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COMPARATIVE ANALYSIS OF FUZZY DELAY DIFFERENTIAL EQUATION

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ABSTRACT. The article focuses to comparing the fuzzy delay differential equation in a fuzzy environment using a fourth order Runge -Kutta method and Adomain Decomposition method. In this paper formulation of the problem is provided. The numerical illustration comparison result is shown by solving a problem with a fuzzy initial value with a trapezoidal fuzzy number. The result obtained is compared with the existing result to point out the conclusion.

1. INTRODUCTION

Much importance has been added in recent years to the role of uncertainity in mathematical biology (fuzzy, interval etc). The fuzzy differential equations appliance is an important way of modeling a dynamic system under potential Zadeh [13] uncertainity. Chang et al [1] first introduced the concept of a fuzzy derivative, followed by Dubois, Prede [2] who established the extension principle and used it. The fuzzy differential equation and the initial value problem where Kaleaetal [4, 5] is routinely viewed. Both derivatives are deliberated in complicated differential equations as either Hukuhara, or generalized derivatives. Using the Standard Euler method, et al [6] establishes the numerical approach used in solving fuzzy differential equations. S.Sindu Devi and K.Ganesan obtained an exact solution using Runge Kutta method. Many researchers have

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suggested various techniques for obtaining the fuzzy differential equation solution [7, 10, 12].

Delay differential equations are used to describe physical phenomena, but they are unique. Fuzzy differential equation plays an important role in a growing number of system models in biology, engineering, physics, and other sciences. Linear fuzzy first-order differential equations are one of the simplest fuzzy differential equations which may occur in many applications. Fuzzy differential equations under certainity and vagueness provide a proper way to modeling dynamical systems. The word Fuzzy Delay Differential Equations is used to refer to Differential Equations with Fuzzy Coefficients, Delay Differential Equations with Fuzzy Initial Values or Fuzzy Boundary Values, which discuss functions in Fuzzy Interval Space.

The paper is structured as follows: Section 2, reviews some basic definitions of fuzzy numbers. Section 3 introduces problem description and process analysis and offers a fitting numerical illustration as well. The paper ends with section 4.

2. Preliminaries

Definition 2.1. *A fuzzy set is an extension of a classical set, whose elements have a degree of membership (i.e) it enables the gradual evaluation of element membership in a set. If U is the universal set, then a fuzzy set $\tilde{T}$ in U is defined as a set of ordered pairs,*

$$\widetilde{T} = [(x, \mu_{\widetilde{T}}(x))/x \in U],$$

where $\mu_{\widetilde{T}}(x)$ is the membership function of x in $\mu_{\widetilde{T}} : U \to [0, 1]$

Definition 2.2. *A fuzzy number is a fuzzy set on the real line that satisfies the conditions of normality and convexity. A fuzzy number is a mapping such that $\mu_{\tilde{T}} : R \to [0, 1]$.*

Definition 2.3. *A fuzzy set $\tilde{T}$ is called normal if there is at least one point x ∈ R with $\mu_{\tilde{T}}(x) = 1$.*

Definition 2.4. *A fuzzy set $\tilde{T}$ of R is convex if for any$x, y \in R$ and $\gamma \in [0, 1]$, we have $\mu_{\widetilde{T}}(\gamma x + (1 - \gamma)y) \geq \min\{\mu_{\widetilde{T}}(x), \mu_{\widetilde{T}}(y)\}$.*

Definition 2.5. *An α-cut of fuzzy set F is a crisp set T is a crisp set α_T or α_{+T} that contains all the elements of the universal set U that have a membership grade in T greater than or equal to the specified value of α,*

$$\alpha_T = \{x/T(x) \geq \alpha\},$$

$$\alpha_{+T} = \{x/T(x) > \alpha\}.$$

Definition 2.6. *A trapezoidal fuzzy number is represented by a quadruplet $\widetilde{T} = (a_1, a_2, a_3, a_4)$, where $a_1, a_2, a_3, a_4 \in R$, with the membership function,*

$$\mu_{\tilde{T}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, a_1 < x < a_2 \\ 1, a_2 < x < a_3 \\ \frac{a_4-x}{a_4-a_3}, a_3 < x < a_4 \\ 0, otherwise \end{cases}.$$

Definition 2.7. *Let $\widetilde{T} = (a_1, a_2, a_3, a_4)$,be a trapezoidal fuzzy number. $\widetilde{T}$is said to be non negative if $a_i \geq 0$, i = 1,2,3,4 and $\widetilde{T}$ is said to be non positive if $a_i \leq 0$ i = 1,2,3,4*

Definition 2.8. *Let $\widetilde{A} = (a_1, a_2, a_3, a_4)$ and $\widetilde{B} = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy numbers then*

(i) *Addition:* $\widetilde{A} \bigoplus \widetilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$.

