

PYTHAGOREAN NANO CONTINUITY

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ABSTRACT. The objective of this paper is to introduce the Pythagorean nano continuous function between Pythagorean nano topological spaces and develop their corresponding characterizations.

1. INTRODUCTION

L.A. Zadeh in [1] established the idea of fuzzy in 1965, which is a generalization of usual set using fuzzy where each element has a membership degree in $[0,1]$. The subsequent advancement of fuzzy subsets was the intuitionistic fuzzy set published by Atanassov, [2] in 1983, which has elements having membership and non-membership degree.

In 1968, Chang in [3] defined fuzzy topological space and fundamental results such as continuity, open and closed set. Following this, Lowen in [4] defined fuzzy topological space in other form. Coker introduced the idea of intuitionistic fuzzy topological space with few properties, see [5].

The concept of Pythagorean fuzzy subset which is atypical fuzzy subset was presented by Yager, see [6, 7]. Pythagorean fuzzy topological space was introduced by Olgun in [8] by taking the lead as from Chang.

L. Thivagar, [9] established the idea of nano topological spaces. Intuitionistic nano topology was introduced by Ramachandran and Stephan, [10] in

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2017. In addition, L.Thivagar introduced nano topology through neutrosophic units, [11]. The concept of Pythagorean nano topology with its properties were established by D. Ajay and J. J. Charisma, [12].

2. PRELIMINARIES

A subset of a non-void set X is said to be a fuzzy subset if it is as $A = \{\langle a, \rho_A(a) \rangle, a \in X\}$ with $\rho_A : X \rightarrow [0, 1]$ (membership degree).

A subset $B = \{\langle b, \vartheta_B(b), \mu_B(b) \rangle, b \in X\}$ with $\vartheta_B : X \rightarrow [0, 1], \mu_B : X \rightarrow [0, 1]$ (membership and nonmembership degree) is intuitionistic fuzzy subset satisfying $\vartheta(b) + \mu(b) \leq 1$.

A non-empty subset C of X of the form $C = \{\langle c, \vartheta_C(c), \mu_C(c) \rangle, c \in X\}$ with $\vartheta_C : X \rightarrow [0, 1], \mu_C : X \rightarrow [0, 1]$ as membership and non-membership degree satisfying $\vartheta^2(c) + \mu^2(c) \leq 1$ is named as Pythagorean fuzzy subset.

The duo (U, R) is said as approximation space where U is a non-void set called as universe and R an equivalence relation. Let $K \subseteq U$.

The lower approximation of K with respect to R is denoted by $L_R(x)$. $L_R(x) = R(x) : R(x) \subseteq K, x \in U$, $R(x)$ is the equivalence relation determined by way of x .

The upper approximation is denoted as $U_R(x)$ and described as $U_R(x) = \cup R(x) : R(x) \cap K \neq \emptyset, x \in U$. The boundary of K , $B_R(x) = U_R(x) - L_R(x)$.

Let U be Universe with equivalence relation

R , $\tau_R(K) = \{\emptyset, U, L_R(x), U_R(x), B_R(x)\}$ in which $K \subseteq U$. $\tau_R(K)$ satisfies the axioms:

- (1) $U, \emptyset \in \tau_R(K)$;
- (2) union of elements of $\tau_R(K)$ is in $\tau_R(K)$;
- (3) intersection of finite sub-collection of $\tau_R(K)$ is in $\tau_R(K)$.

$(U, \tau_R(K))$ is termed the Nano topological space.

A non-empty set J with the collection of Pythagorean fuzzy subsets ρ is called Pythagorean fuzzy topological space if the following axioms are satisfied:

- (1) $1_X, 0_X \in \rho$;
- (2) for any $S_1, S_2 \in \rho$, $S_1 \cap S_2 \in \rho$;
- (3) for any $S_i \in \rho, i \in I$ for all i , $\cup S_i \in \rho$ where I is an arbitrary index set.

