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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF FORCED FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

N. NAGAJOTHI[1] AND V. SADHASIVAM

ABSTRACT. This paper deals with the boundedness of nonoscillatory solutions of forced fractional partial differential equations subject to the Robin and Dirichlet boundary conditions. The technique used in obtaining their results will apply related fractional differential equations with Psi-Hilfer derivative. The main results are illustrated with an example.

1. INTRODUCTION

Fractional calculus has gained importance during the past three decades due to its applicability in diverse fields science and engineering. The origin of fractional calculus traces back to Newton and Leibniz in the seventeenth century. The fractional differential equations find numerous applications in the field of feed back amplifiers, visco-elasticity, electrical circuits, neuron modeling encompassing different branches of physics, fractional multi poles, electro analytical chemistry and biological sciences. It has allowed the operations of differentiation and integration to any fractional order. The order may take on any real or imaginary value. Since the beginning of the fractional calculus, there are numerous definitions of integrals and fractional derivatives, and over time, new

[1]*corresponding author*

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derivatives and fractional integrals arise. These integrals and fractional derivatives have a different kernel and this makes the number of definitions wide, see the references [2, 5, 8, 9, 11, 15, 16, 18–20, 22–25] and those cited there in.

Recently, the research on the theory of fractional partial differential equations is a very interesting topic and some results are established. We refer the articles [14, 21] for fractional partial differential equations.

Results on the oscillatory and asymptotic behavior of solutions of fractional and integro-differential equations are relatively scarce in the literature; some results can be found, for example, in [1, 3, 4, 6, 7, 10, 12, 17] and the references cited therein. Currently there does not appear any such results for forced fractional partial differential equations of type (1.1). Motivated by this gap, we propose the following model, which obviously generalizes the previous models.

Now, we consider the forced fractional partial differential equation of the form

$$ {}_cD^{\alpha,\beta;\psi}_{+,t}y(x,t)+f(t,u(x,t)) = b(t)\Delta u(x,t)+e(x,t), \;\; c>1, \;\; (x,t)\in\Omega\times\mathbb{R}^+ \tag{1.1}$$

where $y(x,t) = \frac{\partial}{\partial t}a(t)\frac{\partial}{\partial t}u(x,t)$, ${}_cD^{\alpha,\beta;\psi}_{+,t}$ is the ψ - Hilfer fractional partial derivative of order α $(0<\alpha<1)$ and type β $(0\le\beta\le1)$. Here Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$ and Δ is the Laplacian operator in the Euclidean N-space $\mathbb{R}^N$.

Eq.(1.1) is supplemented with the boundary condition

$$\frac{\partial u(x,t)}{\partial\gamma} + g(x,t)u(x,t) = 0, \;\; (x,t)\in\partial\Omega\times\mathbb{R}^+, \tag{1.2}$$

where γ is the unit exterior normal vector to $\partial\Omega$ and $g(x,t)$ is non-negative continuous function on $\partial\Omega\times\mathbb{R}^+$ and

$$u(x,t) = 0, \;\; (x,t)\in\partial\Omega\times\mathbb{R}^+. \tag{1.3}$$

In what follows, we always assume without mentioning that

(A_1) $a(t)\in C^{\alpha+1}([c,\infty),\mathbb{R}^+), \mathbb{R}^+ = (0,\infty)$;

(A_2) $b\in C(\mathbb{R}^+,\mathbb{R}^+)$;

(A_3) $f:[c,\infty)\times\mathbb{R}\to\mathbb{R}$ are convex in $(0,\infty)$ and there exist a continuous function $k:[c,\infty)\to(0,\infty)$ and a real number μ with $0<\mu<1$ such that $xf(t,x) > k(t)|x|^{\mu+1}$ for $x\ne0, t\ge c$;

(A_4) $e(x,t)\in C(\bar{G},\mathbb{R})$.

By a solution of (1.1), (1.2) or (1.1), (1.3) we mean a function $u(x,t) \in C^{2+\alpha}(G) \cap C(\bar{G})$ which satisfies (1.1) on G and the associated boundary condition (1.2) (or (1.3)). A nontrivial solution $u(x,t)$ of (1.1), (1.2) is said to be oscillatory in G if it has arbitrarily large zeros, otherwise, it is nonoscillatory. An equation (1.1) is called oscillatory if all its solutions are oscillatory.

In this paper, we begin with some preliminaries in Section 2. In Section 3, we prove the sufficient conditions for every nonoscillatory solution $u(x,t)$ of equations (1.1), (1.2) ((1.1), (1.3)) to be bounded. In Section 4, we present an example that apply the results established. Finally, some conclusions are presented at the end of this article.

Define

$$v(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx,\ where|\Omega| = \int_\Omega dx, \tag{1.4}$$

$$E(t) = \frac{1}{|\Omega|}\int_\Omega e(x,t)dx. \tag{1.5}$$

2. Preliminaries

Definition 2.1. *The left-sided fractional integral of a function g with respect to another function ψ on $[a,b]$ is defined by*

$$(I^{\alpha;\psi}g)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} g(s)ds,\ t > a.$$

Definition 2.2. *Let $\psi'(t) \neq 0$ $(-\infty \leq a < t < b \leq \infty)$ and $\alpha > 0$, $n \in \mathbb{N}$. The Riemann-Liouville fractional derivative of a function g with respect to ψ of order α correspondent to the Riemann-Liouville is defind by*

$$(D^{\alpha;\psi}g)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{n-\alpha-1} g(s)ds,$$

where $n = [\alpha]+1$.

