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TESTING BHNBUE CLASS OF LIFE-TIME DISTRIBUTION BASED ON MOMENT INEQUALITY

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ABSTRACT. In this paper, a new test statistic, for testing BHNBUE class of life-time distribution based on the moment inequality is proposed. This inequality demonstrates that, if the mean life is finite, then all higher order moments exist. Using Monte Carlo method, critical values of the proposed test is calculated for $n = 6(1)40$ and tabulated. Finally, application to real-life data are carried out.

1. INTRODUCTION

Ageing Classes of life-time distributions are defined to categorize the life-time distributions according to their ageing properties. The main aim of constructing new tests is to gain higher efficiencies. Testing bivariate exponentiality against some classes of life-time distributions have been introduced by various researchers from different point of views. Testing bivariate exponentiality against BHNBUE ageing class of life-time distribution has seen a good deal of attention. For testing bivariate exponentiality against BHNBUE alternative can be found in the work of Kanwar Sen and Madhu Bala Jain in [2]. Now we propose a test statistic testing BHNBUE class against bivariate exponential distribution, based on the moment inequality.

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The rest of paper is arranged as follows; In section 2, the preliminaries required for further discussion is given. In section 3, moment inequality for the Bivariate Harmonic New Better than Used in Expectation (BHNBUE) class of life-time distribution is derived. A new test statistic for testing BHNBUE class against bivariate exponential distribution, based on the moment inequality is proposed in section 4. Using Monte Carlo Method critical values of the proposed test statistic are calculated for $n = 6(1)40$ and tabulated in section 5. The application of the proposed test to real data sets is discussed in section 6. Finally, conclusion is given in section 7.

2. DEFINITIONS

Let (X, Y) denote the survival time of a device having a joint distribution function $F(x, y)$. The bivariate joint survival function is given by

$$\overline{F}(x,y) \;=\; P(X > x, Y > y), \qquad\qquad x, y \geq 0,$$

where it is assumed that $\overline{F}(0,0) = 1$.
The following definitions of Bivariate ageing classes of life-time distributions appeared in [5].

Definition 2.1. *A bivariate random variable (X, Y) or its distribution $\overline{F}(t,s)$ is said to have Bivariate Harmonic New Better than Used in Expectation (BHNBUE) ageing class, if*

$$\int_0^x \int_0^y \overline{F}(u,v)dvdu \leq \mu \exp\left[-\frac{x+y}{\mu}\right],$$

for all $x, y, t, s \geq 0$, where $\mu = \int_0^\infty \int_0^\infty \overline{F}(x,y)dydx$ denotes the mean of F and is assumed to be finite.

Definition 2.2. *The r^{th} Moment of a bivariate random variable (X, Y) is*

$$\mu_{(r)} = E(X^r Y^r) = r^2 \int_0^\infty \int_0^\infty (xy)^{r-1} \overline{F}(x,y)dydx.$$

3. MOMENT INEQUALITY

In this section, the moment inequality for BHNBUE class is derived.

Theorem 3.1. *Let F be a BHNBUE ageing class of life-time distribution such that moment of all orders exist and is finite. Then* $\frac{\mu_{(r+2)}}{(r+2)!} \leq \mu^{r+2}(r!)$ *for* $r \geq 0$.

Proof. Let $\overline{G}(x,y) = \int_0^x \int_0^y \overline{F}(u,v)dvdu$. Since F is BHNBUE, we have

$$\overline{G}(x,y) \leq \mu \exp\left[-\frac{x+y}{\mu}\right].$$

Multiplying both sides by $x^r y^r$, for $r \geq 0$ and integrating twice over $[0,+\infty)$ with respect to x and y, we get

$$\begin{aligned}
\int_0^\infty \int_0^\infty x^r y^r \overline{G}(x,y)dydx &\leq \mu \int_0^\infty \int_0^\infty x^r y^r \exp\left[-\frac{x+y}{\mu}\right] dydx \\
&= \mu \int_0^\infty x^r \exp\left(-\frac{x}{\mu}\right) dx \int_0^\infty y^r \exp\left(-\frac{y}{\mu}\right) dy \\
&= \mu^{r+2}(r!)^2
\end{aligned}$$

