ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **9** (2020), no.8, 6221–6227
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.8.89 Special Issue on ICMA-2020

ON FUZZY SUPRA REGULAR BAIRE SPACES

E. POONGOTHAI[1] AND A. CELINE

ABSTRACT. In this paper, the concepts of FSR nowhere dense sets are introduced and studied. By means of FSR nowhere dense sets, the concepts of FSR Baire Space(fuzzy supra regular) are introduced and Characterizations of FSR Baire space are studied.

1. INTRODUCTION

In 1937, Stone [6] introduced the notations of regular open sets in topological spaces, In 1981, Azad [2] introduced the concept of FR open sets in FT Spaces. In 1984, Mashhour et al [3] introduced the concept of FST Spaces. In 1987, Abd El-Monsef et al [1] introduced FSRT Space and defined basic notions of FST Space. The concept of Baire Space was first advanced by Baire (1899). The study of Baire Spaces and related concepts became popular among topologists interested in the completeness. In 2013, the concept of Baire Space in fuzzy setting was introduced and studied by G.Thangaraj and S.Anjalmose [7]. In this paper, Several characterizations of FSR Baire Space are obtained.

2. PRELIMINARIES

We give some basic notions and results used in the sequel. In this work by (X, T^*), we will denote a FST Spaces.

[1]*Corresponding author*

2010 *Mathematics Subject Classification.* 94D05, 03E72.

Key words and phrases. FSR nowhere dense set, FSR dense set, FSR first category, FSR second category and FSR Baire space.

Definition 2.1. *[4] A collection δ^* of fuzzy sets in a set X is called FST on X if the following conditions are satisfied:*

(i) *0 and 1 belongs to δ^*.*
(ii) *$g_\chi \in \delta^*$ for each $\chi \in \wedge$ implies $(\vee_{\chi\in\wedge} g_\chi) \in \delta^*$.*

The pair (X, δ^*) is called a FST Space. The elements of δ^* are called FS open sets and the complement of a FS open set is called FS closed set.

Lemma 2.1. *[5] For any fuzzy set χ in a FST space X,*

(i) $1 - cl^*(\chi) = int^*(1-\chi)$,
(ii) $1 - int^*(\chi) = cl^*(1-\chi)$.

Theorem 2.1. *[4] For any two fuzzy sets g and h in a FST Space (X, δ^*),*

(1) *g is a FS closed set if and only if $cl^*(g) = g$.*
(2) *g is a FS open set if and only if $int^*(g) = (g)$.*
(3) *$g \leq h$ implies $int^*(g) \leq int^*(h)$ and $cl^*(g) \leq cl^*(h)$.*
(4) *$cl^*(cl^*(g)) = cl^*(g)$ and $int^*(int^*(g)) = int^*(g)$.*
(5) $cl^*(g \vee h) \geq cl^*(g) \vee cl^*(h)$.
(6) $cl^*(g \wedge h) \leq cl^*(g) \wedge cl^*(h)$.
(7) $int^*(g \vee h) \geq int^*(g) \vee int^*(h)$.
(8) $int^*(g \wedge h) \leq int^*(g) \wedge int^*(h)$.
(9) $cl^*(g^c) = [int^*(g)]^c$.
(10) $int^*(g^c) = [cl^*(g)]^c$.

Definition 2.2. *[5] Let (X, T^*) be a FST Space. A fuzzy set g is called a FSR open set if $int^*(cl^*(g)) = g$ and a fuzzy set g is called a FSR closed set if $cl^*(int^*(g)) = g$.*

Definition 2.3. *[5] Let (X, T^*) be a FST Space and g be a fuzzy set in X, then the FSR closure and FSR interior of g is denoted and defined respectively as*

$$rcl^*(g) = \wedge\ (\ h \mid h \text{ is a FSR closed set in } X \text{ and } g \leq h)$$
$$rint^*(g) = \vee\ (\ h \mid h \text{ is a FSR open set in } X \text{ and } h \leq g).$$

3. Fuzzy Supra Regular Nowhere Dense Sets

Definition 3.1. *Let (X, T^*) be a FST Space. A fuzzy set χ in (X, T^*) is called a FSR nowhere dense set if there exists no non-zero FSR open set μ in (X, T^*) such that $\mu < cl^*(\chi)$. That is, $int^*cl^*(\chi) = 0$, in (X, T^*).*

Example 1. *Let* $X = \{a, b, c\}$. *The fuzzy sets* χ, μ, γ *and* δ *are defined on* X *as follows:*