Definition 2.3. *Let $\alpha > 0$, $n \in \mathbb{N}$, $I = [a,b]$ is the interval $(-\infty \leq a < t < b < \infty)$, $g, \psi \in C^n([a,b],\mathbb{R})$ two functions such that ψ is increasing and $\psi' \neq 0$, for all $x \in I$. The left ψ - Caputo derivative of g of order α is given by*

$$(D^{\alpha;\psi}g)(t) = I^{n-\alpha;\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n g(t),$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$ and $\alpha = n$ for $\alpha \in \mathbb{N}$.

Definition 2.4. *The ψ - Hilfer fractional derivative of a function g of order α is given by*

$$(D^{\alpha,\beta;\psi}g)(t) = I^{\beta(1-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)I^{(1-\beta)(1-\alpha);\psi}g(t).$$

The ψ - Hilfer fractional derivatives as above defined, can be written in the following

$$D^{\alpha,\beta;\psi}g(t) = I^{\gamma-\alpha;\psi}D^{\gamma;\psi}g(t).$$

Definition 2.5. *The ψ - Hilfer fractional partial derivative of a function $u(x,t)$ of order α is given by*

$$(D^{\alpha,\beta;\psi}_{+,t}u)(x,t) = I^{\beta(1-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{\partial}{\partial t}\right)I^{(1-\beta)(1-\alpha);\psi}u(x,t).$$

In the definitons above $\Gamma(x)$ is the usual Gamma function given by

$$\Gamma(x) = \int_0^\infty s^{x-1}e^{-s}ds, \quad x > 0.$$

Lemma 2.1. *Let α and p be positive constants such that*

$$p(\alpha - 1) + 1 > 0.$$

Then $\int_0^\infty (t-s)^{p(\alpha-1)}e^{ps}ds \leq Qe^{pt}, \; t \geq 0$, *where*

$$Q = \frac{\Gamma(1 + p(\alpha - 1))}{p^{1+p(\alpha-1)}}.$$

Lemma 2.2. *[13] If X and Y are nonnegative and $0 < \mu < 1$, then*

$$X^\mu - (1-\mu)Y^\mu - \mu XY^{\mu-1} \leq 0,$$

where inequality holds if and only if $X = Y$.

3. Main results

Theorem 3.1. *If $u(x,t)$ is a solution of* (1.1), (1.2) *for which $u(x,t) > 0$ in G, then the function $v(t)$ defined by* (1.4) *satisfy the fractional differential inequality*

$$ {}_cD^{\alpha,\beta;\psi}_{+}Y(t) + f(t, v(t))) \leq E(t). \tag{3.1}$$

Proof. Suppose that $u(x,t)$ is a nonoscillatory solution of (1.1), (1.2). Without loss of generality, we may assume that the solution $u(x,t) > 0$ in $G \times [t_0, \infty)$ for $t \geq t_0$ for some $t_0 \geq c$.
Integrating (1.1) over Ω, we obtain

$$\int_\Omega {}_cD_{+,t}^{\alpha,\beta;\psi} y(x,t)dx + \int_\Omega f(t,u(x,t))dx = \int_\Omega b(t)\Delta u(x,t)dx + \int_\Omega e(x,t)dx. \tag{3.2}$$

Using Green's formula, it is obvious that

$$\int_\Omega \Delta u(x,t)dx = 0,\ t \geq t_1. \tag{3.3}$$

By applying, Jensen's inequality, we have

$$\int_\Omega f(t,u(x,t))dx \geq f(t, \int_\Omega u(x,t)dx) \geq f(t,v(t)). \tag{3.4}$$

Combining (3.2)-(3.4) and using (1.5), we get

$${}_cD_{+}^{\alpha,\beta;\psi} Y(t) + f(t,v(t))) \leq E(t). \tag{3.5}$$

□

The above equation is equivalent to the nonlinear Volterra type integral equation

$$Y(t) \leq c_0 + \frac{1}{\Gamma(\alpha)} \int_c^t \psi'(\xi)(\psi(t) - \psi(\xi))^{\alpha-1}[E(\xi) - f(\xi, v(\xi))]d\xi, c > 1,$$

where $\alpha > 0$. By taking the limit as $\beta \to 1$ and choosing $\psi(t) = t$, the equation (3.5) reduces to usual Caputo fractional differential equations. So our newly obtained oscillation criteria can be applied to those class of Caputo fractional differential equations [12] also and in addition to that for a different choices of ψ a wider class of differential equations can be covered.

