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EXTORIAL FUNCTION AND ITS PROPERTIES IN DISCRETE CALCULUS

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ABSTRACT. In this paper, by developing certain properties of the newly defined extorial function, we arrive solution of higher order difference equation with constant coefficients using the extorial function in discrete calculus. Suitable examples are inserted to validate our finding.

1. INTRODUCTION

The difference of two successive values of some sequence of numbers or function is the definition of the Δ. This concept is developed by the difference operator:

$$\Delta u(k) = u(k+1) - u(k), \quad \text{where } \ k \in \mathbb{R}.$$

The applications of difference operator and difference equations have been developed and applied many of the areas such as Astrology, Engineering, Weather proofing and Artificial intelligence etc. The generalized difference operator Δ_ℓ, and its properties have been derived. Authors in [2–4] have established inverse difference operator, generalized version of Lebinitz theorem, Newtons formula, summation of consecutive integers, Binomial theorem etc. In this paper, we apply theory of extorial function to obtain solution of higher order ℓ-difference equation with constant coefficients

Foe a positive integer 'n' and a real ℓ, the factorial polynomial is defined as

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$K_\ell^{(n)} = \prod_{r=0}^{n-1}(k - r\ell)$. This factorial polynomial is used to define new extorial function. This extorial function is obtained by replacing polynomials into factorial polynomials in the expansion of exponential functions [4]. This extorial function satisfies the higher order difference equation of the form $\Delta_\ell^m u(k) = v(k)$ and linear difference equation with constant coefficients.

2. Basic Concept of Delta operator

In this section we present basic concept of Delta operator, which will be used in the subsequent sections.

Definition 2.1. *[1] Let $\ell \neq 0$ be any real and u(k) be any real valued function and c is constant. Then, the generalized difference operator on u(k) defined as*

$$\Delta_\ell u(k) = u(k+\ell) - u(k). \tag{2.1}$$

If $\Delta_\ell v(k) = u(k)$, then

$$v(k) = \Delta_\ell^{-1} u(k) + c. \tag{2.2}$$

For example, if $u(k) = 2e^k + k$, then (2.1) becomes

$$\Delta_\ell u(k) = \Delta_\ell(2e^k + k) = 2(e^{(k+\ell)} + k + \ell) - 2(e^k + k).$$

The generalized n^{th} order delta operator on the function $u(k)$ is given by $\Delta_\ell^n u(k) = \Delta_\ell(\Delta_\ell^{n-1} u(k))$.

Definition 2.2. *The generalized polynomial factorial is defined as*

$$k_\ell^{(n)} = k(k-\ell)(k-2\ell)\cdots(k-(n-1)\ell). \tag{2.3}$$

Lemma 2.3. *We obtain the following identities easily using (2.1) and (2.2). For a fixed $n \in \mathbb{N}$ and $k \in \mathbb{R}$, we have*

(i) $\Delta_\ell k_\ell^{(n)} = n\ell k_\ell^{(n-1)}$,
(ii) $\Delta_\ell^r k_\ell^{(n)} = \ell^k n(n-1)(n-2)\cdots(n-(r-1))k_\ell^{(n-r)}$,
(iii) $\Delta_\ell^n k_\ell^{(n)} = n!\ell^n$,
(iv) $\Delta_\ell \dfrac{1}{k_\ell^{(n)}} = \dfrac{-n\ell}{(k+\ell)_\ell^{(n+1)}}$,
(v) $\Delta_{-\ell} \dfrac{1}{k_\ell^{(n)}} = \dfrac{n\ell}{k_\ell^{(n+1)}}$.

Lemma 2.4. *[1] If $\ell, n \in \mathbb{N}$ and k is positive, then we have*

$$\Delta_\ell^{-1} k_\ell^{(n)} = \frac{k_\ell^{(n+1)}}{\ell(n+1)} + c. \tag{2.4}$$

3. The ℓ - Extorial function

The newly defined ℓ-Extorial function is arrived by replacing the polynomial k^n by polynomial factorial function $k_\ell^{(n)}$ in the exponential function e^k. The formal definition of extorial function is given below.