$\chi : X \to [0,1]$ *defined as* $\chi(a) = 0.5;\ \ \chi(b) = 0.7;\ \ \chi(c) = 0.3$

$\mu : X \to [0,1]$ *defined as* $\mu(a) = 0.6;\ \ \mu(b) = 0.5;\ \ \mu(c) = 0.8$

$\gamma : X \to [0,1]$ *defined as* $\gamma(a) = 0.4;\ \ \gamma(b) = 0.4;\ \ \gamma(c) = 0.7$

$\delta : X \to [0,1]$ *defined as* $\delta(a) = 0.7;\ \ \delta(b) = 0.3;\ \ \delta(c) = 0.6$ *Then* $T^* = \{0, \chi, \mu, \gamma, \delta, (\chi \vee \mu), (\gamma \vee \delta), (\chi \vee \delta), (\chi \vee \gamma), (\mu \vee \delta), (\chi \wedge \mu), (\gamma \wedge \delta), (\chi \wedge \delta), (\chi \wedge \gamma), (\mu \wedge \delta), 1\}$ *is a FST on* X. *The non-zero FSR open sets in* (X, T^*) *are* $\chi, \mu, \gamma, \delta, (\chi \vee \mu), (\gamma \vee \delta), (\chi \vee \delta), (\chi \vee \gamma), (\mu \vee \delta), (\chi \wedge \mu), (\gamma \wedge \delta), (\chi \wedge \delta), (\mu \wedge \delta)$. *Now the fuzzy sets* $int^*cl^*(\chi) = \chi \neq 0$ *and* $int^*cl^*(\gamma) = int[1 - (\chi \vee \mu)] = 0$ *in* (X, T^*). *Hence* χ *is not a FSR nowhere dense set in* (X, T^*) *and* γ *in a FSR nowhere dense set in* (X, T^*).

Remark 3.1.

(i) *If* α *and* β *are FSR nowhere dense sets in a FST Space* (X, T^*), *then* $\alpha \vee \beta$ *need not be a FSR nowhere dense set in* (X, T^*).

(ii) *The complement of a FSR nowhere dense set in a FST Space need not be a FSR nowhere dense set. For consider the following example.*

Example 2. *Let* $X = \{a, b, c\}$. *The fuzzy sets* χ, μ *and* γ *are defined on* X *as follows:*

$\chi : X \to [0,1]$ *defined as* $\chi(a) = 0.3;\ \ \chi(b) = 0.1;\ \ \chi(c) = 0.2$

$\mu : X \to [0,1]$ *defined as* $\mu(a) = 0.7;\ \ \mu(b) = 0.4;\ \ \mu(c) = 0.1$

$\gamma : X \to [0,1]$ *defined as* $\gamma(a) = 0.1;\ \ \gamma(b) = 0.2;\ \ \gamma(c) = 0.7.$

Then $T^* = \{0, \chi, \mu, \gamma, (\mu \vee \gamma), (\mu \wedge \gamma), 1\}$ *is a FST on* X. *On Computation, one can see that FSR nowhere dense sets in* (X, T^*) *are* $1 - \mu$, $1 - \gamma$ *and* $1 - \mu \vee \gamma$. *Now* $(1 - \mu) \vee (1 - \gamma) = 1 - \mu \wedge \gamma$ *and* $int^*cl^*[(1 - \mu) \vee (1 - \gamma)] = int^*(1 - \mu \wedge \gamma) = \chi \neq 0$ *and hence* $(1 - \mu) \vee (1 - \gamma)$ *is not a FSR nowhere dense set in* (X, T^*) *and* $1 - \mu$ *is a FSR nowhere dense set in* (X, T^*) *where as* χ *is not a FSR nowhere dense set in* (X, T^*).

Definition 3.2. *A fuzzy set* χ *in a FST Space* (X, T^*) *is called a FSR dense if there exists no FSR closed set* μ *in* (X, T^*) *such that* $\chi < \mu < 1$. *That is,* $cl^*(\chi) = 1$, *in* (X, T^*).

Proposition 3.1. *If* χ *is a FSR nowhere dense set in a FST Space* (X, T^*), *then* $1 - \chi$ *is a FSR dense set in* (X, T^*).

