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PLANAR EMBEDDING OF $\Gamma(Z_N)$ IN ONE-PAGE-BOOK EMBEDDING

MARIA SAGAYA NATHAN[1] AND J. RAVI SANKAR

ABSTRACT. The notion of zero divisors was started in 1988 by Beck. He introduced this idea to coloring a commutative ring by using simple graphs and also included zero to the set vertices of zero divisors. Few years later, that is in 1999 Anderson and Livingston applied slight modification to Beck's definition by restricting the vertices set only to the nonzero zero divisors of the ring. In this paper we discuss about the embedding of the planar graphs with the help of zero divisor graphs.

1. PRELIMINARIES

In this paper we consider simple, undirected and finite graphs. A graph is said to be planar if it can be embedded in the plane in which no two of its edges intersect. A null graph is a graph with no edges. A nonempty set R is said to a ***ring***, if in R there are defined two operations, denoted by $+$ and $\bullet$ respectively, such that for all a, b, c in R satisfy abelian group under addition, a semi group under multiplication and the both $+$ and $\bullet$ should satisfy the distributive law. A ring that has finite number of elements is called ***finite*** ring . A ring with commutative property under multiplication is called ***commutative ring***. That is, if the multiplication of Z_n is such that $a.b = b.a$ for every a, b in Z_n , then we call Z_n a commutative ring. If a and b are two non-zero elements of a ring Z_n such that $a.b = 0$, then $'a'$ and $'b'$ are the ***zero divisors*** of commutative ring

[1]*Corresponding author*

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Z_n. In particular, $'a'$ is a left zero divisor and $'b'$ is a right zero divisor. If a commutative ring has no zero divisors then it is called an integral domain. If a commutative ring has zero divisors then it is known as non-integral domain. If the non-zero elements of a ring is able to form a group under multiplication then that ring is known as division ring. Let $G = (V, E)$ be a graph. The vertices of G are placed in the outerplanar and labeled such that $v_i < v_j$ where $i < j$ for all i, j. Then the ***merging number*** is the minimum number of crossing between the edges $e \in E$ of G and the edge $e^{'} \in E^{'}$ of its complement $G^{'} = (V, E^{'})$ and it is denoted by $mg(G)$.

2. Planar embedding of $\Gamma(Z_n)$ in one-page-book embedding

Theorem 2.1. *For any prime $p > 5$, book embedding of $\Gamma(Z_{3p}) \geq 3r$ where r is the pre prime number of p, $3 < r < p$.*

Proof. The vertices of $\Gamma(Z_{3p})$ are placed along the spine of the book $v_i < v_j.i < j$ where v_i and v_j are vertices and i and j are integers. Since, the book contains only one page, all the edges of the graph $\Gamma(Z_{3p})$ are placed either on the spine or on the page.

Case (i): When $p = 7$.

The vertices of $\Gamma(Z_{21})$ are placed along the spine as {3,6,7,9,12,14,15,18} then the edges between 6 and 7,7 and 9,12 and 14, and 14 and 15 lies on the spine and the remaining edges lies on the page. The crossing number will be zero when all the edges of vertex 7 are drawn. Similarly, by drawing all the edges of vertex 14 the crossing number will be $3+2+1+1 = 7$. Thus the book embedding of $\Gamma(Z_{21})$ in page one is 7. That is, $bk(\Gamma(Z_{21})) = 7 \geq 3r$ where r is the pre-prime number.

Case (ii): When $p = 11$.

The vertices of $\Gamma(Z_{33})$ are placed along the spine as $\{3, 6, 9, 11, 12, 15, 18, 21, 22, 24, 27, 30\}$ then the edges between 9 and 11,11 and 12,21 and 22, and 22 and 24 lies on the spine and the remaining edges lies on the page. The crossing number will be zero when all the edges of vertex 11 are drawn. Similarly, by drawing all the edges of vertex 22 the crossing number will be $5+4+3+2+1+1+2+3 = 21$. Thus the book embedding of $\Gamma(Z_{33})$ in page one is 21. That is, $bk(\Gamma(Z_{33})) = 21 \geq 3r$ where r is the pre-prime number.

Case (iii): When $p = 13$.