The pair (V, R) is said to be a Pythagorean approximation space with D a Pythagorean set in V having θ_D, ω_D as membership and non-membership degree.

The Pythagorean nano lower, Pythagorean nano upper and Boundary of D are:

$$PNL_R(D) = \{\langle x, \theta_{LD}, \omega_{LD}/z \in [x]_R, x \in V \rangle\},$$

$$PNU_R(D) = \{\langle x, \theta_{RD}, \omega_{RD}/z \in [x]_R, x \in V \rangle\},$$

$$PNB_R(D) = PNU_R(D) - PNL_R(D)$$

where

$$\theta_{LD}(x) = \bigwedge_{y \in [x]_R} \theta_{LD}(y), \omega_{LD}(x) = \bigvee_{y \in [x]_R} \omega_{LD}(y) \text{ and } \theta_{RD}(x) = \bigvee_{y \in [x]_R} \theta_{RD}(y), \\ \omega_{RD}(x) = \bigwedge_{y \in [x]_R} \omega_{RD}(y), \text{ respectively, see [12].}$$

The pair (V, R) along with $\tau_R(Y) = \{\emptyset_P, V_P, PNL_R(Y), PNU_R(Y), PNB_R(Y)\}$ is said to be Pythagorean nano topological space while satisfying these conditions $\emptyset_P, V_P \in \tau_R(Y)$, if $A_i \in \tau_R(Y)$ for $i=1,2,\dots$ then arbitrary union of A_i is in $\tau_R(Y)$, if $A_i \in \tau_R(Y)$ for $i=1,2,\dots,n$ then finite intersection of A_i is in $\tau_R(Y)$ where $\emptyset_P = \{\langle x, 0, 1/\forall x \in V \rangle\}, V_P = \{\langle x, 1, 0/\forall x \in V \rangle\}$, [12].

3. PYTHAGOREAN NANO CONTINUITY

Definition 3.1. Let A and U be two Pythagorean Fuzzy subsets (PF subsets) of a Pythagorean Nano Topological space (PNT space), then U is said to be a neighbourhood of A if there exists a Pythagorean fuzzy open subset E such that $A \subset E \subset U$.

Definition 3.2. Let (U, R) and (V, R') be the approximation spaces, X and Y be PF subsets of U and V and let $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(Y))$ be a function. The membership and non-membership functions of image of X with respect to f is defined as

$$\mu_{f(X)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \mu_X(x), & \text{if } f^{-1}(y) \text{ is non-empty} \\ 0, & \text{otherwise} \end{cases}$$

$$v_{f(X)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} v_X(x), & \text{if } f^{-1}(y) \text{ is non-empty} \\ 0, & \text{otherwise} \end{cases}$$

The membership and non-membership functions of pre-image of Y with respect to f is denoted by $f^{-1}(Y)$ and is defined as

$$\mu_{f^{-1}(X)}(y) = \mu_Y(f(y))$$

$$v_{f^{-1}(X)}(y) = v_Y(f(y))$$

Proposition 3.1. Let (U, R) and (V, R') be the approximation spaces and $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(X))$, then

- (1.) $f^{-1}(Y') = f^{-1}(Y)'$ for any PN subset Y of V .
- (2.) $f^{-1}(X') \subset f^{-1}(Y)'$ for any PN subset X of U .
- (3.) If $Y_1 \subset Y_2$ then $f^{-1}(Y_1) \subset f^{-1}(Y_2)$ where Y_1 and Y_2 are PN subsets of V .
- (4.) If $X_1 \subset X_2$ then $f^{-1}(X_1) \subset f^{-1}(X_2)$ where X_1 and X_2 are PN subsets of U .
- (5.) $f(f^{-1}(Y)) \subset Y$ for any PN subset Y of V .
- (6.) $X \subset f(f^{-1}(X))$ for any PN subset X of U .