Theorem 3.2. *Let us assume the Conditions $(A_1) - (A_4)$ hold and suppose that $\psi'(\xi) \geq \lambda$ for some $\lambda > 0$ and for all $\xi \neq 0$. Also assume that there exist real number $p > 1$ and $0 < \alpha < 1$ such that $p(\alpha - 1) + 1 > 0$, there are numbers $S > 0$ and $\sigma > 1$ such that*

$$\frac{\psi(t)}{a(t)} \leq Se^{-\sigma t} \tag{3.6}$$

and there exists a continuous function $\eta : [c, \infty) \to (0, \infty)$ such that

$$\int_c^\infty e^{-q\xi}\eta^q(\xi)d\xi < \infty,\ where\ q = \frac{p}{p-1}. \tag{3.7}$$

If

$$\limsup_{t\to\infty} \frac{1}{t}\int_c^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}E(\xi)d\xi < \infty, \tag{3.8}$$

$$\liminf_{t\to\infty} \frac{1}{t}\int_c^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}E(\xi)d\xi > -\infty, \tag{3.9}$$

where

$$\lim_{t\to\infty} \frac{1}{t}\int_c^t \int_{t_1}^{\zeta} \psi'(\xi)(\psi(\zeta)-\psi(\xi))^{\alpha-1}H_\mu(\xi)d\xi d\zeta < \infty, \tag{3.10}$$

then any non-oscillatory solution $u(x,t)$ *of* (1.1), (1.2) *are bounded.*

Proof. Let us suppose that $v(t)$ be a non-oscillatory solution of (3.5). We may assume that $v(t) > 0$ for $t \geq t_1$ for some $t_1 > c$. We let $F(t) = f(t, v(t))$ and we use $(A_1) - (A_4)$. We see that the equation (3.5) can be written as

$$\begin{aligned}(a(t)v'(t))' \leq c_0 &+ \frac{1}{\Gamma(\alpha)}\int_c^{t_1} \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}|E(\xi)|d\xi \\ &- \frac{1}{\Gamma(\alpha)}\int_c^{t_1} \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}|F(\xi)|d\xi \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t} \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}E(\xi)d\xi \\ &- \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t} \psi'(\xi)[k(\xi)v^{\mu}(\xi)-\eta(\xi)v(\xi)](\psi(t)-\psi(\xi))^{\alpha-1}d\xi \\ &- \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t} \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}\eta(\xi)v(\xi)d\xi.\end{aligned} \tag{3.11}$$

Using the fact that $(\psi(t)-\psi(\xi))^{\alpha-1} \leq (\psi(t_1)-\psi(\xi))^{\alpha-1}$ in the first and second integrals in (3.11), we get

$$\begin{aligned}(a(t)v'(t))' &\leq c_1 + \tfrac{1}{\Gamma(\alpha)}\textstyle\int_{t_1}^{t} \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}E(\xi)d\xi \\ &-\tfrac{1}{\Gamma(\alpha)}\textstyle\int_{t_1}^{t} \psi'(\xi)[k(\xi)v^{\mu}(\xi)-\eta(\xi)v(\xi)](\psi(t)-\psi(\xi))^{\alpha-1}d\xi \\ &-\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t} \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}\eta(\xi)v(\xi)d\xi,\end{aligned} \tag{3.12}$$

where $c_1 = c_0 + \frac{1}{\Gamma(\alpha)}\int_c^{t_1} \psi'(\xi)(\psi(t_1)-\psi(\xi))^{\alpha-1}|E(\xi)|d\xi$

$$-\frac{1}{\Gamma(\alpha)}\int_c^{t_1} \psi'(\xi)(\psi(t_1)-\psi(\xi))^{\alpha-1}|F(\xi)|d\xi.$$

Applying Lemma 2.2, with

$$X = k^{\frac{1}{\mu}}(\xi)v(\xi), \quad Y = \left(\frac{1}{\mu}\eta(\xi)k^{\frac{-1}{\mu}}(\xi)\right)^{\frac{1}{\mu-1}},$$

we obtain $k(\xi)v^{\mu}(\xi) - \eta(\xi)v(\xi) \leq (1-\mu)\eta^{\frac{\mu}{\mu-1}}(\xi)k^{\frac{1}{1-\mu}}(\xi)\mu^{\frac{\mu}{1-\mu}} := H_{\mu}(\xi)$.
and substituting this into (3.11), we have

$$\begin{aligned}(a(t)v'(t))' \leq c_1 + &\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}E(\xi)d\xi \\ &- \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}H_{\mu}(\xi)d\xi \\ &- \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}\eta(\xi)v(\xi)d\xi.\end{aligned} \tag{3.13}$$

An integrating of (3.12) from t_1 to t, we have

$$\begin{aligned}a(t)v'(t) \leq & a(t_1)v'(t_1) + c_1(t-t_1) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\int_{t_1}^{\zeta}\psi'(\xi)(\psi(\zeta)-\psi(\xi))^{\alpha-1}E(\xi)d\xi d\zeta \\ & - \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\int_{t_1}^{\zeta}\psi'(\xi)(\psi(\zeta)-\psi(\xi))^{\alpha-1}H_{\mu}(\xi)d\xi d\zeta \\ & - \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\int_{t_1}^{\zeta}\psi'(\xi)(\psi(\zeta)-\psi(\xi))^{\alpha-1}\eta(\xi)v(\xi)d\xi d\zeta \\ \leq & a(t_1)v'(t_1) + c_1(t-t_1) - \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\int_{t_1}^{\zeta}\psi'(\xi)(\psi(\zeta)-\psi(\xi))^{\alpha-1}H_{\mu}(\xi)d\xi d\zeta \\ & + \frac{1}{\Gamma(\alpha+1)}\int_{t_1}^{t}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha}E(\xi)d\xi \\ & - \frac{\psi(t)}{\Gamma(\alpha+1)}\int_{t_1}^{t}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}\eta(\xi)v(\xi)d\xi.\end{aligned}$$