Proof. Let χ be a FSR nowhere dense set in (X, T^*). Then $int^*cl^*(\chi) = 0$, in (X, T^*). Now $1 - int^*cl^*(\chi) = 1 - 0 = 1$ and hence by lemma 2.1, $cl^*int^*(1-\chi) = 1$ in (X, T^*). But $cl^*int^*(1-\chi) \leq cl^*(1-\chi)$ implies that $1 \leq cl^*(1-\chi)$. That is, $cl^*(1-\chi) = 1$ in (X, T^*). Therefore $1 - \chi$ is a FSR dense set in (X, T^*). □

Proposition 3.2. *If χ is a FSR closed set with $int^*(\chi) = 0$, in a FST Space (X, T^*), then χ is a FSR nowhere dense set (X, T^*).*

Proof. Let χ be a FSR closed set with $int^*(\chi) = 0$, in (X, T^*). Then $cl^*(\chi) = \chi$ and $int^*(\chi) = 0$ and hence $int^*cl^*(\chi) = int^*(\chi) = 0$ in (X, T^*). Therefore χ is a FSR nowhere dense set (X, T^*). □

Proposition 3.3. *If χ is a non-zero fuzzy set in a FST Space (X, T^*), then $\chi \vee int^*cl^*int^*\chi \leq cl^*\chi$, in (X, T^*).*

Proof. Since $cl^*(\chi)$ is a FSR closed set in (X, T^*), $int^*cl^*int^*(cl^*(\chi)) \leq cl^*(\chi)$. Then $int^*cl^*int^*(\chi) \leq int^*cl^*int^*(cl^*(\chi))$, $\chi \leq cl^*(\chi)$ and hence $\chi \vee int^*cl^*int^*(\chi) \leq cl^*(\chi)$, in (X, T^*). □

Proposition 3.4. *If χ is a FSR nowhere dense set in a FSR topological space (X, T^*), then $int^*(\chi) = 0$ in (X, T^*).*

Proof. Let χ be a FSR nowhere dense set in (X, T^*). Then $int^*cl^*(\chi) = 0$ in (X, T^*). Now $int^*(\chi) \leq int^*cl^*(\chi)$, implies that $int^*(\chi) \leq 0$, in (X, T^*). That is, $int^*(\chi) = 0$ in (X, T^*). □

Proposition 3.5. *If $\chi \leq \mu$ and μ is a FSR nowhere dense set in FST space (X, T^*), than χ is also a FSR nowhere dense set in (X, T^*).*

Proof. Now $\chi \leq \mu$, implies that $cl^*(\chi) \leq int^*cl^*(\mu)$ in (X, T^*). Since μ is a FSR nowhere dense set in (X, T^*). That is $int^*cl^*(\mu) = 0$. Then $int^*cl^*(\chi) \leq 0$ in (X, T^*). That is $int^*cl^*(\chi) = 0$ in (X, T^*). Therefore χ is a FSR nowhere dense set in (X, T^*). □

Proposition 3.6. *If either χ or μ is a FSR nowhere dense set in FST space (X, T^*), than $\chi \wedge \mu$ is a FSR nowhere dense set in (X, T^*).*

Proof. $Int^*cl^*(\chi \wedge \mu) \leq int^*[cl*(\chi) \wedge cl^*(\mu)] = int^*cl^*(\chi) \wedge int^*cl^*(\mu)$ in (X, T^*). If either χ or μ is a FSR nowhere dense set in (X, T^*), then either $int^*cl^*(\chi) = 0$ or $int^*cl^*(\mu) = 0$. Then either $int^*cl^*(\chi \wedge \mu) \leq 0 \wedge int^*cl^*(\mu) = 0$ (or) $int^*cl^*(\chi \wedge \mu) \leq int^*cl^*(\chi) \wedge 0 = 0$. Therefore, $\chi \wedge \mu$ is a FSR nowhere dense set in (X, T^*). □

Proposition 3.7. *If χ is a FSR dense set in FST space (X, T^*), than χ is a FSR open set in (X, T^*).*

Proof. Let χ be a FSR dense set in (X, T^*). Then $cl^*(\chi) = 1$, in (X, T^*). Since $cl^*(\chi) \leq cl^*(\chi), 1 \leq cl^*(\chi)$. That is $cl^*(\chi) = 1$, in (X, T^*). Now, $cl^*int^*cl^*(\chi) = cl^*int^*(\chi) = cl^*(\chi) = 1$ and hence $\chi \leq cl^*int^*cl^*(\chi)$ in (X, T^*). Therefore, χ is a FSR open set in (X, T^*). □

Definition 3.3. *A fuzzy set χ in a FST space (X, T^*) is called a FSR first category set if $\chi = \vee_{i=1}^{\infty}(\chi_i)$, where (χ_i)'s are FSR nowhere dense set in (X, T^*). Any other fuzzy set in (X, T^*) is said to be FSR second category.*

Definition 3.4. *Let χ be a FSR first category set in a FST space (X, T^*). Then $1-\chi$ is called a FSR residual set in (X, T^*).*