Definition 3.3. Let $(U, \tau_R(X))$ and $(V, \tau_{R'}(X))$ be two PNT spaces and let $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(Y))$ be a function. Then f is said to Pythagorean Nano Continuous (PNCN) on U if the inverse image of every PNO in V is PNO in U .

Theorem 3.1. A function $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(Y))$ is PNCN iff the inverse image of every Pythagorean Nano closed (PNC) set in V is PNC in U .

Proof. Let f be a PNCN function and C be a PNC set in V . We know that $V - C$ is PNO in V . Since f is PNCN, $f^{-1}(V - C)$ is PNO in U , $U - f^{-1}(C)$ is PNO in U . Therefore $f^{-1}(C)$ is PNC in U .

Conversely let us assume that the inverse image of every PNC set in V is PNC in U . Let A be a PNO set in V , then $V - A$ is PNC in V . By assumption we have $f^{-1}(V - A)$ is PNC in U . Thus $U - f^{-1}(A)$ is PNC in U . Therefore $f^{-1}(A)$ is PNO in U . Thus, the inverse image of every PNO set in V is PNO in U . Hence f is PNCN on U . $\square$

Theorem 3.2. A function $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(Y))$ is PNCN if and only if $f(PNcl(A)) \subseteq PNcl(f(A))$ for every subset A of U .

Proof. Let f be a PNCN function and $A \subseteq U$. Since f is PNCN and $PNC(f(A))$ is PNC in V , $f^{-1}(PNC(f(A)))$ is PNC in U . Since $f(A) \subseteq PNC(f(A))$, $A \subseteq f^{-1}(PNC(f(A)))$. Thus $f^{-1}(PNC(f(A)))$ is a PNC containing A . $PNC(f(A))$ is the smallest PNC containing A . Therefore $PNC(A) \subseteq f^{-1}(PNC(f(A)))$. That is, $f(PNC(A)) \subseteq PNC(f(A))$.

Conversely, let $f(PNC(A)) \subseteq PNC(f(A))$ for every subset A of U . If F is a PNC in V , since $f^{-1}(F) \subseteq U$, $f(PNC(f^{-1}(F))) \subseteq PNC(f(f^{-1}(F))) = PNC(F)$. But $f^{-1}(F) \subseteq PNC(f^{-1}(F))$, thus $PNC(f^{-1}(F)) = f^{-1}(F)$. Therefore $f^{-1}(F)$ is a PNC in U for every PNC F in V . Therefore f is PNCN. $\square$

Theorem 3.3. A function $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(Y))$ is PNCN iff $PNcl(f^{-1}(B)) \subseteq f^{-1}(PNcl(B))$ for every subset B of V .

Theorem 3.4. A function $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(Y))$ is PNCN on U iff $f^{-1}(PNint(B)) \subseteq PNint(f^{-1}(B))$ for every subset B of V .

Proof. Let f be a PNCN function and $B \subseteq V$. Then $PNint(B)$ is PNO in $(V, \tau_{R'}(Y))$. Therefore $f^{-1}(PNint(B))$ is PNO in $(U, \tau_R(X))$, and $f^{-1}(PNint(B)) = PNint(f^{-1}(PNint(B)))$. We have $f^{-1}(PNint(B)) \subseteq f^{-1}(B)$. Thus, $PNint(f^{-1}(PNint(B))) \subseteq PNint(f^{-1}(B))$. Therefore, $f^{-1}(PNint(B)) \subseteq PNint(f^{-1}(B))$.

Conversely, let us assume $f^{-1}(PNint(B)) \subseteq PNint(f^{-1}(B))$ for every subset B of V . If B is a PNO in V , $PNint(B) = B$, and also $f^{-1}(PNint(B)) \subseteq PNint(f^{-1}(B))$, thus we get $f^{-1}(B) \subseteq PNint(f^{-1}(B))$. But $PNint(f^{-1}(B)) \subseteq f^{-1}(B)$, thus $f^{-1}(B) = PNint(f^{-1}(B))$. Thus $f^{-1}(B)$ is PNO in U for every PNO B in V . Therefore f is PNCN. $\square$

Definition 3.4. A subset A of a PNT space $(U, \tau_R(X))$ is called as Pythagorean nano dense (PND) if $PNcl(A) = U$.