In view of (3.8)-(3.10), the last inequality implies

$$a(t)v'(t) \leq c_2 + c_3 t - \frac{\psi(t)}{\Gamma(\alpha+1)}\int_{t_1}^{t}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}\eta(\xi)v(\xi)d\xi$$

for some positive constants c_2 and c_3. Integrating (3.13) from t_1 to t and noting condition (3.6), we see that

$$\begin{aligned} v(t) \leq & v(t_1) + c_2 \int_{t_1}^{t} \frac{1}{a(\xi)} d\xi + c_3 \int_{t_1}^{t} \frac{\xi}{a(\xi)} d\xi \\ & - \frac{1}{\Gamma(\alpha+1)} \int_{t_1}^{t} \frac{\psi'(\xi)\psi(\zeta)}{a(\zeta)} \int_{t_1}^{\zeta} (\psi(\zeta)-\psi(\xi))^{\alpha-1} \eta(\xi) v(\xi) d\xi d\zeta \\ & \leq c_4 - \frac{1}{\Gamma(\alpha+1)} \int_{t_1}^{t} \frac{\psi'(\xi)\psi(\zeta)}{a(\zeta)} \int_{t_1}^{\zeta} (\psi(\zeta)-\psi(\xi))^{\alpha-1} \eta(\xi) v(\xi) d\xi d\zeta, \end{aligned}$$

for some constants $c_4 > 0$. From the mean value theorem,

$$v(t) \leq c_4 + \frac{\lambda^{\alpha}}{\Gamma(\alpha+1)} \int_{t_1}^{t} \frac{\psi(\zeta)}{a(\zeta)} \int_{t_1}^{\zeta} (\zeta-\xi)^{\alpha-1} \eta(\xi) v(\xi) d\xi d\zeta \tag{3.14}$$

Applying Hölder's inequality and Lemma 2.1, we obtain

$$\begin{aligned} \int_{t_1}^{\zeta} ((\zeta-\xi)^{\alpha-1} e^{\xi})(e^{-\xi}\eta(\xi)v(\xi)) d\xi & \leq \left(\int_{t_1}^{\zeta} (\zeta-\xi)^{p(\alpha-1)} e^{p\xi} d\xi\right)^{\frac{1}{p}} \left(\int_{t_1}^{\zeta} e^{-q\xi} \eta^q(\xi) v^q(\xi) d\xi\right)^{\frac{1}{q}} \\ & \leq \left(\int_{0}^{\zeta} (\zeta-\xi)^{p(\alpha-1)} e^{p\xi} d\xi\right)^{\frac{1}{p}} \left(\int_{t_1}^{\zeta} e^{-q\xi} \eta^q(\xi) v^q(\xi) d\xi\right)^{\frac{1}{q}} \\ & \leq \left(Q e^{p\zeta}\right) \left(\int_{t_1}^{\zeta} e^{-q\xi} \eta^q(\xi) v^q(\xi) d\xi\right)^{\frac{1}{q}}. \end{aligned} \tag{3.15}$$

From (3.6), (3.14) and (3.15),

$$\begin{aligned} v(t) \leq & c_4 + \frac{Q^{\frac{1}{p}} \lambda^{\alpha}}{\Gamma(\alpha+1)} \int_{t_1}^{t} \frac{\psi(\zeta) e^{\zeta}}{a(\zeta)} \left(\int_{t_1}^{\zeta} e^{-q\xi} \eta^q(\xi) v^q(\xi) d\xi\right)^{\frac{1}{q}} d\zeta \\ & \leq c_4 + \frac{Q^{\frac{1}{p}} \lambda^{\alpha} S}{\Gamma(\alpha+1)} \int_{t_1}^{t} e^{-(\sigma-1)\zeta} \left(\int_{t_1}^{\zeta} e^{-q\xi} \eta^q(\xi) v^q(\xi) d\xi\right)^{\frac{1}{q}} d\zeta. \end{aligned} \tag{3.16}$$

Since $\sigma > 1$ and the integral on the far right in (3.16) is increasing, we obtain the estimate

$$v(t) \leq 1 + c_4 + K \left(\int_{t_1}^{\zeta} e^{-q\xi} \eta^q(\xi) v^q(\xi) d\xi\right)^{\frac{1}{q}} \tag{3.17}$$

where $K = \frac{Q^{\frac{1}{p}} S \lambda^{\alpha}}{(\sigma-1)\Gamma(\alpha+1)}$.