Definition 3.1. *The ℓ-extorial function denoted as $e(k_\ell^{(n)})$ is defined as*

$$e(k_\ell^{(n)}) = 1 + \frac{k_\ell^{(n)}}{1!} + \frac{k_\ell^{(2n)}}{2!} + \frac{k_\ell^{(3n)}}{3!} + \cdots + \infty, \tag{3.1}$$

where $|\ell| \le 1$ and $n, k \in \mathbb{R}$.

Lemma 3.2. *[4] If $|\ell| \le 1$ and k real variable then the following holds.*
(i) $e(k_0^{(1)})=e^k$,
(ii) $e((-k)_1^{(1)}) = -\infty$,
(iii) $e(k_{-1^{(1)}}) = \infty$,
(iv) $e((-k)_\ell^{(1)}) = 1 - \frac{k_{-\ell}^{(1)}}{1!} + \frac{k_{-\ell}^{(2)}}{2!} - \frac{k_{-\ell}^{(3)}}{3!} + \cdots + \infty$,
(v) $e((-k)_{-\ell}^{(1)}) = 1 - \frac{k_\ell^{(1)}}{1!} + \frac{k_\ell^{(2)}}{2!} - \frac{k_\ell^{(3)}}{3!} + \cdots + \infty$,
(vi) $\Delta_\ell e(k_\ell^{(1)}) = \ell e(k_\ell^{(1)})$,
(vii) $\Delta_\ell^n e(k_\ell^{(n)}) = \ell^n e(k_\ell^{(1)})$.

Lemma 3.3. *[4] Let k be the multiple of ℓ. Then $e(k_\ell^{(1)})$ can be expressed as finite series such that $e(k_\ell^{(1)}) = \sum\limits_{r=0}^{a} \frac{k_\ell^{(r)}}{r!}$.*

Lemma 3.4. *[4] For any $\ell \in \mathbb{N}$, $e(-\ell)_{(-\ell)}^{(1)} = 1 - \ell$.*

Lemma 3.5. *[4] For $k_1, k_2 \in \mathbb{R}$ and $\ell \in (0,1)$, we have*

$$e(k_1 + k_2)_\ell^{(1)} = e(k_1)_\ell^{(1)} e(k_2)_\ell^{(1)}. \tag{3.2}$$

Definition 3.6. *If $k_\ell^{(rn)} \neq 0$ for $n > 0$ and $r \in \mathbb{N}$, then the negative index extorial function is defind as*

$$e(k_\ell^{(-n)}) = 1 + \frac{1}{1!}\frac{1}{k_\ell^{(n)}} + \frac{1}{2!}\frac{1}{k_\ell^{(2n)}} + \frac{1}{3!}\frac{1}{k_\ell^{(3n)}} + \cdots \infty. \tag{3.3}$$

Remark 3.7.

(i) $e(1_{-1}^{(-1)}) = \sum_{r=0}^{\infty} \frac{1}{(r!)^2}$,

(ii) $e(-1_{1}^{(-1)}) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{(r!)^2}$,

(iii) $e((mk)_{(m\ell)}^{(1)}) = 1 + \frac{(mk)_{(m\ell)}^{(1)}}{1!} + \frac{(mk)_{(m\ell)}^{(2)}}{2!} + \frac{(mk)_{(m\ell)}^{(3)}}{3!} + \cdots + \infty.$

Lemma 3.8. *Let $k_\ell^{(rn)} \neq 0$, where $n \in \mathbb{N}$, $|\ell| < 1$ and $k_\ell^{(-n)} = \frac{1}{k_\ell^{(n)}}$. Then,*

$$\Delta_\ell e(k_\ell^{(-n)}) = \frac{-n\ell}{(k+\ell)_\ell^{(n+1)}} e((k-n\ell)_\ell^{(-n)}). \tag{3.4}$$