Definition 3.5. A function $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(Y))$ is PNO if the image of every PNO set in U is PNO in V .

The function is said to be Pythagorean nano closed (PNC) if the image of every PNC set in U is PNC in V .

Theorem 3.5. A function $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(Y))$ is PNC if and only if $PNcl(f(A)) \subseteq f(PNcl(A))$ for every subset A of U .

Proof. If f is PNC, then $f(PNcl(A))$ is also PNC in V , since $PNcl(A)$ is PNC in U . Since $A \subseteq PNcl(A)$, $f(A) \subseteq f(PNcl(A))$. Thus $f(PNcl(A))$ is PNC set containing $f(A)$. Therefore $PNcl(f(A)) \subseteq f(PNcl(A))$.

Conversely, let us consider $PNcl(f(A)) \subseteq f(PNcl(A))$ for every subset A of U and if F is PNC in U , then $PNcl(F) = F$ and hence $f(F) \subseteq PNcl(F) \subseteq f(PNcl(A)) = f(F)$. Thus $f(F) = PNcl(f(F))$ and hence $f(F)$ is PNC in V . $\square$

Theorem 3.6. A function $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(Y))$ is PNO if and only if $PNcl(f(A)) \subseteq f(PNcl(A))$ for every subset A of U .

Definition 3.6. A function $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(Y))$ is named as Pythagorean nano homeomorphism (PNH) if f is bijective and Pythagorean bi-continuous.

Theorem 3.7. Let $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(Y))$ be a bijective function. Then f is PNH iff f is PNC and PNCN.

Proof. Let f be a PNH. By the definition, f is PNCN. Let F be an arbitrary PNC set in $(U, \tau_R(X))$. Since F is PNC, $U - F$ is PNO. f is PNO, thus $f(U - F)$ is PN open in V . Thus $f(F)$ is PNC in V . The image of every PNC set in U is PNC in V . Hence f is Pythagorean nano closed.

Conversely let f be PNC and PNCN. Let G be a PNO set in U . Then $U - G$ is PNC in U . Since f is PNC, $f(U - G) = V - f(G)$ is PNC in V . Thus $f(G)$ is PNO in V . Thus f is PNO and hence f is PNH. $\square$

Theorem 3.8. An one-one function f of $(U, \tau_R(X))$ onto $(V, \tau_{R'}(Y))$ is a PNH iff $f(PNcl(A)) = PNcl(f(A))$ for every subset A of U .

Proof. If f is PNH, f is PNCN and PNC. If $A \subseteq U$, f is a PNCN, $f(PNcl(A)) \subset PNcl(f(A))$. Since $PNcl(A)$ is PNC in U and f is PNC, $f(PNcl(A))$ is PNC in V . This implies that f is PNH.

Conversely, assume that $f(PNcl(A)) = PNcl(f(A))$ for every subset A of U , then f is PNCN. If A is PNC in U , $PNcl(A) = A$ which implies $f(PNcl(A)) = f(A)$. Hence f is PNCN. Therefore f is a PNH. $\square$

4. EXTREMALLY DISCONNECTED PYTHAGOREAN NANO TOPOLOGY

In this section, we define extremely disconnected PNT spaces in terms of Pythagorean Nano closure ($PNclosure$) of a PNO set and derive the necessary and sufficient conditions for a PNT space to be extremely disconnected.

Definition 4.1. A Pythagorean nano topological space $(U, \tau_R(X))$ is extremely disconnected, if the PN closure of each Pythagorean nano open set is Pythagorean nano open in U .