Applying the following inequality:
$(x+y)^q \le 2^{q-1}(x^q+y^q)$ for $x,y\ge 0$ and $q>1$, to (3.17) gives:

$$v^q(t) \le 2^{q-1}(1+c_4)^q + 2^{q-1}K^q\left(\int_{t_1}^{\zeta} e^{-q\xi}\eta^q(\xi)v^q(\xi)d\xi\right).$$

Setting $A = 2^{q-1}(1+c_4)^q$, $B = 2^{q-1}K^q$ and $W(t) = v^q(t)$ so that $v(t) = W^{\frac{1}{q}}(t)$, equation (3.17) becomes

$$W(t) \le A + B\left(\int_{t_1}^{\zeta} e^{-q\xi}\eta^q(\xi)W(\xi)d\xi\right)$$

for $t \ge t_1$. By Grounwall's inequality and condition (3.7), we see that $W(t)$ is bounded, and so $v(t)$ is bounded. Clearly, a similar argument holds if $v(t)$ is an eventually negative solution of (1.1), (1.2). □

Next, we consider the forced fractional partial differential equation
(3.18)

$${}_cD_{+,t}^{\alpha,\beta;\psi}y(x,t) + f(t,u(x,t)) = b(t)\Delta u(x,t) + e(x,t),\ \ c>1,\ \ (x,t)\in\Omega\times\mathbb{R}^+ = G,$$

where $y(x,t) = a(t)\frac{\partial}{\partial t}u(x,t)$. We now give sufficient conditions under which any non-oscillatory solution $u(x,t)$ of (3.18), (1.2) is bounded.

Theorem 3.3. *If $u(x,t)$ is a solution of* (3.18), (1.2) *for which $u(x,t) > 0$ in G, then the function $v(t)$ defined by* (1.4) *satisfies the fractional differential inequality*

$$(3.19)\qquad {}_cD_{+}^{\alpha,\beta;\psi}Y(t) + f(t,v(t))) \le E(t).$$

Proof. This proof is the same as that of Theorem 3.1 and hence is omitted. □

Theorem 3.4. *Let Conditions $(A_1)-(A_4)$ hold and assume that $\psi'(\xi) \ge \lambda$ for some $\lambda > 0$ and for all $\xi \ne 0$. Also assume that there exist real number $p > 1$ and $0 < \alpha < 1$ such that $p(\alpha-1)+1 > 0$. Suppose that there exists a continuous function $\eta : [c,\infty) \to (0,\infty)$ such that* (3.7) *holds and*

$$\frac{1}{a(t)} \le Se^{-\sigma t}$$

for some $S > 0$ and $\sigma > 1$. If

$$\limsup_{t\to\infty}\int_c^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}E(\xi)d\xi < \infty,$$

$$\liminf_{t\to\infty}\int_c^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}E(\xi)d\xi > -\infty,$$

$$\limsup_{t\to\infty}\int_c^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}H_\mu(\xi)d\xi<\infty,$$

then any nonoscillatory solution $u(x,t)$ *of* (3.18)*,* (1.2) *is bounded.*

Proof. Suppose that $v(t)$ be a nonoscillatory solution of (3.19). We may assume that $v(t)>0$ for $t\geq t_1$ for some $t_1>c$. We let $F(t)=f(t,v(t))$ and we use $(A_1)-(A_4)$. We see that the equation (3.19) can be written as

$$\begin{aligned}
a(t)v'(t) &\leq c_0+\frac{1}{\Gamma(\alpha)}\int_c^{t_1}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}|E(\xi)|d\xi\\
&\quad-\frac{1}{\Gamma(\alpha)}\int_c^{t_1}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}|F(\xi)|d\xi\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}E(\xi)d\xi\\
&\quad-\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\psi'(\xi)[k(\xi)v^\mu(\xi)-\eta(\xi)v(\xi)](\psi(t)-\psi(\xi))^{\alpha-1}d\xi\\
&\quad-\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}\eta(\xi)v(\xi)d\xi\\
&\leq M-\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}\eta(\xi)v(\xi)d\xi,
\end{aligned}\tag{3.20}$$

for some positive constant M. An integration of (3.20) from t_1 to t yields

$$v(t)\leq v(t_1)-M\int_{t_1}^{t}\frac{1}{a(\xi)}d\xi-\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\frac{1}{a(\zeta)}\int_{t_1}^{\zeta}\psi'(\xi)(\psi(\zeta)-\psi(\xi))^{\alpha-1}\eta(\xi)v(\xi)d\xi d\zeta.$$

The rest part of the proof is similar to that of Theorem 3.2 and hence is omitted. □

Similar reasoning to that used in the sublinear case guarantees the following theorems for the integro-differential equations (1.1), (1.2) and (3.18), (1.2) in case $\mu=1$.

Theorem 3.5. *Let* $\mu=1$ *and the hypotheses of Theorem 3.2 and Theorem 3.4 hold with* $m(t)=k(t)$*. Then the conclusion of Theorems 3.2 and Theorem 3.4 holds.*

Next, we establish sufficient conditions under which any non-oscillatory solution $u(x,t)$ of (3.18), (1.3) is bounded. For this we need the following:
The smallest eigen value β_0 of the Dirichlet problem

$$\Delta\omega(x)+\beta\omega(x)=0 \;\; in \;\; \Omega$$

$$\omega(x) = 0 \ \ on \ \ \partial\Omega,$$

is positive and the corresponding eigen function $\phi(x)$ is positive in Ω.

Theorem 3.6. *Let all the conditions of Theorem 3.4 hold. Then any non-oscillatory solution $u(x,t)$ of* (3.18)*,* (1.3) *is bounded.*

On the other hand, one can deduce a wider class of fractional partial differential equations by choosing various function for ψ and taking the limit of the parameter α and β. Now, we deduce some new results for the class of Katugampola and Hadamard fractional partial differential equations and state them as following Corollaries.

