k_m))}{2[f(k_m)-f(k_m-f(k_m))]f^2(k_m)}$$

$m = 0, 1, 2$. By Theorem 3 in [4] we obtain

$$\lim_{m\to\infty} \frac{h_{m+1}}{h_m^3} = \left[\frac{f'''}{6f'}\left(f'^2 - 1\right) + \frac{f''^2}{4f'^2}\left(1 - f'\right)\right].$$

This iteration is cubically convergent.

Now consider the iteration

(3.7) $$k_{m+1} = k_m - \frac{u_m f^2\left(k_m\right)}{s\left(k_m\right) f^2\left(k_m\right) + f\left(k_m\right) - f\left(k_m - u_m f\left(k_m\right)\right)},$$

$m = 0, 1, 2$. for any (u_m) with a new

(3.8)
$$s\left(k_m\right) = -u_m \frac{f\left(k_m - u_m f\left(k_m\right)\right)\left[f\left(k_m - f\left(k_m\right)\right) + f\left(k_m + f\left(k_m\right)\right) - 2f\left(k_m\right)\right)}{2\left[f\left(K_m\right) - f\left(k_m - f\left(k_m\right)\right)\right]^t\left(k_m\right)}.$$

Now, we accelerate the convergence of CRF method at m^{th} step by using formula

(3.9) $$K_{m+1} = k_m - \frac{s_m\left(q_m^1 - p_m^1\right)\left|f^d\left(k_m\right)\right|}{s\left(k_m\right) f^2\left(k_m\right) + f\left(k_m\right) - f\left(y_m\right)},$$

where $m = 0, 1, 2$ and

$$Ym = p_m^1 - f\left(p_m^1\right) u_m$$
$$u_m = \frac{q_m^1 - p_m^1}{f(q_m^1) - f(p_m^1)} = \frac{q_m^1 - p_m^1}{|f(k_m)|} s_m$$

with $s_{\text{ma}} = \frac{|f(u_m)|}{f(q_m^2)-f(p_m^2)}$.

3.2. **Algorithm for Improved method.** Let $[p_0^1, q_0^1]$ be the basic intervals containing a zero of $(1), \epsilon_1$ and ϵ_2 be the exactness wanted. N is the number of iterations taken by the algorithm to converge to a zero of (1.1) with wanted exactness.

1. [Initialization] $m = 0, k_m = p_m^1$ (or q_m^1)

2. [Regula Falsi Iteration] $ym = p_m^1 - f\left(p_m^1\right) \frac{p_m^1 - q_m^1}{f(p_m^1) - f(q_m^1)}$

3. [Convergence Test] If $|f\left(x_m\right)| \leq \epsilon_1$, then print y_m as a zero of (1) Stop.

4. If $f\left(p_m^1\right) f\left(x_m\right) < 0$ $\overline{p_m^1} = p_m^1, \overline{q_m^1} = y_m$. Else $\overline{p_m^1} = y_m, \overline{q_m^1} = q_m^1$

5. [Improved method] $L_m = k_m - \frac{s_m\left(a_m^1 - p_m^1\right)|f(k_m)|}{s(k_m)f^2(k_m)+f(k_m)-f(y_m)}$

6. If $L_m \in [\overline{p_m^1, \overline{q_m}^1}]$, then $k_{m+1} = L_m$.
If $f(\overline{p_m^1})f\left(L_m\right) < 0$, then $p_{m+1}^1 = \overline{p_m^1}, q_{m+1}^1 = L_m;$
else
$p_{m+1}^1 = L_m, q_{m+1}^1 = \overline{q_m^1}$

7. If $L_n \notin [\overline{p_m^1}, \overline{q_m^1}]$, then $p_{m+1}^1 = \overline{p_m^1}, \quad q_{m+1}^1 = \overline{q_m^1}$
If $L_m < \overline{p_m^1}$, then $k_{m+1} = \overline{p_m^1};$
else
$k_{m+1} = \overline{q_m^1}.$

8. [Convergence Test]
If $|f(k_{m+1})| \leq \epsilon_1$ or $|q_(m+1)^1 - p_(m+1)^1| \leq \epsilon_2$, print k_{m+1} is a zero and stop;
else
increase m, go to step 2. End.