$$\begin{aligned}
&\int_0^\infty \int_0^\infty x^r y^r \overline{G}(x,y)dydx \\
&= E\Big[\int_0^\infty \int_0^\infty x^r y^r (X-x)(Y-y) \times I(X>x)I(Y>y)dydx\Big] \\
&= E\Big[\Big(\int_0^\infty x^r (X-x)I(X>x)dx\Big) \times \Big(\int_0^\infty y^r (Y-y)I(Y>y)dy\Big)\Big] \\
&= E\Big[\Big(X\int_0^X x^r dx - \int_0^X x^{r+1}dx\Big) \times \Big(Y\int_0^Y y^r dy - \int_0^Y y^{r+1}dy\Big)\Big] \\
&= E\left[\left(X\frac{x^{r+1}}{r+1}\Big|_0^X - \frac{x^{r+2}}{r+2}\Big|_0^X\right)\left(Y\frac{y^{r+1}}{r+1}\Big|_0^Y - \frac{y^{r+2}}{r+2}\Big|_0^Y\right)\right] \\
&= E\left[X^{r+2}\left(\frac{1}{r+1} - \frac{1}{r+2}\right) Y^{r+2}\left(\frac{1}{r+1} - \frac{1}{r+2}\right)\right] \\
&= E\left[X^{r+2}Y^{r+2}\right]\left[\frac{r+2-r-1}{(r+1)(r+2)}\right] \\
&= \frac{\mu_{(r+2)}}{(r+1)(r+2)}.
\end{aligned}$$

Therefore $\frac{\mu_{(r+2)}}{(r+1)(r+2)} \leq \mu^{r+2}(r!)$ (or) $\frac{\mu_{(r+2)}}{(r+2)!} \leq \mu^{r+2}(r!)$.
This completes the proof of theorem. □

Remark 3.1. *For $r = 0$, the above inequality reduces to* $\frac{\mu_{(2)}}{2!} \leq \mu^2$,*(or)* $\mu_{(2)} \leq 2\mu^2$.

4. Testing against BHNBUE Alternatives

Using the above inequality, we test the null hypothesis
$H_0 : F$ is bivariate exponential aganist
$H_1 : F$ is BHNBUE and not bivariate exponential.
Consider the bivariate exponential distribution introduced by Marshall and Olkin in [4], given by
$\overline{F}(x, y) = \exp(-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x, y))$, for all $x, y, \lambda_1, \lambda_2 > 0$ *and* $\lambda_{12} \geq 0$, where

$$\lambda_1 = \frac{\mu_1 + \mu_2}{\mu_{12}} - \frac{1}{\mu_2}, \quad \lambda_2 = \frac{\mu_1 + \mu_2}{\mu_{12}} - \frac{1}{\mu_1}, \lambda_{12} = \left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right) \frac{\mu_{12} - \mu_1\mu_2}{\mu_{12}}$$

$$\mu_1 = \int_0^\infty \overline{F}(x, 0)dx, \mu_2 = \int_0^\infty \overline{F}(0, y)dy \quad \text{and} \quad \mu_{12} = \int_0^\infty \int_0^\infty \overline{F}(x, y)dxdy.$$

Define $\delta_E = 2\mu^2 - \mu_{(2)} \geq 0$.
Note that under H_0, $\delta_E = 0$, while under $H_1, \delta_E > 0$. Let $(X_1, X_2), \ldots, (X_{i-1}, X_i)$ $, \ldots, (X_n, X_{n+1})$ be a random sample from a distribution F. The empirical estimate $\widehat{\delta}_E$ of δ_E can be obtained as

$$\widehat{\delta}_E = \frac{1}{n^2} \sum_i \sum_j \{2X_i^2 - X_j^2\} = \frac{1}{n^2} \sum_i \sum_j \phi(X_i, X_j)$$

where $\phi(X_i, X_j) = 2X_i^2 - X_j^2$, to make the test Statistic scale invariant, let

$$\widehat{\Delta} = \frac{\widehat{\delta}_E}{\overline{X}^2} .$$

Define the symmetric kernel $\eta(X_i, X_j) = \frac{1}{2!} \sum \phi(X_i, X_j)$, where the sum is taken over all arrangement of X_i, X_j. Then $\widehat{\Delta}$ is equivalent to the classical U-statistic, [3] and is given by $U_n = \frac{1}{\binom{n}{2}} \sum_{i<j} \eta(X_i, X_j)$. The asymptotic normality of $\widehat{\Delta}$ is summarized in the following theorem.