Let $\psi(t) = t^{\rho}$ and taking the limit $\beta \to 0$, then the equation (1.1) reduces to the Katugampola fractional partial differential equation of the form

(3.21)

$$_{c}D^{\alpha,\beta;t^{\rho}}_{+,t} y(x,t) + f(t,u(x,t)) = b(t)\Delta u(x,t) + e(x,t), \ \ c > 1, \ \ (x,t) \in \Omega \times \mathbb{R}^{+} = G.$$

together with the boundary condition (1.2). After reducing the multi dimensional problem to one dimensional problem, (3.21) reduces the following fractional differential inequality of the form

$$_{c}D^{\alpha,\beta;t^{\rho}}_{+} Y(t) + f(t,v(t))) \leq E(t).$$

It's equivalent to the nonlinear Voltera type integral equation

$$Y(t) \leq c_0 + \frac{\rho}{\Gamma(\alpha)} \int_c^t \xi^{\rho-1}(t^{\rho} - \xi^{\rho})^{\alpha-1}[E(\xi) - f(\xi, v(\xi))]d\xi, \ \ c > 1, \ \ \alpha > 0.$$

Corollary 3.1. *Let Conditions $(A_1) - (A_4)$ hold and assume that $\xi^{\rho-1} \geq \frac{\lambda}{\rho}$ for some $\lambda > 0$ and for all $\xi, \rho \neq 0$. Also assume that there exist real number $p > 1$ and $0 < \alpha < 1$ such that $p(\alpha - 1) + 1 > 0$, there are numbers $S > 0$ and $\sigma > 1$ such that*

$$\frac{t^{\rho}}{a(t)} \leq Se^{-\sigma t}$$

and the condition (3.7) *of Theorem 3.2 holds. If*

$$\limsup_{t\to\infty} \frac{\rho}{t} \int_c^t \xi^{\rho-1}(t^{\rho} - \xi^{\rho})^{\alpha-1}E(\xi)d\xi < \infty,$$

$$\liminf_{t\to\infty} \frac{\rho}{t} \int_c^t \xi^{\rho-1}(t^{\rho} - \xi^{\rho})^{\alpha-1}E(\xi)d\xi > -\infty,$$

$$\lim_{t\to\infty} \frac{\rho}{t} \int_c^t \int_{t_1}^{\zeta} \xi^{\rho-1}(t^{\rho} - \xi^{\rho})^{\alpha-1}H_{\mu}(\xi)d\xi d\zeta < \infty,$$

then any non-oscillatory solution $u(x,t)$ *of* (3.18)*,* (1.2) *are bounded.*

Next, we consider the forced fractional partial differential equation

$$ {}_{c}D_{+,t}^{\alpha,\beta;t^\rho} y(x,t) + f(t,u(x,t)) = b(t)\Delta u(x,t) + e(x,t), \;\; c>1, \;\; (x,t)\in\Omega\times\mathbb{R}^+ = G, \tag{3.22}$$

where $y(x,t) = a(t)\frac{\partial}{\partial t}u(x,t)$. After reducing the multi dimensional problem to one dimensional problem, we obtain the following fractional differential inequality

$$ {}_{c}D_{+}^{\alpha,\beta;t^\rho} Y(t) + f(t,v(t))) \le E(t). $$

Corollary 3.2. *Let Conditions* $(A_1)-(A_4)$ *hold and assume that* $\xi^{\rho-1} \ge \frac{\lambda}{\rho}$ *for some* $\lambda > 0$ *and for all* $\xi, \rho \neq 0$*. Also assume that there exist real number* $p>1$ *and* $0<\alpha<1$ *such that* $p(\alpha-1)+1>0$*. Suppose that there exists a continuous function* $\eta:[c,\infty)\to(0,\infty)$ *such that* (3.7) *holds and*

$$ \frac{1}{a(t)} \le Se^{-\sigma t} \tag{3.23}$$

for some $S>0$ *and* $\sigma>1$*. If*

$$ \limsup_{t\to\infty} \rho\int_c^t \xi^{\rho-1}(t^\rho-\xi^\rho)^{\alpha-1}E(\xi)d\xi < \infty, \tag{3.24}$$

$$ \liminf_{t\to\infty} \rho\int_c^t \xi^{\rho-1}(t^\rho-\xi^\rho)^{\alpha-1}E(\xi)d\xi > -\infty, $$

$$ \limsup_{t\to\infty} \rho\int_c^t \xi^{\rho-1}(t^\rho-\xi^\rho)^{\alpha-1}H_\mu(\xi)d\xi < \infty, \tag{3.25}$$

then any non-oscillatory solution $u(x,t)$ *of* (3.22)*,* (1.2) *is bounded.*

Let $\psi(t) = \ln(t)$ and taking the limit $\beta\to 0$, then the equation (1.1) reduces to the Hadamard fractional partial differential equation of the following form

$$ {}_{c}D_{+,t}^{\alpha,\beta;\ln(t)} y(x,t) + f(t,u(x,t)) = b(t)\Delta u(x,t) + e(x,t), \;\; c>1, \;\; (x,t)\in\Omega\times\mathbb{R}^+ = G. \tag{3.26}$$

together with the boundary condition (1.2). After reducing the multi dimensional problem to one dimensional problem, (3.26) reduces the following fractional partial inequality of the form

$$ {}_{c}D_{+}^{\alpha,\beta;\ln(t)} Y(t) + f(t,v(t))) \le E(t). $$

It's equivalent to the nonlinear Voltera type integral equation is

$$Y(t) \leq c_0 + \frac{1}{\Gamma(\alpha)} \int_c^t (\ln(t) - \ln(\xi))^{\alpha-1}[E(\xi) - f(s, v(\xi))]\frac{d\xi}{\xi}, \;\; c > 1, \;\; \alpha > 0.$$