The vertices of $\Gamma(Z_{39})$ are placed along the spine as $\{3, 6, 9, 12, 13, 15, 18, 21, 24, 26, 27, 30, 33, 36\}$ then the edges between 12 and 13,13 and 15,24 and 26, and 26 and 27 lies on the spine and the remaining edges lies on the page. The crossing number will be zero when all the edges of vertex 13 are drawn. Similarly, by drawing all the edges of vertex 26 the crossing number will be $7 + 6 + 5 + 4 + 3 + 2 + 1 + 1 + 2 + 3 = 34$. Thus the book embedding of $\Gamma(Z_{39})$ in page one is 34. That is, $bk(\Gamma(Z_{39})) = 34 \geq 3r$ where r is the pre-prime number.

Case (iv): When $p > 13$.

In general, for any prime $p > 13$ the vertices of $\Gamma(Z_{3p})$ is $\{p, 2p, 3, 6, 9, \dots, 3(p-1)\}$ are placed along the spine such that $v_i < v_j.i < j$ where v_i and v_j are vertices and i and j are integers. Then the book embedding of $\Gamma(Z_{3p})$ in one page is greater than $3r$,$3 < r < p$, where r is the pre-prime number. □

3. Planar embedding of $\Gamma(Z_n)$ in k pages

Theorem 3.1. *For any prime $q > p$, the planar embedding of $\Gamma(Z_{pq}) = p - 1$.*

Proof. To embed the graph $\Gamma(Z_{pq})$ in a book first place all the vertices of the graph on the spine in a definite order. This proof can be given by method of induction.

Case (i): When $p = 2$.

Now $\Gamma(Z_{pq})$ is $\Gamma(Z_{2q})$. The vertex set of $\Gamma(Z_{2q})$ is $\{2, 4, 6, \dots, 2(q-1), q\}$. Let M_1 be the set of all vertices with maximum degree. That is $M_1 = q$. Draw all the incident edges of q in the first half plane of the book. Since the cardinality of $M_1 = 1$, then all the edgeswill be exhausted or all of its vertices can be placed on the first half plane of the book. Therefore $bt(\Gamma(Z_{2q})) = 1 = 2 - 1 = p - 1$.

Case (ii): When $p = 3$.

Now $\Gamma(Z_{pq})$ is $\Gamma(Z_{3q})$. The vertex set of $\Gamma(Z_{3q})$ is $\{3, 6, 9, \dots, 3(q-1), q, 2q\}$. Let M_2 be the set of all vertices with maximum degree. That is $M_2 = \{q, 2q\}$. Draw all the incident edges of q in the first half plane of the book. Next draw all the incident edges of $2q$ in the second half plane of the book since the incident edges of q crosses with the incident edges of $2q$. Moreover the cardinality of $M_2 = 2$ and then all the edges will be exhaused or all of its edges can be placed on either first half plane or second half plane of the book. Therefore $bt(\Gamma(Z_{3q})) = 2 = 3 - 1 = p - 1$.

Case (iii): When $p = 5$.

Now $\Gamma(Z_{pq})$ is $\Gamma(Z_{5q})$. The vertex set of $\Gamma(Z_{5q})$ is $\{5, 10, 15, \ldots, 5(q-1), q, 2q, 3q, 4q\}$. Let M_3 be the set of all vertices with maximum degree of the vertex set of $\Gamma(Z_{5q})$ is $M_3 = \{q, 2q, 3q, 4q\}$. Draw all the incident edges of q in the first half plane of the book. Next draw all the incident edges of $2q$ in the second half plane of the book since the incident edges of $2q$ crosses with the incident edges of q. Continuing the above process till all the edges are placed on either one of the half plane of the book. Since the cardinality of $M_3 = 4$ then all the edges will be exhaused or all of its edges can be placed on any one of the half plane of the book. Therefore $bt(\Gamma(Z_{5q})) = 4 = 5 - 1 = p - 1$.