(ii) *Subtraction:* $\widetilde{A} \ominus \widetilde{B} = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$.

(iii) *Scalar Multiplication:* $k\widetilde{A} = (ka_1, ka_2, ka_3, ka_4)$ *if* $k > 0$ *and* $k\widetilde{A} = (ka_4, ka_3, ka_2, ka_1)$ *if k < 0.*

3. Theory behind Solving Fuzzy Initial Value Problem

In this section we define a first order fuzzy initial value problem (FTVP) of differentiability than we replace it by its parametric form. Let us consider ordinary differential equation:

$$w^{'}(s) = g(s, w(s));$$

$s_0 \leq s \leq s_0 + a$, where $a > 0$, and initial condition $w(s_0) = w^0$, where $f : [s_0, s_0 + a] \times R \rightarrow R$ is a continuous real valued function, $w^0 \in R$ and s_0 and a are finite constant with $a > 0$.

Now in order to solve above equation at the same time we write the fuzzy function w(t) in its α cut representation to get,

$$[w(s)]^{\alpha} = [\underline{w}_{\alpha}(s), \overline{w}_{\alpha}(s)].$$

And the initial condition in α cut form,

$$[w(0)^{\alpha}] = [\underline{w}_{\alpha}(0), \overline{w}_{\alpha}(0)]$$

or

$$[\underline{w}_{\alpha}{}^{0}, \overline{w}_{\alpha}{}^{0}].$$

The Zadeh extension theory leads to the following description of $g(s, w(s))$ where $w(s)$ is a fuzzy number,

$$g(s, w(s))(z) = Sup\{w(s)(\alpha) = g(s, \alpha), z \in R\}.$$

According to Nguyen theorem, it follows that,

$$[g(s, w(s))]^{\alpha} = [\underline{g}_{\alpha}(s, w(s)), \overline{g}_{\alpha}(s, w(s))] = g(s, [w(s)]^{\alpha})$$

$$[g(s, w(s))]^{\alpha} = \{g(s, u) : u \in [w(s)]^{\alpha}$$

$$[g(s, w(s))]^{\alpha} = [g_{1,\alpha}(s, \underline{w}_{\alpha}(s), \overline{w}_{\alpha}(s)), g_{2,\alpha}(s, \underline{w}_{\alpha}(s), \overline{w}_{\alpha}(s))],$$

where the two term end point function are given as,

$$g_{1,\alpha}(s, \underline{w}_{\alpha}(s), \overline{w}_{\alpha}(s)) = \min\{g(s, u)/u \in [w(s)]^{\alpha}\}$$

$$g_{2,\alpha}(s, \underline{w}_{\alpha}(s), \overline{w}_{\alpha}(s)) = \max\{g(s, u)/u \in [w(s)]^{\alpha}\}$$

Analysis of the Method

Consider the differential equation:

$$Pz + Qz + Rz = b(y), \tag{3.1}$$

where P is the highest order derivative which is assumed to be easily invertible, Q is a linear differential operator of order less than P, R represents the nonlinear terms, and b is the source term. If differential equations describe by i order, where the differential operation P is given [3]:

$$P(.) = \frac{d^{i}(.)}{dy^{i}}, \tag{3.2}$$

P^{-1} is the inverse operator. Here we consider i-fold integral operator which is given as,

$$P^{-1}(*) = \int_0^z \int_0^z \cdots \int_0^z (*)dz....dz$$

then

$$z = P^{-1}(*)(b(y) - Rz). \tag{3.3}$$

The Adomain method consists of the solution of (3.3) as an infinite series,

$$z = \sum_{i=0}^{\infty} z_i(y) \tag{3.4}$$

and the nonlinear operator R decomposes as,

$$Rz = \sum_{i=0}^{\infty} M_i(y), \tag{3.5}$$

where M_i are adomain polynomials of $z_0, z_1, z_2, \ldots, z_i$,

$$M_i = \frac{1}{i}\frac{d^i}{d\phi^i}[N(\sum_{n=0}^{\infty} \phi^n z_n)]_{\phi=0}, \tag{3.6}$$

$N = 0, 1, 2,.$ Substituting the derivatives (3.4), (3.5) and the above equation in (3.3) which yields,

$$\sum_{i=0}^{\infty} z_i(y) = P^{-1}(b(t) - P^{-1}(\sum_{i=0}^{\infty} M_i)). \tag{3.7}$$

Thus, $z_0 = P^{-1}(b(t))$, $z_{i+1} = P^{-1}(\sum\limits_{i=0}^{\infty} M_i)$, $z_{i+1} = M_i(z_0, z_1, z_2, ...z_i)$, $i = 0, 1, 2, \ldots$.