Corollary 3.3. *Let Conditions* $(A_1) - (A_4)$ *hold and assume that* $\frac{1}{\xi} \geq \lambda$ *for some* $\lambda > 0$ *and for all* $\xi \neq 0$*. Also assume that there exist real number* $p > 1$ *and* $0 < \alpha < 1$ *such that* $p(\alpha - 1) + 1 > 0$*, there are numbers* $S > 0$ *and* $\sigma > 1$ *such that*

$$\frac{\ln(t)}{a(t)} \leq Se^{-\sigma t} \tag{3.27}$$

and the condition (3.7) *of Theorem 3.2 holds. If*

$$\limsup_{t\to\infty} \frac{1}{t} \int_c^t (\ln(t) - \ln(\xi))^{\alpha-1} E(\xi)\frac{d\xi}{\xi} < \infty,$$
$$\liminf_{t\to\infty} \frac{1}{t} \int_c^t (\ln(t) - \ln(\xi))^{\alpha-1} E(\xi)\frac{d\xi}{\xi} > -\infty,$$
$$\lim_{t\to\infty} \frac{1}{t} \int_c^t \int_{t_1}^{\zeta} (\ln(t) - \ln(\xi))^{\alpha-1} H_\mu(\xi)\frac{d\xi}{\xi} d\zeta < \infty,$$

then any nonoscillatory solution $u(x,t)$ *of* (3.26)*,* (1.2) *is bounded.*

Next, we consider the forced fractional partial differential equation

$$ {}_cD^{\alpha,\beta;\ln(t)}_{+,t} y(x,t) + f(t, u(x,t)) = b(t)\Delta u(x,t) + e(x,t), \;\; c > 1, \;\; (x,t) \in \Omega\times\mathbb{R}^+ = G, \tag{3.28}$$

where $y(x,t) = a(t)\frac{\partial}{\partial t}u(x,t)$. After reducing the multi dimensional problem to one dimensional problem, we obtain the following fractional differential inequality

$$ {}_cD^{\alpha,\beta;\ln(t)}_{+} Y(t) + f(t, v(t))) \leq E(t).$$

Corollary 3.4. *Let the conditions* $(A_1) - (A_4)$ *hold and assume that* $\frac{1}{\xi} \geq \lambda$ *for some* $\lambda > 0$ *and for all* $\xi \neq 0$*. Also assume that there exist real number* $p > 1$ *and* $0 < \alpha < 1$ *such that* $p(\alpha - 1) + 1 > 0$*. Suppose that there exists a continuous function* $\eta : [c, \infty) \to (0, \infty)$ *such that* (3.7) *holds and*

$$\frac{1}{a(t)} \leq Se^{-\sigma t}$$

for some $S > 0$ *and* $\sigma > 1$*. If*

$$\limsup_{t\to\infty} \int_c^t (\ln(t) - \ln(\xi))^{\alpha-1} E(\xi)\frac{d\xi}{\xi} < \infty,$$

$$\liminf_{t\to\infty}\int_c^t(\ln(t)-\ln(\xi))^{\alpha-1}E(\xi)\frac{d\xi}{\xi}>-\infty,$$

$$\limsup_{t\to\infty}\int_c^t(\ln(t)-\ln(\xi))^{\alpha-1}H_\mu(\xi)\frac{d\xi}{\xi}<\infty,$$

then any nonoscillatory solution $u(x,t)$ of (3.28)*,* (1.2) *is bounded.*

4. Examples

Example 1. *Consider the Katugampola fractional partial differential equation of the form* ${}_cD^{\alpha,\beta;t^\rho}_{+,t}y(x,t)+k(t)|u(x,t)|^{\mu-1}u(x,t)$

(4.1) $$=t\Delta u(x,t)+e^{-(4t)^\rho},\;\; c>1,(x,t)\in\Omega\times\mathbb{R}^+=G,$$

which satisfies the boundary condition (1.2)*, then the corresponding nonlinear Volterra integral equation is*

$$Y(t)\le c_0+\frac{\rho}{\Gamma(\alpha)}\int_c^t\xi^{\rho-1}(t^\rho-\xi^\rho)^{-\frac{1}{2}}[e^-(4t)^\rho-k(t)|v(t)|^{\mu-1}v(t)]d\xi,\;\; c>1,\;\;\alpha>0.$$