Proof. From (3.3),

$$e(k_\ell^{(-n)}) = 1 + \frac{1}{1!}\frac{1}{k_\ell^{(n)}} + \frac{1}{2!}\frac{1}{k_\ell^{(2n)}} + \frac{1}{3!}\frac{1}{k_\ell^{(3n)}} + \cdots + \infty$$

$$\begin{aligned}
\Delta_\ell(e(k_\ell^{(-n)})) &= \Delta_\ell\left(1 + \frac{1}{1!}\frac{1}{k_\ell^{(n)}} + \frac{1}{2!}\frac{1}{k_\ell^{(2n)}} + \frac{1}{3!}\frac{1}{k_\ell^{(3n)}} + \cdots + \infty\right) \\
&= (1-1) + \Delta_\ell \frac{1}{k_\ell^{(n)}} + \Delta_\ell \frac{1}{2!}\frac{1}{k_\ell^{(2n)}} + \Delta_\ell \frac{1}{3!}\frac{1}{k_\ell^{(3n)}} + \cdots \\
&= \frac{1}{1!}\frac{-n\ell}{(k+\ell)_\ell^{(n+1)}} + \frac{1}{2!}\frac{-2n\ell}{(k+\ell)_\ell^{(2n+1)}} + \frac{1}{3!}\frac{-3n\ell}{(k+\ell)_\ell^{(3n+1)}} + \cdots \\
&= \frac{-n\ell}{(k+\ell)_\ell^{(n+1)}}\left(1 + \frac{1}{1!}\frac{1}{(k-n\ell)_\ell^{(n)}} + \frac{1}{2!}\frac{1}{(k-n\ell)_\ell^{(2n)}} + \cdots\right),
\end{aligned}$$

which gives (3.4). □

Lemma 3.9. *For any positive k and $\ell \in \mathbb{N}$, we have*

$$e(-k_\ell^{(-1)}) = 1 - \frac{1}{1!}\frac{1}{k_\ell^{(-1)}} + \frac{1}{2!}\frac{1}{k_\ell^{(-2)}} - \frac{1}{3!}\frac{1}{-k_\ell^{(-3)}} + \cdots \infty.$$

The proof follows from the definition of extorial function.

Definition 3.10. *For $\ell \in (-1,1)$ and $k \in \mathbb{R}$, the n^{th} order ℓ-extorial function denoted as $e_n(k_\ell)$ is defined as*

$$e_n(k_\ell) = 1 + \frac{k_\ell^{(n)}}{n!} + \frac{k_\ell^{(2n)}}{(2n)!} + \frac{k_\ell^{(3n)}}{(3n)!} + \cdots + \infty. \tag{3.5}$$

From the definition extorial function, we obtain following lemma.

Lemma 3.11. *For any real k and $\ell, n \in \mathbb{N}$, then*

$$\text{(i) } e_n(-k_\ell) = \begin{cases} e_n(k_{(-\ell)}) & \text{if } n \text{ is even} \\ 1 - \dfrac{k_{(-\ell)}^{(n)}}{n!} + \dfrac{k_{(-\ell)}^{(2n)}}{(2n)!} - \dfrac{k_{(-\ell)}^{(3n)}}{(3n)!} + \cdots & \text{if } n \text{ is odd} \end{cases};$$

$$\text{(ii) } e_n(-k_{(-\ell)}) = \begin{cases} e_n(k_{(\ell)}) & \text{if } n \text{ is even} \\ 1 - \dfrac{(k)_\ell^{(n)}}{n!} + \dfrac{(k)_\ell^{(2n)}}{2n!} - \dfrac{(k)_\ell^{(3n)}}{3n!} + \cdots & \text{if } n \text{ is odd} \end{cases}.$$

Lemma 3.12. *Let $k \in \mathbb{R}$ and $n, \ell \in \mathbb{N}$. Then, we have*

$$\Delta_\ell e_n(k_\ell) = \ell \sum_{m=1}^{\infty} \frac{k_\ell^{(mn-1)}}{(mn-1)!}, \ nm \neq 1.$$