Case (iv): When $p > 5$,

In general, The vertex set of $\Gamma(Z_{pq})$ is $\{p, 2p, \ldots, p(q-1), q, 2q, \ldots, q(p-1)\}$. Let M be the set of all vertices with maximum degree of the vertex set of $\Gamma(Z_{pq})$ is $M = \{q, 2q, \ldots, q(p-1)\}$. Draw all the incident edges of $q, 2q, \ldots, q(p-1)$ on any one of the half plane of the book such that no two edges crosses each other in any half plane as book embedding does not allow crossing. Then all the edges can be placed in $(p-1)$ half plane of the book and moreover the cardinality of $M = p - 1$. Therefore $bt(\Gamma(Z_{pq})) = p - 1$. □

Theorem 3.2. *For any prime $p > 2$, the planar embedding of $\Gamma(Z_{2^n p}) = 2^n - 1$ where n is any positive integer.*

Proof. The vertex set of $\Gamma(Z_{2^n p})$ is $\{2, 4, \ldots, 2(2^{n-1}p - 1), p, 2p, \ldots, (2^n - 1)p\}$. To embed the graph $\Gamma(Z_{2^n p})$ in a book first place all the vertices of $\Gamma(Z_{2^n p})$ on the spine in a definite order. Let M be the set of all vertices with maximum degree of the vertex set of $\Gamma(Z_{2^n p})$ is $\{p, 2p, \ldots (2^n - 1)p\}$. Draw all the incident edges of q in the first half plane of the book. Next draw all the incident edges of $2q$ in the second half plane of the book since book embedding does not allow crossing. That is all the incident edges of q should not intersect with the incident edges of $2q$. Continuing the above process till all the edges are placed on any one of the half plane of the book. Then all the edges will be exhaused or all of its edges can be placed on any one of the $(2^n - 1)$ half plane since cardinality of $M = 2^{n-1}$. Therefore $bt(\Gamma(Z_{2^n p})) = 2^n - 1$ where n is any positive integer. □

Theorem 3.3. *For any prime $p > 2$, the planar embedding of* $\Gamma(Z_{2p^2}) = p$.

Proof. The vertex set of $\Gamma(Z_{2p^2})$ is $\{2, 4, \ldots, 2(p^2 - 1), p, 2p, \ldots, p(2p - 1)\}$. To embed the graph $\Gamma(Z_{2p^2})$ in a book first place all the vertices of $\Gamma(Z_{2p^2})$ on the spine in a definite order. Let M be the set of all vertices with maximum degree of the vertex set of $\Gamma(Z_{2p^2})$ is $\{2p, 4p, \ldots 2p(p-1), p^2\}$. Now draw all the incident edges of $2p$ in the first half plane of the book. Next draw all the incident edges of $4p$ in the second half plane of the book since book embedding does not allow crossing. That is all the incident edges of $2p$ will not intersect with the incident edges of $4p$. The above process is continued till all the edges of M are placed on any one of the half plane of the book. Since the cardinality of M is p then all the edges will be exhaused or all of its edges can be placed on any one of the p half plane of the book. Therefore $bt(\Gamma(Z_{2p^2}) = p$. □

Theorem 3.4. *For any distinct primes p, q, r, the planar embedding of* $\Gamma(Z_{pqr}) = pq - 1$.

Proof. The vertex set of $\Gamma(Z_{pqr})$ is $\{p, 2p, \ldots, p(qr-1), q, 2q, \ldots q(pr-1), r, 2r, \ldots, r(pq-1)\}$. For embedding the graph $\Gamma(Z_{pqr})$ in a book first place all the vertices of $\Gamma(Z_{pqr})$ on the spine in a definite order. Let M be the set of all vertices with maximum degree of the vertex set of $\Gamma(Z_{pqr})$ is $\{r, 2r, \ldots r(pq-1)\}$. Now draw all the incident edges of r in the first half plane of the book. Next draw all the incident edges of $2r$ in the second half plane of the book since all the incident edges of r crosses with all the incident edges of $2r$. The above process is continued till all the edges of M are placed on any one of the half plane of the book. That is all the edges will be exhaused. Moreover the cardinality of $M = pq - 1$. Then all the edges will be placed in any of $pq - 1$ half plane of the book. Therefore $bt(\Gamma(Z_{pqr})) = pq - 1$ where p, q, r are distinct primes. □

Theorem 3.5. *For any prime $p > 3$, the planar embedding of* $\Gamma(Z_{3p^2}) = p + 1$.