4. Fuzzy Supra Regular Baire Spaces

Definition 4.1. *Let (X, T^*) be a FST space. Then (X, T^*) is called a FSR Baire space $int^*(\vee_{i=1}^{\infty}(\chi_i)) = 0$, where (χ_i)'s are FSR nowhere dense set in (X, T^*).*

Example 3. *Let $X = \{a, b, c\}$. The fuzzy sets $\chi, \mu, \gamma, \alpha, \beta, \delta$ and η are defined on X as follows:*

$\chi : X \to [0, 1]$ *defined as* $\chi(a) = 0.5$; $\chi(b) = 0.4$; $\chi(c) = 0.6$
$\mu : X \to [0, 1]$ *defined as* $\mu(a) = 0.4$; $\mu(b) = 0.6$; $\mu(c) = 0.3$.
$\gamma : X \to [0, 1]$ *defined as* $\gamma(a) = 0.6$; $\gamma(b) = 0.5$; $\gamma(c) = 0.7$
$\alpha : X \to [0, 1]$ *defined as* $\alpha(a) = 0.3$; $\alpha(b) = 0.1$; $\alpha(c) = 0.7$
$\beta : X \to [0, 1]$ *defined as* $\beta(a) = 0.6$; $\beta(b) = 0.6$; $\beta(c) = 0.7$
$\delta : X \to [0, 1]$ *defined as* $\delta(a) = 0.5$; $\delta(b) = 0.6$; $\delta(c) = 0.6$
$\eta : X \to [0, 1]$ *defined as* $\eta(a) = 0.5$; $\eta(b) = 0.5$; $\eta(c) = 0.6$

Then $T^* = \{0, \chi, \mu, \gamma, \alpha, \beta, \delta, \eta, 1\}$ is a FST on X. On computations, one can see that the non-zero FSR open set in (X, T^*) are $\chi, \gamma, 1 - \mu, \alpha, 1 - \alpha, \delta, \eta, 1$. Now the FSR nowhere dense set in (X, T^*) are $1-\delta, 1-\eta, 1-\beta, \mu, 1-\chi$. On computations, $int^*[(1 - \delta) \vee (1 - \eta) \vee (1 - \beta) \vee (\mu) \vee (1 - \chi)] = int^*(1 - \chi) = 0$. Hence the FST Space (X, T^*) is a FSR Baire Space.

Proposition 4.1. *If $int^*(\vee_{i=1}^{\infty}(\chi_i)) = 0$, where $int^*(\chi_i) = 0$ and (χ_i)'s are FSR closed sets in a FST space (X, T^*), then (X, T^*) is a FSR Baire space.*

Proof. Let (χ_i)'s be FSR closed sets in (X, T^*). Since $int^*(\chi_i) = 0$, by proposition 3.2, the (χ_i)'s are FSR nowhere dense sets in (X, T^*). Therefore $int^*(\vee_{i=1}^{\infty}(\chi_i)) = 0$, where (χ_i)'s are FSR nowhere dense in (X, T^*), implies that (X, T^*) is a FSR Baire space. □

Proposition 4.2. *If $cl^*(\wedge_{i=1}^{\infty}(\chi_i)) = 1$, where (χ_i)'s are FSR dense and FSR open sets in a FST space (X, T^*), then (X, T^*) is a FSR Baire space.*

Proof. Now, $cl^*(\wedge_{i=1}^{\infty}(\chi_i)) = 1$, implies $1 - cl^*(\wedge_{i=1}^{\infty}(\chi_i)) = 0$. Then $int^*[1 - \wedge_{i=1}^{\infty}(\chi_i)] = 0$ in (X, T^*). This implies that $int^*[(\vee_{i=1}^{\infty}(1 - \chi_i)] = 0$. Since (χ_i)'s are FSR dense in (X, T^*), $cl^*(\chi_i) = 1$ and $int^*(1 - \chi_i) = 1 - cl^*(\chi_i) = 1 - 1 = 0$. Hence $int^*(\vee_{i=1}^{\infty}(1-\chi_i)) = 0$, where $int^*(1-\chi_i) = 0$ and $(1-\chi_i)$'s are FSR closed sets in (X, T^*). Then by proposition 4.1, the FST space (X, T^*) is a FSR Baire space. □

Proposition 4.3. *Let (X, T^*) be a FST space. Then the following are equivalent.*

(1) *(X, T^*) is a FSR Baire space.*
(2) *$int^*(\chi) = 0$ for every FSR first category set χ in (X, T^*).*
(3) *$cl^*(\mu) = 1$ for every FSR residual set μ in (X, T^*).*

Proof.