Proof. We shall prove this by induction method

$$e_2(k_\ell) = 1 + \frac{k_\ell^{(2)}}{2!} + \frac{k_\ell^{(4)}}{4!} + \frac{k_\ell^{(6)}}{6!} + \cdots + \infty$$

$$\Delta_\ell e_2(k_\ell) = \Delta_\ell \frac{k_\ell^{(2)}}{2!} + \Delta_\ell \frac{k_\ell^{(4)}}{4!} + \Delta_\ell \frac{k_\ell^{(6)}}{6!} + \cdots + \infty = \ell \left[\frac{k_\ell^{(1)}}{1!} + \frac{k_\ell^{(3)}}{3!} + \frac{k_\ell^{(5)}}{5!} + \cdots \right]$$

$$e_3(k_\ell) = 1 + \frac{k_\ell^{(3)}}{3!} + \frac{k_\ell^{(6)}}{6!} + \frac{k_\ell^{(9)}}{9!} + \cdots + \infty$$

$$\Delta_\ell e_3(k_\ell) = \Delta_\ell \frac{k_\ell^{(3)}}{3!} + \Delta_\ell \frac{k_\ell^{(6)}}{6!} + \Delta_\ell \frac{k_\ell^{(9)}}{9!} + \cdots + \infty = \ell \left[\frac{k_\ell^{(2)}}{2!} + \frac{k_\ell^{(5)}}{5!} + \frac{k_\ell^{(8)}}{8!} + \cdots \right]$$

In general, we find

$$\Delta_\ell e_n(k_\ell) = \ell \left[\frac{k_\ell^{(n-1)}}{(n-1)!} + \frac{k_\ell^{(2n-1)}}{(2n-1)!} + \frac{k_\ell^{(3n-1)}}{(3n-1)!} + \cdots \right] = \ell \sum_{m=1}^{\infty} \frac{k_\ell^{(mn-1)}}{(mn-1)!}. \qquad \square$$

Lemma 3.13. *For any positive integer m, we have $\Delta_\ell^m e_m(k_\ell) = \ell^m e_m(k_\ell)$.*

Proof.

$$\Delta_\ell e_1(k_\ell) = 0 + \Delta_\ell \frac{k_\ell^{(1)}}{1!} + \Delta_\ell \frac{k_\ell^{(2)}}{2!} + \Delta_\ell \frac{k_\ell^{(3)}}{3!} + \cdots = \ell e_1(k_\ell).$$

$$\Delta_\ell e_2(k_\ell) = 0 + \Delta_\ell \frac{k_\ell^{(2)}}{2!} + \Delta_\ell \frac{k_\ell^{(4)}}{4!} + \Delta_\ell \frac{k_\ell^{(6)}}{6!} + \cdots = \frac{2\ell k_\ell(1)}{2!} + \frac{4\ell k_\ell(3)}{4!} + \frac{6\ell k_\ell(5)}{6!} + \cdots$$

$$\Delta_\ell^2 e_2(k_\ell) = \frac{2\ell(\ell k_\ell^{(0)})}{2!} + \frac{4\ell(3\ell k_\ell^{(2)})}{4!} + \frac{6\ell(5\ell k_\ell^{(4)})}{6!} + \cdots = \ell^2 e_2(k_\ell),$$

which yields $\Delta_\ell^m e_m(k_\ell) = \ell^m e_m(k_\ell)$. $\square$

Lemma 3.14. *For positive m and real k, we have $\Delta_\ell^{(-m)} e_m(k_\ell) = \dfrac{e_m(k_\ell)}{\ell^m}$, $\ell \in \mathbb{N}$.*

Proof. From Lemma 3.13, we find $\Delta_\ell^m e_m(k_\ell) = \ell^m e_m(k_\ell)$. Taking Δ_ℓ^{-m} on both sides, we get $\Delta_\ell^{-m}(\Delta_\ell^m e_m(k_\ell)) = \Delta_\ell^{-m}(\ell^m e_m(k_\ell))$, which gives $\Delta_\ell^{(-m)} e_m(k_\ell) = \dfrac{e_m(k_\ell)}{\ell^m}$. □