Proof. The vertex set of $\Gamma(Z_{3p^2})$ is $\{3, 6, ..3(p^2-1), p, 2p, \ldots p(3p^2-1)\}$. To embed the graph $\Gamma(Z_{3p^2})$ in a book first place all the vertices of $\Gamma(Z_{3p^2})$ on the spine in a definite order. Let M be the set of all vertices with maximum degree of the vertex set of $\Gamma(Z_{3p^2})$ is $\{3p, 6p, \ldots 3p(p-1), p^2, 2p^2\}$. Now draw all the incident edges of $3p$ in the first half plane of the book. Next draw all the incident edges of $6p$ in the second half plane of the book since book embedding does not allow crossing. That is all the incident edges of $3p$ will not intersect with the incident edges of $6p$ on the first half plane of the book. The above process is continued till all the incident edges of the set of vertices M are placed on any one of the half plane of the book. Moreover the cardinality of M is $p+1$ then all the edges will be exhaused or all of its edges can be placed on any one of the $p+1$ half plane of the book. Therefore $bt(\Gamma(Z_{3p^2}) = p+1$ where p is any prime. $\square$

Theorem 3.6. *For any prime $p \geq 3$, the planar embedding of $\Gamma(Z_{p^2}) = p-3$.*

Proof. The vertex set of $\Gamma(Z_{p^2})$ is $\{p, 2p, \ldots, p(p-1)\}$. Clearly p is adjacent to all the vertices. Embed the graph $\Gamma(Z_{p^2})$ in a book, wherein all the vertices taken in definite order are placed in the spine of the book. As the graph $\Gamma(Z_{p^2})$ is complete and the degree of all the vertices is $p-2$, so any vertices are arbitrarily chosen and its incident edges are drawn on the first half plane as no crossing asises. Now choose the vertex $2p$ as one incident edges of $2p$ is adjacent with the already chosen vertex p. Hence degree of $2p$ is $(p-3)$. As all the incident edges of $2p$ crosses with the incident edges of p, Now move to second half plane. Continuing the above process all the incident edges must be drawn without allowing edges crossing in any half plane. $\square$

Theorem 3.7. *For any $p \geq 5$ the page number of $\Gamma(Z_{p^2}) = \frac{p-1}{2}$.*

Proof. There are $\frac{(p-1)(p-2)}{2}$edges in a complete graph for $p-1$ vertices. A book embedding of such a graph would have $p-1$ edges on the spine. Then the number of internal edges will be $\frac{(p-1)(p-2)}{2} - (p-1) = \frac{(p-1)(p-2-2)}{2} = \frac{(p-1)(p-4)}{2}$. Let there be k-page book where all the $\frac{(p-1)(p-4)}{2}$ internal edges will be embedding. Then for any k book embedding we require that $4(p-1) \geq \frac{(p-1)(p-4)}{2}$. This implies the maximum number of pages in which we could embed all the edges is $\frac{p-1}{2}$ Now we will show that there exist book-embeddings of $\Gamma(Z_{p^2})$ with $\frac{p-1}{2}$ pages. As $p-1$ is an even number then we write $p-1 = 2k$. Then we will show that $\Gamma(Z_{(2k)^2})$ has a book embedding with k pages. Consider the vertices as being in a circular layout with the vertices ordering $v_0, v_1, v_2, \ldots, v_{n-1}$. For each of $0 \leq i \leq k-1$, we define $E_i = \{v_a v_b : \lceil \frac{a+b}{2} \rceil \equiv imod\frac{p-1}{2}\}$ so that no pair of edges in the same set cross, as that would give two edges $v_a v_b$ and $v_i v_j$ such that $a \leq i \leq b \leq j$ so $a+b \leq i+j+2$ that is two edges can not be in the same set by the construction. Since $k = \frac{p-1}{2}$ we can see that each of these k sets are distinct. That is each of these k disjoint partitions of the edges of $\Gamma(Z_{(2k)^2})$ contains $p-4$ internal edges which can be embedded without crossing on a single page. By adding the external edges to any of these pages we get a k-page book-embedding of $\Gamma(Z_{p^2})$. □

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DEPARTMENT OF MATHEMATICS
VELLORE INSTITUTE OF TECHNOLOGY, VELLORE, TAMILNADU,INDIA
Email address: sagayanathan.m@vit.ac.in, sagajack12@gmail.com

DEPARTMENT OF MATHEMATICS
VELLORE INSTITUTE OF TECHNOLOGY, VELLORE, TAMILNADU,INDIA
Email address: ravisankar.j@vit.ac.in, ravisankar.maths@gmail.com