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INTRODUCTION TO OPEN HUB POLYNOMIAL OF GRAPHS

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ABSTRACT. In this paper we introduce the open hub polynomial of a connected graph G. The open hub polynomial of a connected graph G of order n is the polynomial $H_O G(x) = \sum_{i=h_O(G)}^{n} h_O(G,i)x^i$ where $h_O(G,i)$ denotes the number of open hub sets of G of cardinality i and $h_O(G)$ is the open hub number of G. We obtain the open hub polynomial of some special classes of graphs. Also we obtain open hub polynomial of join of two graphs.

1. INTRODUCTION

By a graph $G = (V, E)$ we mean a finite ordered graph with no loops and no multiple edges. For graph theoretic terminology we refer [1]. All the graphs considered in this paper are connected ,unless otherwise stated.The concept of hub set was Introduced by M. Walsh [2] . A subset H of V is called a hub set of G if for any two distinct vertices $u, v \in V - H$, either u and v are adjacent or there exists a u-v path P in G such that all the internal vertices of P are in H. The minimum cardinality of a hub set of G is called the hub number of G and is denoted by $h(G)$. In [3] we have defined the concept of open hub set of a Graph G as follows.

Definition 1.1. *A hub set H of a graph G is called an open hub set if the induced sub graph , $< H >$ has no isolated vertices.The minimum cardinality of an open hub set of G is called the open hub number of G and is denoted by $h_O(G)$.*

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In this paper we introduce the open hub polynomial of a Graph G.

Definition 1.2. *The open hub polynomial of a graph G of order n is the polynomial* $H_OG(x) = \sum_{i=h_O(G)}^{n} h_O(G,i)x^i$ *where* $h_O(G,i)$ *denotes the number of open hub sets of cardinality i.*

2. Main Results

From the very definition of open hub polynomial we obtain the following result.

Theorem 2.1. *Let G be a connected graph of order n,Then*

(1) $h_O(G,n) = 1$
(2) $h_O(G,i) = 0$ *if and only if* $i \leq h_O(G) - 1$ *or* $i \geq n+1$.
(3) *If* G_1 *is any sub graph of G,then* $deg(H_OG(x)) \geq deg(H_OG_1(x))$.
(4) *zero is a root of* $H_OG(x)$ *of multiplicity* $h_O(G)$ *for all graph G.*

Now we will find the open hub polynomial of some well known graphs. For the complete graph K_n, $h(K_n) = 0$.But $h_O(G) \geq 2$ fo any graph G. As every two element sub sets of vertex set of K_n is an open hub set we have $h_O(K_n) = 2$. Hence $H_OK_n(x) = (1+x)^n - 1 - nx$.

Theorem 2.2. *The open hub polynomial of the star graph* $K_{1,n}, n \geq 3$ *is* $H_OK_{1,n}(x) = x[(1+x)^n - 1]$.

Proof. Let u be the central vertex of $K_{1,n}$ and $u_1, u_2....u_n$ are the pendent vertices. Clearly, $h_O(K_{1,n}) = 2$. For $2 \leq i \leq n$, every open hub set of cardinality i must include the central vertex u,hence the number of open hub sets of cardinality i is $\binom{n}{i-1}$, $2 \leq i \leq n$, so that $H_OK_{1,n}(x) = x[(1+x)^n - 1]$ □

Theorem 2.3. *The open hub polynomial of* $K_{2,n}$ *is* $H_OK_{2,n}(x) = x(x+2)[(1+x)^n - 1]$.

Proof. Let $\{v_1, v_2\}$,$\{u_1, u_2, \dots, u_n\}$ are the bipartition of vertex set of $K_{2,n}$. Clearly, $h_O(K_{2,n}) = 2$. For $2 \leq i \leq n$ there are $\binom{n}{i-1}$ open hub sets of cardinality i ,which contains v_1 but not v_2, $\binom{n}{i-1}$ open hub sets of cardinality i, which contains v_2 but not v_1, $\binom{n}{i-2}$ open hub sets of cardinality i,which contain boh v_1 and v_2. The sets, $\{v_1, u_1, u_2, \dots, u_n\}$, $\{v_2, u_1, u_2, \dots, u_n\}$, $\{v_1, v_2, u_1, u_2, , \dots, u_{n-1}\}$, $\{v_1, v_2, u_1, u_2, , \dots, u_{n-2}, u_n\}$, $\{v_1, v_2, u_2, \dots, u_n\}$ are the open hub set of cardinality $n+1$.

Hence $H_OK_{2,n}(x) = 2\binom{n}{1}x^2 + (2\binom{n}{2} + \binom{n}{1})x^3 + (2\binom{n}{3} + \binom{n}{2})x^4 + \cdots + (2\binom{n}{n-1} + \binom{n}{n-2})x^n + (\binom{n}{1} + 2)x^{n+1} + x^{n+2} = x(x+2)[(1+x)^n - 1]$. □

Next we find the open hub polynomial of the double star graph . A double star graph $S_{m,n}$ is a tree obtained from the graph K_2 with two vertices u and v by attaching m pendedant edges in u and n pendant edges in v.