Definition 3.15. *For $|\ell| < 1$, the negative order extorial function for $\ell \in (-1,1)$ is defined as*

$$e_{(-n)}(k_\ell) = 1 + \frac{1}{n!}\frac{1}{k_\ell^{(n)}} + \frac{1}{(2n)!}\frac{1}{k_\ell^{(2n)}} + \frac{1}{(3n)!}\frac{1}{k_\ell^{(3n)}} + \cdots + \infty. \tag{3.6}$$

Lemma 3.16. *For $\ell \in (-1,1)$ and positive k, we have*

$$\Delta_\ell e_{(-n)}(k_\ell) = -\ell\Big[\frac{1}{(n-1)!}\frac{1}{(k+\ell)_\ell^{(n+1)}} + \frac{1}{(2n-1)!}\frac{1}{(k+\ell)_\ell^{(2n+1)}} + \frac{1}{(3n-1)!}\frac{1}{(k+\ell)_\ell^{(3n+1)}} + \cdots\Big]$$

Proof. Putting $n = 1$ in (3.6), we get

$$e_{(-1)}(k_\ell) = 1 + \frac{1}{1!}\frac{1}{k_\ell^{(1)}} + \frac{1}{2!}\frac{1}{k_\ell^{(2)}} + \frac{1}{3!}\frac{1}{k_\ell^{(3)}} + \cdots + \infty$$

$$\Delta_\ell e_{(-1)}(k_\ell) = 1 + \Delta_\ell\frac{1}{1!}\frac{1}{k_\ell^{(1)}} + \Delta_\ell\frac{1}{2!}\frac{1}{k_\ell^{(2)}} + \Delta_\ell\frac{1}{3!}\frac{1}{k_\ell^{(3)}} + \cdots + \infty$$

$$= -\ell\left[\frac{1}{(k+\ell)_\ell^{(2)}} + \frac{1}{1!}\frac{1}{(k+\ell)_\ell^{(3)}} + \frac{1}{2!}\frac{1}{(k+\ell)_\ell^{(4)}} + \cdots\right].$$

Putting $n = 2$ in (3.6), we get

$$e_{(-2)}(k_\ell) = 1 + \frac{1}{2!}\frac{1}{k_\ell^{(2)}} + \frac{1}{4!}\frac{1}{k_\ell^{(4)}} + \frac{1}{6!}\frac{1}{k_\ell^{(6)}} + \cdots + \infty$$

$$\Delta_\ell e_{(-2)}(k_\ell) = 1 + \Delta_\ell\frac{1}{2!}\frac{1}{k_\ell^{(2)}} + \Delta_\ell\frac{1}{4!}\frac{1}{k_\ell^{(4)}} + \Delta_\ell\frac{1}{6!}\frac{1}{k_\ell^{(6)}} + \cdots + \infty$$

$$= -\ell\left[\frac{1}{1!}\frac{1}{(k+\ell)_\ell^{(3)}} + \frac{1}{3!}\frac{1}{(k+\ell)_\ell^{(5)}} + \frac{1}{5!}\frac{1}{(k+\ell)_\ell^{(7)}} + \cdots\right].$$

Putting $n = 3$ in (3.6), we get

$$e_{(-3)}(k_\ell) = 1 + \frac{1}{3!}\frac{1}{k_\ell^{(3)}} + \frac{1}{6!}\frac{1}{k_\ell^{(6)}} + \frac{1}{9!}\frac{1}{k_\ell^{(9)}} + \cdots + \infty$$

$$\Delta_\ell e_{(-3)}(k_\ell) = 1 + \Delta_\ell\frac{1}{3!}\frac{1}{k_\ell^{(3)}} + \Delta_\ell\frac{1}{6!}\frac{1}{k_\ell^{(6)}} + \Delta_\ell\frac{1}{9!}\frac{1}{k_\ell^{(9)}} + \cdots + \infty$$

$$= -\ell\left[\frac{1}{2!}\frac{1}{(k+\ell)_\ell^{(4)}} + \frac{1}{5!}\frac{1}{(k+\ell)_\ell^{(7)}} + \frac{1}{8!}\frac{1}{(k+\ell)_\ell^{(10)}} + \cdots\right]$$