$(\mathbf{1}) \Rightarrow (\mathbf{2})$ Let χ be a FSR first category set in (X, T^*). Then $\chi = \vee_{i=1}^{\infty}(\chi_i)$, where (χ_i)'s are FSR nowhere dense set in (X, T^*). Now $int^*(\chi) = int^*(\vee_{i=1}^{\infty}(\chi_i))$ (Since (X, T^*). is a FSR Baire space). Therefore $int^*(\chi) = 0$, in (X, T^*).

$(\mathbf{2}) \Rightarrow (\mathbf{3})$ Let μ be a FSR residual set in (X, T^*). Then, $1 - \mu$ is a FSR first category set in (X, T^*). By hypothesis, $int^*(1 - \mu) = 0$, in (X, T^*). This implies that $1 - cl^*(\mu) = 0$ and hence $cl^*(\mu) = 1$, in (X, T^*).

$(\mathbf{3}) \Rightarrow (\mathbf{1})$ Let χ be a FSR first category set in (X, T^*). Then $\chi = \vee_{i=1}^{\infty}(\chi_i)$, where (χ_i)'s are FSR nowhere dense set in (X, T^*). Since χ is a FSR first category set in (X, T^*), $1-\chi$ is a FSR residual set in (X, T^*). By hypothesis, $cl^*(1-\chi) = 1$. Then $1 - int^*(\chi) = 1$, in (X, T^*). This implies that $int^*(\chi) = 0$, in (X, T^*). Hence $int^*(\vee_{i=1}^{\infty}(\chi_i)) = 0$, where (χ_i)'s are FSR nowhere dense in (X, T^*), implies that (X, T^*) is a FSR Baire space. □

Definition 4.2. *A FST space (X, T^*) is said to be of FSR first category space if $\vee_{i=1}^{\infty}(\chi_i) = 1_X$, where (χ_i)'s are FSR nowhere dense set in (X, T^*). Otherwise (X, T^*) will be called a FSR second category space.*

Proposition 4.4. *If a FST space (X, T^*) is a FSR Baire space, then (X, T^*) is a FSR second category space.*

Proof. Let (X, T^*) is a FSR Baire space. Then,$int^*(\vee_{i=1}^{\infty}(\chi_i)) = 0$, where (χ_i)'s are FSR nowhere dense in (X, T^*). Then $\vee_{i=1}^{\infty}(\chi_i) \neq 1_X$ [Otherwise, $\vee_{i=1}^{\infty}(\chi_i) = 1_X$, implies that $int^*(\vee_{i=1}^{\infty}(\chi_i)) = int^*(1_X) = 1_X \neq 0$,a contradiction]. Hence (X, T^*) is a FSR second category space. □

REFERENCES

[1] M.E. ABD EL-MONSEF, A.E. RAMADAN: *On Fuzzy Supra Topological Spaces*, Indian J. Pure Appl.Math.,**18**(4) (1987), 322–329.

[2] K.K. AZAD: *On fuzzy semi continuity, fuzzy almost continuity and fuzzy weakly continuity* , Journal of Mathematical Analysis and Applications, **82** (1981), 14–32.

[3] A.S. MASHHOUR A.A.ALLAM, F.S. MAHMOUD, F.H. KHEDR: *On supra topological spaces*, Indian J. Pure and Appl. Math., **4**(14) (1983), 502–510.

[4] S. AHMED, B. CHANDRA CHETIA: *On Certain Properties of Fuzzy Supra Semi open Sets*, International Journal of Fuzzy Mathematics and Systems, **4**(1) (2014), 93–98.

[5] J. SRIKIRUTHIKA, A. KALAISELVI: *Fuzzy Supra Regular Open Sets*, International Journal of Pure and Applied Mathematics, **119**(15) (2018), 731–735.

[6] M. STONE: *Application of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc., **41** (1937), 374–481.

[7] G. THANGARAJ, S. ANJALMOSE: *On Fuzzy Baire Spaces*, Journal of Fuzzy Mathematics, **21**(3) (2013), 667–676.

[8] L.A. ZADEH: *Fuzzy sets*, Informatics and Control, **8** (1965), 338–353.

PG AND RESEARCH DEPARTMENT OF MATHEMATICS
SHANMUGA INDUSTRIES ARTS AND SCIENCE COLLEGE,
TIRUVANNAMALAI - 606603, TAMIL NADU, INDIA.
Email address: epoongothai5@gmail.com

Email address: celine.archana@gmail.com