Theorem 2.4. *The open hub polynomial of the double star $S_{m,n}$ is $H_OS_{m,n}(x) = x^2(1+x)^{m+n}$.*

Proof. Let $U = \{u_1, u_2, \ldots, u_m\}$,$V = \{v_1, v_2, \ldots, v_n\}$ and $\{u, v\}$ be the vertices of $S_{m,n}$ such that u and v are adjacent, every vertices in U are adjacent to u and every vertices in V are adjacent to v. Clearly the open hub number of $S_{m,n}$ is two and $\{u, v\}$ is the only open hub set of cardinality 2. For $3 \leq i \leq m+n+2$,every open hub set must contain both the vertices u and v. An open hub set of cardinality i must contin $i-2$ vertices from $\{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$. Hence there are $\binom{m+n}{i-2}$ such open hub sets. Hence $H_OS_{m,n}(x) = x^2(1+x)^{m+n}$. □

Next we find the open hub polynomial of some graph construction. A lollipop graph $L_{(m,n)}$ is the graph obtained by joining the complete graph K_m to a path graph P_n with a bridge.

Theorem 2.5. *The open hub polynomial of the lollipop graph $L_{(m,1)}$ is $H_OL_{(m,1)}(x) = x[(1+x)^m - 1] + x^{m-1}$.*

Proof. Let $v_1, v_2, ...v_m$ be the vertices of the complete graph and let v_1 is adjacent to v. Clearly, $h_O(L_{(m,1)}) = 2$. Then $\{v_1, v\}, \{v_1, v_2\}, \ldots, \{v_1, v_m\}$ are the open hub sets of cardinality two. For $3 \leq r \leq m-2$,as every open hub set must contain v_1, number of open hub sets, which contains v, of cardinality r is equal to number of sub sets of $\{v_2, v_3, \ldots, v_m\}$ havig cardinality $r-2$ and the number of open hub sets,which does not contain v, of cardinality r is equal to number of sub sets of $\{v_2, v_3, \ldots, v_m\}$ havig cardinality $r-1$. For $r = m-1$,number of open hub sets which contain both v and v_1 is equal to number of sub sets of $\{v_2, v_3, \ldots, v_m\}$ of cardinality m-3.

Also $\{v_1, v_2, \ldots, v_{m-1}\}, \{v_1, v_2, \ldots, v_{m-2}, v_m\}, \ldots, \{v_2, v_3, \ldots, v_m\}$ are open hub sets. The sets $\{v_1, v_2, ..., v_m\}$, $\{v, v_1, v_3, \ldots, v_m\}$, $\ldots, \{v, v_1, v_2, \ldots, v_{m-1}\}$ are the open hub sets of cardinality m. Hence $H_OL_{(m,1)}(x) = x[(1+x)^m - 1] + x^{m-1}$. □

Theorem 2.6. *The open hub polynomial of the lollipop graph* $L_{(3,n)}, n \neq 1,2$ *is,* $H_O L_{(3,n)}(x) = (n-1)x^n + (\frac{n^2+n+4}{2})x^{n+1} + (n+2)x^{n+2} + x^{n+3}$.

Proof. Let v_1, v_2, v_3 be the vertices of the complete graph K_3 and $u_1, u_2, \dots, u_n$ be the vertices of the path graph P_n. Let v_1 is adjacent u_1. Then $h_O(L_{(3,n)}) = n$. Then a hub set of cardinality n is a sub set of $V(P_n) \cup \{v_1\}$ of cardinality n. Among these $n+1$ hub sets only $\{v_1, u_1, u_2, \dots, u_{n-2}, u_n\}$ and $\{v_1, u_2, u_3, \dots, u_n\}$ are not open hub sets. All sub sets of $V(L_{(3,n)})$ of cardinality $n+1$ is a hub set. Let v be a vertex of degree 2 in $L_{(3,n)}$. Let the open neighbourhood of v is, $N(v) = \{v', v''\}$. Then $V(L_{(3,n)}) - \{v', v''\}$ is not an open hub set. Number of such hub sets = number of vertices of degree 2. Again $V(L_{(3,n)}) - \{v, u_{n-1}\}$ where $v \in \{v_1, v_2, v_3, u_1, u_2, \dots, u_{n-4}, u_{n-2}\}$ is also not an open hub set. Hence $h_{O,L(3,n),n+1} = \binom{n+3}{2} - (n+1) - n = \frac{n^2+n+4}{2}$. Any sub set of cardinality (n+2) which contains the vertex u_n is an open hub set.Hence the result follows. □

Remark 2.1.

(1) *The open hub polynomial of* $L_{(3,1)}$ *is* $H_O L_{(3,1)}(x) = 4x^2 + 3x^3 + x^4$.
(2) *The open hub polynomial of* $L_{(3,2)}$ *is* $H_O L_{(3,2)}(x) = 2x^2 + 4x^3 + 4x^4 + x^5$.

The Dutch windmill graph D_3^m is the graph obtained by selecting one vertex in each of m triangles and identifying them.