Order of convergence of Newton's Raphson Method

Let us assume that ω is a root of equation $f(k) = 0$. The general formula of N-R method is

$$k_{m+1} = k_m - (f(k_m))/(f'(k_m)). \tag{3.10}$$

If h_m and h_{m+1} are the errors in the approximations k_m and k_{m+1} respectively then

$$h_m = km - \omega \text{ and } h_{m+1} = k_{m+1} - \omega$$

or

$$h_m = \omega + h_m \text{ and } h_{m+1} = \omega + h_{m+1}.$$

Substituting these values in equations (3.10), we have

$$h_{m+1} = h_m - \frac{f(\omega+k_m)}{f'(\omega+k_m)},$$

and expanding $f\left(\omega + h_m\right)$ and $f'\left(\omega + h_m\right)$ using Taylor's series about ω and using $f(\omega) = 0$, we get

$$h_{m+1} = h_m - \left[h_m + \frac{1}{2}h_m^2\frac{f^m(\omega)}{f'(\omega)} + \ldots\right]\left[1 + hm\frac{f''(\omega)}{f'(\omega)} + \ldots\ldots\ldots..\right]^{-1}.$$

Using binomial theorem for any index, we have

$$h_{m+1} = \frac{1}{2}h_m^2\frac{f''(\omega)}{f'(\omega)}$$

and

$$h_{m+1} = Ah_m^2, \text{ where } A = \frac{1}{2}\frac{f^m(\omega)}{f'(\omega)}.$$

It is clear that N-R method is of order 2 [7].

3.3. **Algorithm (working rule).**
Step 1. Set the functions by $f\left(k_m\right), f'\left(k_m\right)$.
Step 2. Let us take the initial approximation of the zero k_m and take $m = 0$.
Step 3. find $k_{m+1} = k_m - \frac{f(k_m)}{f'(k_m)}$.
Step 4. If $|k_{m+1} - k_m| < h, h$ being the precision.
Step 5. If $f\left(k_{m+1}\right) = 0$, at that point k_{m+1} is itself a root of $f(k) = 0$.
If $f\left(k_{m+1}\right) \neq 0$, at that point continue as in step 3

4. NUMERICAL RESULT

In this area we will show the aftereffects of CRF method and IRF strategy. For the numerical examination the accompanying test models have been utilized.

Stopping criteria is $|k_{m+1} - k_m| < 10 - 100$. Here equations are classified as:

$f_1(t) = t^3 - t^2 - 3t - 3 = 0$
$f_2(t) = t^3 - 3t + 4 = 0$
$f_3(t) = t^4 + t^3 - 7t^2 - t + 5 = 0$
$f_4(t) = t^2 - e^t - 3t + 2 = 0$
$f_5(t) = ln(t^2 + t + 2) - t + 1 = 0$
$f_6(t) = t^3 - 10 = 0$
$f_7(t) = t^3 + 4t^2 - 10 = 0$

The following table shows the results at initial interval:

Equation	Initial interval	CRF	IRF	NR ω
t^3- $t^2 - 3t - 3 = 0$	[2]	208	8	62.59
$t^3 - 3t + 4 = 0$	[-2]	208	6	-2.19
t^4+$t^3 - 7t^2 - t + 5 = 0$	[0.9]	207	6	-1.00
$t^2 - e^t - 3t + 2$= 0	[1]	209	7	50.2575
$\ln(t^2 + t + 2) - t + 1 = 0$	[2]	208	5	54.15
$t^3 - 10 = 0,$	[2]	208	6	- 2.15
$t^3 + 4t^2 - 10 = 0$	[1]	207	7	51.36

5. Conclusion

As we observe from the above table that:

(1) from equation 1, 4, 7, we conclude that N-R gives better results.
(2) from equation 2, 3, 6, we conclude that IRF gives results but N-R does not converge.
(3) from equation 5, we conclude that IRF and N-R gives same results.

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