Here $b(t)=t, f(t,v(t))=k(t)|v(t)|^{\mu-1}v(t), a(t)=\frac{e^{4t}}{S}, S>0, k(t)=e^{(-2t)^\rho}, E(t)=e(^{-4t})^\rho, \alpha=\frac{1}{2}, p=\frac{3}{2}>1$. *Then* $q=\frac{p}{p-1}=3$ *and* $p(\alpha-1)+1=\frac{1}{4}>0,\sigma=4,c=m_o=4^\rho$ *and* $k(t)=\eta(t)$ *thus the conditions* (3.23) *and* (3.7) *become*

$$\frac{1}{a(t)}=\frac{S}{e^{4t}}\le Se^{-4t}$$

and

$$\int_c^t e^{-q\xi}\eta^q(\xi)d\xi=\int_{m_o}^t e^{-3s}e(^{-2\xi})^{3\rho}d\xi=\frac{e^{3m_0}(1+2\rho)}{3(1+2\rho)}<\infty.$$

With $k(t)=\eta(t)$*, we have*

$$\int_{m_0}^t\rho\xi^{\rho-1}(t^\rho-\xi^\rho)^{\alpha-1}\eta(\xi)(1-\mu)\mu^{\frac{\mu}{1-\mu}}d\xi=\rho(1-\mu)\mu^{\frac{\mu}{1-\mu}}\int_{m_0}^t\xi^{\rho-1}(t^\rho-\xi^\rho)^{-\frac{1}{2}}e^{(-2\xi)^\rho}d\xi.$$

Letting $\zeta=t^\rho-\xi^\rho+4^\rho$*, we get*

$$\begin{aligned}
&\int_{m_0}^{t} \xi^{\rho-1}(t^\rho - \xi^\rho)^{-\frac{1}{2}} e^{(-2\xi)^\rho} d\xi = \int_{t^\rho}^{4^\rho} (\zeta - 4^\rho)^{\alpha-1} e^{-(2\zeta)^\rho} (\frac{-d\zeta}{\rho}) \\
&= \frac{1}{\rho} \int_{4^\rho}^{t^\rho} (\zeta - 4^\rho)^{-\frac{1}{2}} e^{-2^\rho(t^\rho + 4^\rho - \zeta)} d\zeta \\
&= \frac{1}{\rho e^{2^\rho(t^\rho+4^\rho)}} \int_{4^\rho}^{8^\rho} (\zeta - 4^\rho)^{-\frac{1}{2}} e^{2^\rho \zeta} d\zeta + \frac{1}{\rho e^{2^\rho(t^\rho+4^\rho)}} \int_{8^\rho}^{t^\rho} (\zeta - 4^\rho)^{-\frac{1}{2}} e^{2^\rho \zeta} d\zeta \\
&= 2e^{(3.5)^\rho} (8^\rho - 4^\rho)^{\frac{1}{2}} + \frac{4^{-\frac{\rho}{2}}}{2^\rho} (e^{2^\rho(t^\rho - 8^\rho)}).
\end{aligned}$$

So (3.25) *holds. Finally,*

$$\int_{m_0}^{t} \xi^{\rho-1}(t^\rho - \xi^\rho)^{-\frac{1}{2}} E(\xi) d\xi = \int_{m_0}^{t} \xi^{\rho-1}(t^\rho - \xi^\rho)^{-\frac{1}{2}} e^{(-4\xi)^\rho} d\xi < \infty.$$

So (3.24) *satisfied. Hence by Corollary 3.2, every nonoscillatory solution u(x,t) with the boundary condition* (1.2) *of the equation* (4.1) *is bounded.*

Remark. By taking $\psi(t) = \ln t$ and taking the $\lim \beta \to 0$, we get a another class of Hadamard fractional partial differential equations of the form
${}_cD^{\alpha,\beta;\ln t}_{+,t} y(x,t) + k(t)|u(x,t)|^{\mu-1}u(x,t)$

$$= t^2 \Delta u(x,t) + \frac{(lnt - \ln \xi)^{\frac{1}{4}}}{\xi}, \;\; c > 1, (x,t) \in \Omega \times \mathbb{R}^+ = G,$$

with the boundary condition (1.2). Let $\alpha = \frac{3}{4}, p = \frac{5}{4}, q = 5, k(t) = \eta(t) = e^{-t}, c = 2, f(t, x(t)) = k(t)|v(t)|^{\mu-1}v(t), a(t) = \frac{e^{2t}}{S}, S > 0, b(t) = t^2$ and one can obtain the similar conclusion by verifying the conditions as stated as in the Corollary 3.2.

5. Conclusion

In this article, we have obtained some new sufficient conditions for the boundedness of nonoscillatory solutions of ψ - Hilfer fractional partial differential equations which extend, generalize and give a broad outlook of known results in the existing literature.

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POST GRADUATE AND RESEARCH DEPARTMENT OF MATHEMATICS
THIRUVALLUVAR GOVERNMENT ARTS COLLEGE (AFFLI. TO PERIYAR UNIVERSITY)
RASIPURAM - 637 401, NAMAKKAL DT.
TAMIL NADU, INDIA.
Email address: nagajothi006@gmail.com,ovsadha@gmail.com

POST GRADUATE AND RESEARCH DEPARTMENT OF MATHEMATICS
THIRUVALLUVAR GOVERNMENT ARTS COLLEGE (AFFLI. TO PERIYAR UNIVERSITY)
RASIPURAM - 637 401, NAMAKKAL DT.
TAMIL NADU, INDIA.