Theorem 2.7. *The open hub polynomial of the Dutch wind mill graph* D_3^m *is* $H_O D_3^m(x) = x[(1+x)^{2m} - 1] + mx^{2m-2} + x^{2m},\ m \geq 2$.

Proof. Let v be the central vertex and $v_1, v_2, \dots, v_{2m}$ are the other vertices of D_3^m so that $v_{2i-1}vv_{2i}$ forms a triangle for $1 \leq i \leq m$. For $2 \leq i \leq 2m-1, i \neq 2m-2$, every open hub set must contain the vertex v and every sub sets containing v must be an open hub set. Therefore number of open hub sets of cardinality i is equal to number of sub sets of $\{v_1, v_2, \dots, v_{2m}\}$ of cardinality $i-1$. For $i = 2m-2$ there are $\binom{2m}{2m-3}$ open hub sets which contains v and there are m open hub sets which doesnot conains v, $\{v_1, v_2, \dots, v_{n-3}, v_{n-2}\}$, $\{v_1, v_2, \dots, v_{n-4}, v_{n-1}, v_n\}$ $, \dots, \{v_3, v_4, ...v_{n-1}, v_n\}$. For $i = 2m$ there are $\binom{2m}{2m-1}$ open hubsets which contains v and one open hub set $\{v_1, v_2, \dots, v_{2m}\}$ which does not contains v.

Therefore $H_O D_3^m(x) = x[(1+x)^{2m} - 1] + mx^{2m-2} + x^{2m}, m \geq 2$. □

The $(m,n)-$ tadpole graph $T_{m,n}$,is the graph obtained by identifying a vertex v_k of the cycle graph C_m with an end vertex of the path graph P_{n+1}. Here we find the open hub polynomial of the tadpole graph $T_{4,n-1}$.

Theorem 2.8. *The open hub polynomial of the tadpole graph* $T_{4,n-1}, n \neq 2,3$ *is* $H_O T_{4,n-1}(x) = (2n-3)x^n + (\frac{n^2+n+6}{2})x^{n+1} + (n+2)x^{n+2} + x^{n+3}$.

Proof. Let v_1, v_2, v_3, v_4 are the vertices of the cycle C_4 and let $u_1, u_2, \ldots, u_n$ are the vertices of the path graph P_n. Let v_4 is identified with u_1. Clearly $h_O(T_{4,n-1}) = n$. Then the subsets of $V(P_n)$ of cardinality $n-1$, which contains both u_1 and u_{n-1},union with $\{v_i\}, i = 1,3$ are open hub sets of cardinality n. The set $\{u_1, u_2, \ldots, u_n\}$ is also an open hub set. Hence $h_O(T_{4,n-1}, n) = 2n-3$. Any subset of $V(T_{4,n-1})$ of cardinality $n+1$ is a hub set. Among these hub sets, a set which contains u_n but not contains u_{n-1} is not an open hub set. Now $\{v_1, v_2, v_3\} \cup V(P_n) - \{u_{k-1}, u_{k+1}\}$, $k = 2, 3, \ldots, n-3, n-1$ are not open hub sets. The sets $\{v_1, v_3, u_2, \ldots, u_n\}$ and $\{v_2, u_1, u_2, \ldots, u_n\}$ are also not open hub sets. Hence $h_o(T_{4,n-1}, n+1) = \binom{n+3}{2} - (n+1) - (n-1) = \frac{n^2+n+6}{2}$. Any subset of $V(T_{4,n-1})$ of cardinality $n+2$ which contains u_{n-1} is an open hub set. Hence the result. □

Remark 2.2.

(1) *The open hub polynomial of* $T_{4,1}$ *is*
$H_O T_{4,1}(x) = 3x^2 + 6x^3 + 4x^4 + x^5$.

(2) *The open hub polynomial of* $T_{4,2}$ *is* $H_O T_{4,2}(x) = 3x^3 + 8x^4 + 5x^5 + x^6$.

3. Open hub polynomial of join of two graphs

Now we find the open hub polynomial of join of two graphs. The join of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv/u \in V(G_1), v \in V(G_2)\}$.

Theorem 3.1. *Let* G_1 *and* G_2 *be two connected graphs of order* n_1 *and* n_2 *respectively and let* $G = G_1 + G_2$.*Then open hub polynomial of G is* $H_O G(x) = [(1+x)^{n_1} - 1][(1+x)^{n_2} - 1] + H_O G_1(x) + H_O G_2(x)$.

Proof. If H is an open hub set of G_j for $j = 1,2$ of cardinality i then H is also an open hub set of G of cardinality i. Also for every $H_1 \subset V(G_1)$ and $H_2 \subset V(G_2)$, $H_1 \cup H_2$ is an open hub set of G of cardinality $i = i_1 + i_2$, where i_j is the cardinality of H_j for $j = 1,2$. Thus $H_O G(x) = [(1+x)^{n_1} - 1][(1+x)^{n_2} - 1] + H_O G_1(x) + H_O G_2(x)$. □

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