Theorem 4.1. A PNT space $(U, \tau_R(X))$ is extremely disconnected if and only if $PNU_R(X) = U_p$.

Proof. Let $(U, \tau_R(X))$ be extremely disconnected. Thus the PN closure of all PNO sets are PNO sets in U . It is known that $U_p, \emptyset_p, PNL_R(X), PNB_R(X)$ are PNC. Thus $PNcl(PNL_R(X)) = PNL_R(X)$ and $PNcl(PNB_R(X)) = PNB_R(X)$. That is, $(PNL_R(X))^c = PNB_R(X)$, $(PNB_R(X))^c = PNL_R(X)$ and

$$\begin{aligned} PNB_R(X) &= PNU_R(X) - PNL_R(X) \Rightarrow \\ (PNL_R(X))^c &= PNU_R(X) - PNL_R(X). \end{aligned}$$

Thus $(PNL_R(X))^c = PNU_R(X) \cap (PNL_R(X))^c \Rightarrow (PNL_R(X))^c \subseteq PNU_R(X)$. Since $PNL_R(X) \subseteq PNU_R(X)$, $(PNU_R(X))^c \subseteq (PNL_R(X))^c$. Therefore, $(PNU_R(X))^c \subseteq (PNL_R(X))^c \subseteq (PNU_R(X))$. This is possible only if $(PNU_R(X))^c = \emptyset_p$. That is, $PNU_R(X) = U_p$.

Conversely, let $PNU_R(X) = U_p$. If $PNL_R(X) = \emptyset_p$, then $\tau_R(X) = \{U_p, \emptyset_p\}$ and PN closure of each PNO set is obviously PNO. If $PNL_R(X) \neq \emptyset_p$, then $\tau_R(X) = \{U_p, \emptyset_p, PNL_R(X), PNB_R(X)\}$ where $(PNL_R(X))^c = PNB_R(X)$ hence $\tau_R(X) = \{U_p, \emptyset_p, PNL_R(X), (PNL_R(X))^c\}$. Therefore, each set in PNT is both PNO and PNC and hence PN closure of each Pythagorean nano open is PNO. Hence, $(U, \tau_R(X))$ is extremely disconnected. $\square$

Theorem 4.2. *If $PNL_R(X) = PNU_R(X)$ where $X \subseteq U$, then $(U, \tau_R(X))$ is extremely disconnected.*

Proof. Since $PNL_R(X) = PNU_R(X)$, $PNB_R(X) = \emptyset_p$ and hence $\tau_R(X) = \{U_p, \emptyset_p, PNL_R(X)\}$ where $PNcl(U_p) = U_p$, $PNcl(\emptyset_p) = \emptyset_p$ and $PNcl(PNL_R(X)) = (PNL_R(X))^c = \emptyset_p$. That is, the PN closure of each PNO set in U is PNO. Thus $(U, \tau_R(X))$ is extremely disconnected. $\square$

Theorem 4.3. *If $PNL_R(X) = \emptyset_p$, then $(U, \tau_R(X))$ is extremely disconnected.*

Proof. Since $PNL_R(X) = \emptyset_p$, $PNB_R(X) = PNU_R(X)$. Therefore, $\tau_R(X) = \{U_p, \emptyset_p, PNU_R(X)\}$ where $PNcl(U_p) = U_p$, $PNcl(\emptyset_p) = \emptyset_p$, and

$$PNcl(PNU_R(X)) = U_p.$$

Thus, the PN closure of each PNO set is PNO in U . Hence, $(U, \tau_R(X))$ is extremely disconnected. $\square$

Definition 4.2. *In a PNT space, a Pythagorean set is called as Pythagorean clopen if it is both Pythagorean nano open and closed.*

5. CONCLUSION

Herein, the continuity of Pythagorean nano topology was studied. And along with it the different characterizations of continuous using Pythagorean nano closure and Pythagorean nano interior were discussed. Furthermore, weakest forms of Pythagorean nano open sets is being studied.

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