Theorem 4.1. *As* $n \to \infty, \sqrt{n}(\widehat{\Delta} - \delta_E)$ *is asymptotically normal with mean 0 and variance* σ^2, *where* $\sigma^2 = Var\{2X^2 + 2\mu_2 - X^2 - \mu_2\}$. *Under* H_0, *the variance reduces to* $\sigma^2 = Var\{X^2 - 2\} = 20$.

Proof. From the standard theory of U-Statistics, [3], $\sigma^2 = Var\{\varsigma(X_i, X_j)\}$, where $\varsigma(X_i, X_j) = E[\phi(X_1, X_2) \mid X_1] + E[\phi(X_1, X_2) \mid X_2]$

$E[\phi(X_1, X_2) \mid X_1] = 2X_1^2 - \mu_2, \quad E[\phi(X_1, X_2) \mid X_2] = 2\mu_2 - X_2^2$ and
$\varsigma(X_i, X_j) = 2X_1^2 + 2\mu_2 - X_2^2 - \mu_2.$
Under H_0, $\varsigma_0(X_i, X_j | X_i = X_j) = X^2 - 2$.
From the above equation, it is clear that $E[\varsigma_0(X_i, X_j)] = 0$ and
$\sigma^2 = E[(\varsigma_0(X_i, X_j))^2] = 20.$ □

Corollary 4.1. *Under H_0, the limiting distribution of U_n is normal with mean $\widehat{\Delta}$. The variance of $\sqrt{n}(U_n)$ is a function of λ_1, λ_2 and λ_{12}.*

Proof. Since the variances of $\sqrt{n}(U_n)$ is very complicated under H_0 and since U_n is a function of U -statistic, jackknifing would not only reduce the bias, but also enable us to estimate the variance of $V(\sqrt{n}U_n)$.
The estimate of $V(\sqrt{n}U_n)$ is $\widehat{V}(\sqrt{n}U_n) = \dfrac{n}{n-1}\sum_{i=1}^{n}[U_{n,i} - U_n^*]^2$, where,

$$U_{n,i} = U_{n-1}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) \text{ and } U_n^* = \frac{1}{n}\sum_{i=1}^{n} U_{n,i}$$

Using the results from [6], $\dfrac{\sqrt{n}(U_n)}{[\widehat{V}(\sqrt{n}U_n)]^{\frac{1}{2}}} \sim N(0,1)$ asymptotically. □

5. Monte Carlo Simulations

In this section the Monte Carlo null distribution critical points of $\widehat{\Delta}$ are simulated based on 10000 generated samples of size $n = 6(1)40$. Table 1 gives the upper percentile points of statistic $\widehat{\Delta}$ for different confidence levels $90\%, 95\%, 98\%$ and 99%.It is clear from Table 1 and Figure 1, that the critical values are increase as the confidence level increases and are almost increases as the sample size increase.