We then define the K term approximate to the solution z by

$$\chi_k(z) = \sum_{i=0}^{\infty} z_i, \quad \lim_{k\to\infty} \chi_k(z) = z.$$

Practical formula for calculating Adomain decomposition polynomials is given in M_i. However, it is not usually possible to determine any term of the series $M_i = \sum\limits_{i=0}^{\infty} z_i$ is approximate to the truncated series of, $\chi_k = z_0 + z_1 + z_2 + \cdots + z_{i-1}$.

Example 1. *Consider the fuzzy initial value problem,* $\{z'(s) = z(s), s \in [0, 1]\}$, $z(0) = (0.8 + 0.125\alpha, 1.1 - 0.1\alpha)$, $0 < \alpha \leq 1$. *The exact solution is given by* $\underline{z}(s, \alpha) = (0.8 + 0.125\alpha)e^s$ *and* $\overline{z}(s, \alpha) = (1.1 - 0.1\alpha)e^s$, $0 < \alpha \leq 1$.

Apply the inverse operator P^{-1} for the given fuzzy initial value problem we obtain $\underline{z}(s,\alpha) = (0.8 + 0.125\alpha) + L^{-1}(\underline{z}(s,\alpha))$. Use the decomposition series of $\underline{z}(s,\alpha)$ we get

$$\sum_{n=0}^{\infty} \underline{z}_n(s,\alpha) = (0.8 + 0.125\alpha) + \int_0^s \underline{z}_{n+1}(s,\alpha)ds$$

$$\underline{z}_0(s,\alpha) = (0.8 + 0.125\alpha)$$

$$\underline{z}_1(s,\alpha) = (0.8 + 0.125\alpha)s$$

$$\underline{z}_2(s,\alpha) = (0.8 + 0.125\alpha)\frac{s^2}{2!}$$

$$\underline{z}_n(s,\alpha) = (0.8 + 0.125\alpha)(1 + s + \frac{s^2}{2!} + \cdots)$$

$$\overline{z}_n(s,\alpha) = (1.1 - 0.1\alpha)(1 + s + \frac{s^2}{2!} + \cdots)$$

The above two equation provides approximate solution of $\underline{z}_n(s,\alpha)$ and $\overline{z}_n(s,\alpha)$ which is identical to the exact solution. The results of the comparison are given in tabular form.

TABLE 1. Example of a table

	Exact solution (t=1)		RKM(h=0.01)		ADM(t=1)	
r	$\underline{z}$	$\overline{z}$	$\underline{z}$	$\overline{z}$	$\underline{z}$	$\overline{z}$
0	2.1744	2.9898	2.174625	2.990110	2.1744	2.9898
0.2	2.24235	2.93544	2.242583	2.935744	2.24235	2.93544
0.4	2.3103	2.88108	2.310540	2.881379	2.3103	2.88108
0.6	2.37825	2.82672	2.378497	2.827013	2.37825	2.82672
0.8	2.4462	2.77236	2.446454	2.772647	2.4462	2.77236
1	2.51415	2.718	2.514411	2.718282	2.51415	2.718

In the above table, we are comparing numerical values of $\underline{z}_n(s,\alpha)$ and $\overline{z}_n(s,\alpha)$ using Fuzzy Adomain decomposition method (ADM) and 4th order Runge-Kutta method (RKM) with the exact solution for different values of α at s = 1.

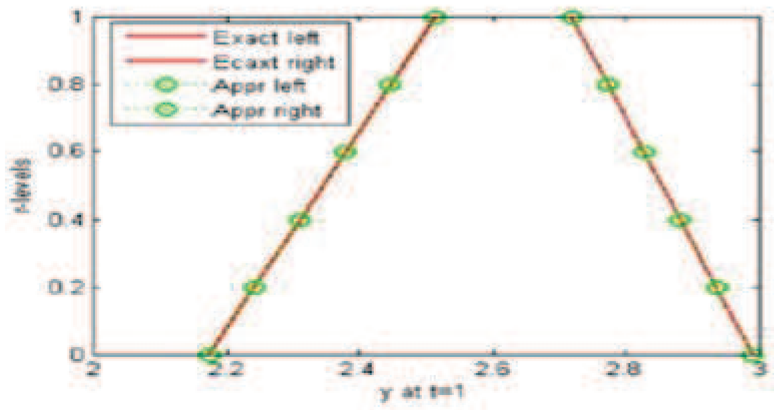

FIGURE 1

In the figure above, the approximate solution of $\underline{z}_n(s, \alpha)$ and $\overline{z}_n(s, \alpha)$ is plotted.

4. CONCLUSION

In this paper, we used the Adomain decomposition method to use a trapezoidal fuzzy number to find a numerical solution of fuzzy differential equations. The solution obtained in this paper is compared with the existing solution of the 4^{th} order Runge $-$ Kutta process which is given in the tabular form. This method is very easy when compared with the other methods to find an approximate solution of a fuzzy differential equation.

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