In general,

$$\Delta_\ell e_{(-n)}(k_\ell) = -\ell\Big[\frac{1}{(n-1)!}\frac{1}{(k+\ell)_\ell^{(n+1)}} + \frac{1}{(2n-1)!}\frac{1}{(k+\ell)_\ell^{(2n+1)}} + \frac{1}{(3n-1)!}\frac{1}{(k+\ell)_\ell^{(3n+1)}} + \cdots\Big]$$

□

4. Extorial Type solution of Difference Equation

In this section, we obtain extorial type solutions of higher order linear ℓ-difference equations with constant coefficients.

Consider the n^{th} order linear difference equation

$$\left(a_n\frac{\Delta_\ell^n}{\ell^n} + a_{n-1}\frac{\Delta_\ell^{n-1}}{\ell^{n-1}} + \cdots + a_0\right)u(k) = e_1(tk)_{t\ell}, \tag{4.1}$$

where $a_i's$ for $i = 1, 2, 3, ..., n$ are constants. Now we consider homogenous equation

$$\left(a_n\frac{\Delta_\ell^n}{\ell^n} + a_{n-1}\frac{\Delta_\ell^{n-1}}{\ell^{n-1}} + \cdots + a_0\right)u(k) = 0. \tag{4.2}$$

Assume that $u(k) = e_1((mk)_{(m\ell)}$ as solution of (4.2). Then we get

(4.3)
$$\left(a_n\frac{\Delta_\ell^n e_1((mk)_{(m\ell)}}{\ell^n} + a_{n-1}\frac{\Delta_\ell^{n-1}e_1((mk)_{(m\ell)}}{\ell^{n-1}} + \cdots + a_0 e_1((mk)_{(m\ell)}\right)u(k) = 0.$$

Now $\Delta_\ell e_1(mk)_{(m\ell)} = m\ell e_1(mk)_{(m\ell)}$, $\Delta_\ell^2 e_1(mk)_{(m\ell)} = (m\ell)^2 e_1(mk)_{(m\ell)}$. In general, $\Delta_\ell^n e_1(mk)_{(m\ell)} = (m\ell)^n e_1(mk)_{(m\ell)}$. Substituting the values in (4.3), we get

$$\frac{a_n}{\ell^n}(m\ell)^n e_1(mk)_{m\ell} + \frac{a_{n_1}}{\ell^{n-1}}(m\ell)^{n-1}e_1(mk)_{m\ell} + \cdots + a_0 e_1(mk)_{m\ell} = 0,$$

which gives

$$\left(\frac{a_n}{\ell^n}(m\ell)^n + \frac{a_{n_1}}{\ell^{n-1}}(m\ell)^{n-1} + \cdots + a_0\right) = 0. \tag{4.4}$$

The auxiliary equation for (4.4) is obtained as

$$a_n m^n + a_{n-1}m^{n-1} + \cdots + a_0 = 0. \tag{4.5}$$

Therefore, suppose that m is a root of (4.5), $e_1(mk)_{(m\ell)}$ is solution of (4.2). To find particular solution, since

$$\Delta_\ell e_1(tk)_{t\ell} = e_1(tk)_{(t\ell)}(\Delta_\ell e_1(tk)_{(t\ell)} - 1), \Delta_\ell^2 e_1(tk)_{(t\ell)} = e_1(tk)_{(t\ell)}(\Delta_\ell e_1(tk)_{(t\ell)} - 1)^2,$$

and in general, $\Delta_\ell^n e_1(tk)_{(t\ell)} = e_1(tk)_{(t\ell)}(\Delta_\ell e_1(tk)_{(t\ell)} - 1)^n$, we get

$$\left[a_n\Delta_\ell^n + a_{n-1}\Delta_\ell^{n-1} + \cdots + a_0\right]$$
$$\left\{\frac{e_1(tk)_{(t\ell)}}{a_n e_1((t\ell)-1)^n + a_{n-1}e_1((t\ell)-1)^{n-1} + \cdots + a_0}\right\} = e_1(tk)_{(t\ell)}.$$