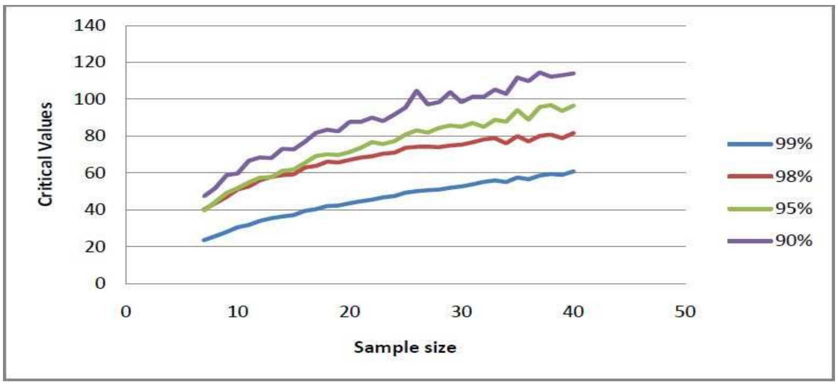


TABLE 1. Critical Values of the statistic $\widehat{\Delta}$

n	90%	95%	98%	99%	n	90%	95%	98%	99%
6	21.194	26.888	35.547	43.401	24	47.839	59.865	77.976	93.944
7	23.155	30.351	39.714	47.533	25	49.597	62.710	80.864	95.969
8	26.380	34.206	45.740	54.679	26	49.790	63.410	81.698	98.888
9	28.327	36.931	48.068	56.343	27	50.356	63.414	81.513	93.798
10	30.512	38.767	52.539	63.335	28	51.799	65.444	85.390	101.14
11	31.950	41.507	54.307	65.870	29	51.747	65.230	84.089	100.56
12	34.145	43.369	56.536	68.492	30	52.912	66.019	84.738	98.442
13	35.012	44.145	57.466	69.062	31	54.294	68.769	87.709	102.760
14	36.084	46.042	60.769	70.627	32	55.056	68.010	86.783	102.150
15	37.097	47.352	62.172	72.584	33	55.595	68.879	87.989	103.096
16	40.113	50.910	66.199	79.778	34	55.583	69.922	89.747	105.302
17	40.624	51.623	68.124	81.570	35	57.483	71.927	91.536	110.573
18	41.736	52.105	66.371	80.325	36	56.969	69.788	90.503	109.840
19	42.325	53.159	69.741	83.958	37	59.238	73.840	96.568	111.364
20	43.748	55.542	71.378	86.869	38	59.836	75.032	95.964	111.711
21	44.372	56.471	74.017	88.644	39	58.415	73.286	93.419	110.487
22	45.816	57.718	73.481	86.673	40	61.239	75.556	96.664	115.024
23	46.500	58.401	75.030	89.678					

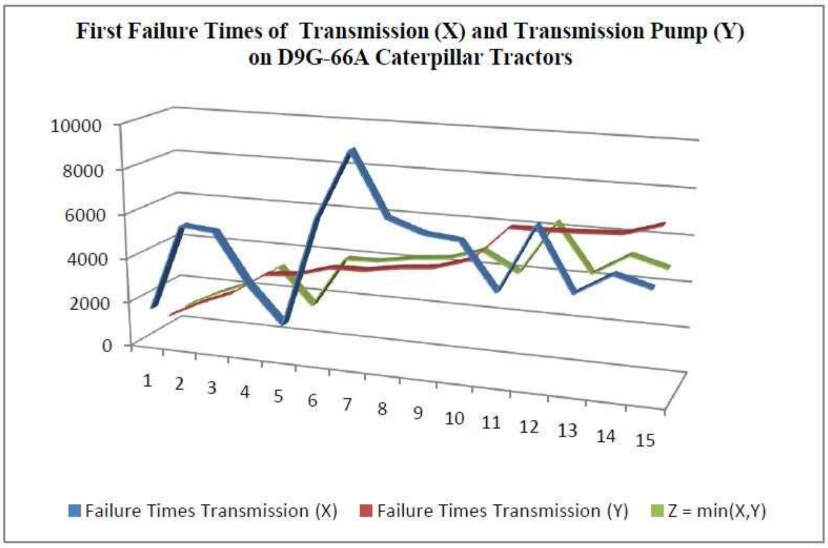
6. APPLICATION TO REAL-LIFE DATA

Here, we present a real life example to illustrate the use of our test statistics $\widehat{\Delta}$. We consider the example given by Barlow and Proschan in [1], which is a list of paired first failure time (in hours) of the transmission and the transmission pump on 15 Caterpillar tractors. We use our test to detect whether these failure times follow a bivariate exponential distribution.

Table 2: First failure times of transmission (X) and transmission pump (Y) on D9G-66A Caterpillar Tractors.

Tractor Number	X	Y	$\min(X, Y)$
1	1641	850	850

2	5556	1607	1607
3	5421	2225	2225
4	3168	3223	3168
5	1534	3379	1534
6	6367	3832	3832
7	9460	3871	3871
8	6679	4142	4142
9	6142	4300	4300
10	5995	4789	4789
11	3953	6310	3953
12	6922	6310	6310
13	4210	6378	4210
14	5161	6449	5161
15	4732	6949	4732



Using the data in Table 2, we obtained, the value of the test statistic $\widehat{\Delta} = 623.4667$.

Thus $\dfrac{\sqrt{n}(U_n)}{[\widehat{V}(\sqrt{n}U_n)]^{\frac{1}{2}}} = 1.219678 \times 10^4$.

Hence, we reject H_0 and conclude in favour of BHNBUE. Then we accept H_1 which shows that the data set has BHNBUE property, but not Bivariate exponential.

7. CONCLUSION

The BHNBUE class of life-time distribution is considered. The moment inequality is derived. A new test statistic for testing BHNBUE class of life-time distributions is proposed based on the moment inequality. Using Monte Carlo Method, critical values of the proposed test is calculated for $n = 6(1)40$ and tabulated. Finally, application to real-life data is carried out.

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