Hence the particular solution of (4.1) is obtained as

$$u(k) = \frac{e_1(tk)_{(t\ell)}}{a_n e_1((t\ell)-1)^n + a_{n-1}e_1((t\ell)-1)^{n-1} + \cdots + a_0}.$$

Case 1 : Suppose zeros are real and different,then the complementary function for (4.1) is $u(k) = A_1e_1(m_1k)_{(m_1\ell)} + A_2e_1(m_2k)_{(m_2\ell)} + \cdots + A_ne_1(m_2k)_{(m_n\ell)}$, where A_i are are constants, for all i=0,1,2,$\cdots n$. Therefore the general solution of (4.1) is

$$\begin{aligned} u(k) = &\left[A_1e_1(m_1k)_{(m_1\ell)} + A_2e_1(m_2k)_{(m_2\ell)} + \cdots + A_ne_1(m_nk)_{(m\ell)}\right] \\ &+\frac{e_1(tk)_{(t\ell)}}{a_n e_1((t\ell)-1)^n + a_{n-1}e_1((t\ell)-1)^{n-1} + \cdots + a_0}. \end{aligned} \tag{4.6}$$

Case 2 : Suppose the roots are real and same then the general solution of (4.1) is

$$\begin{aligned} u(k) = &\left[A_n + A_{n-1}(mk)_{(m\ell)}^{(n-1)} + A_{n-2}(mk)_{(m\ell)}^{(n-2)} + \cdots + A_1(mk)_{(}m\ell)^{(1)}\right] \\ &\cdot e_1(mk)_{(m\ell)} + \frac{e_1(tk)_{(t\ell)}}{a_n e_1((t\ell)-1)^n + a_{n-1}e_1((t\ell)-1)^{n-1} + \cdots + a_0}. \end{aligned} \tag{4.7}$$

The following example illustration (4.6) and (4.7).

Example 4.1. *Consider the linear homogeneous difference equation*

$$\left(\frac{\Delta_\ell^2}{\ell^2} - 4\frac{\Delta\ell}{\ell} + 3\right)u(k) = 0. \tag{4.8}$$

The auxiliary equation is $m^2 - 4m + 3 = (m-1)(m-3) = 0$. *Therefore roots are* $m_1 = 1$ *and* $m_2 = 3$ *and* (4.8) *has a solution. From case 1:*

$$u(k) = Ae_1((k)_\ell) + Be_1((3k)_{(3\ell)}). \tag{4.9}$$

Example 4.2. *Consider the* 3^{rd} *order linear non-homogeneous difference equation*

$$\frac{\Delta_\ell^3u(k)}{\ell^3} - 3\frac{\Delta_\ell^2u(k)}{\ell^2} + 3\frac{\Delta_\ell u(k)}{\ell} - u(k) = e_1(tk)_{t\ell}. \tag{4.10}$$

The auxilary equation of (4.10) *is given by*

$$m^3 - 3m^2 + 3m - 1 = (m-1)^3 = 0.$$

So roots are $m = (1, 1, 1)$ *that is real and same. Therefore the general function is*

$$\begin{aligned} u(k) &= \left[A + B(k)_\ell^{(1)} + C(k)_\ell^{(\nu)}\right] e_1(k_\ell) \\ &+ \frac{e_1(tk)_{(t\ell)}}{a_n e_1((t\ell)-1)^n + a_{n-1} e_1((t\ell)-1)^{n-1} + \cdots + a_0}. \end{aligned}$$

5. Conclusion

The ℓ-extorial function has played an extra ordinary role in the field of difference equation. The ℓ-extorial function and its derivations are discussed through the solution of n^{th} order difference equation. Here, we derived solutions for the integer order difference equation. This research work may be extended to functional order difference equations Discrete Fractional Calculus.

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