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OSCILLATION CRITERIA FOR HIGHER ORDER NONLINEAR NEUTRAL DELAY GENERALIZED $\alpha-$DIFFERENCE EQUATIONS

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ABSTRACT. In the present study, we find oscillation results for the higher order nonlinear neutral delay generalized $\alpha-$difference equation of the form

$$\Delta_{\alpha(\ell)}\left(a(k)\Delta_{\alpha(\ell)}^{m-1}z(k)\right)+q(k)f(x(k-\rho\ell))=0\,,$$

where $z(k)=x(k)+p(k)x(\tau(k))$.

1. INTRODUCTION

The difference equations are based on the operator Δ given in the form of $\Delta u(k)=u(k+1)-u(k),\ k\in\mathbb{N}=\{0,1,2,3,\cdots\}$. The generalized $\alpha-$ difference operator $\Delta_{\alpha(\ell)}$ is defined as $\Delta_{\alpha(\ell)}u(k)=u(k+\ell)-\alpha u(k)$, and its inverse defined as if $\Delta_{\alpha(\ell)}v(k)=u(k)$, then $\Delta_{\alpha(\ell)}^{-1}u(k)=v(k)-\alpha^{\left[\frac{k}{\ell}\right]}v(j)$ where $\alpha>1$ and $k\in\mathbb{N}_\ell(j)$, $j=k-\left[\frac{k}{\ell}\right]\ell$. The most general form is given in [3] by

$$\Delta_{\alpha(\ell)}^{-1}u(k)=\sum_{r=0}^{\left[\frac{k-k_0-j-\ell}{\ell}\right]}\frac{u(k_0+j+r\ell)}{\alpha^{\left\lceil\frac{k_0+j+\ell-k+r\ell}{\ell}\right\rceil}}+\alpha^{\left\lceil\frac{k-k_0}{\ell}\right\rceil}u(k_0+j),$$

for all $k\in\mathbb{N}_\ell(j)$, $j=k-k_0-\left[\frac{k-k_0}{\ell}\right]\ell$.

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In recent years, there is an increasing interest in finding sufficient conditions which ensure that all solutions or all the bounded solutions of difference equation of neutral type are oscillatory and asymptotic behavior of solutions involving operators Δ and Δ_ℓ has been studied and many research articles was available in the literature, see for example [1, 5–11]. But, a similar study on the oscillation of difference equations involving the operator $\Delta_{\alpha(\ell)}$ is rare. Hence we are motivated to present the oscillation of solutions of higher-order nonlinear $\alpha-$difference equation of the form

$$\Delta_{\alpha(\ell)}\left(a(k)\Delta^{m-1}_{\alpha(\ell)}z(k)\right)+q(k)f(x(k-\rho\ell))=0,\quad k\geq k_0, \tag{1.1}$$

where $m\geq 2$ is an integer and $z(k)=x(k)+p(k)x(\tau(k))$. Here, $a(k)$, $q(k)$ are sequence of positive real number, $p(k)$ is a bounded sequence for $k\geq k_0$, $\tau(k)$ is a sequence of integers with $\lim\limits_{n\to\infty}\tau(n)=\infty$ and ρ is a positive integer also $f(x)$ is a continuous real valued function such that $\dfrac{f(x)}{x}>L>0$ for $x\neq 0$ and L is a constant.

Throughout this paper we use the following notations.

(a) $\mathbb{N}=\{0,1,2,3,\dots\},\ \mathbb{N}(a)=\{a,a+1,a+2,\dots\}$,
(b) $\mathbb{N}_\ell(a)=\{a,a+\ell,a+2\ell,\dots\}$.
(c) $\lceil x\rceil$ upper integer part of x, $m_i(k)=\lceil\frac{k-k_i-j-\ell}{\ell}\rceil$.

2. Preliminaries

In this section, we present some lemmas, which will be useful in proving our main results.

Lemma 2.1. *[4] Let $u(k)$ and $v(k)$ be any two functions. Then, for all $k\in[k_0,\infty)$*

$$\Delta_{\alpha(\ell)}\{u(k)v(k)\}=u(k+\ell)\Delta_{\alpha(\ell)}v(k)+u(k+\ell)v(k)(\alpha-1)+v(k)\Delta_{\alpha(\ell)}u(k)\,.$$

Lemma 2.2. *[4] Let $u(k)$ be defined on $[0,\infty)$ and $k_0 \in [0,\infty)$. Then, for all $k \in [k_0,\infty)$, $j = k - k_0 - \left[\frac{k-k_0}{\ell}\right]\ell$ and $0 \le m \le n-1$.*

$$\Delta^m_{\alpha(\ell)} u(k) = \sum_{i=m}^{n-1} \frac{(k-k_0-j)_\ell^{(i-m)}}{(i-m)!\ell^{(i-m)}} \alpha^{\left\lceil \frac{k-k_0-j}{\ell} + m - i \right\rceil} \Delta^i_{\alpha(\ell)} u(k_0+j)$$
$$+ \sum_{r=0}^{\frac{k-k_0-j}{\ell}-n+m} \frac{(k-k_0-j-r\ell-\ell)_\ell^{(n-m-1)} \Delta^n_{\alpha(\ell)} u(k_0+j+r\ell)}{(n-m-1)!\ell^{n-m-1}\alpha^{-\left\lceil \frac{k-k_0-j}{\ell} + m - (n+r)\right\rceil}},$$

where $k_\ell^{(n)} = k(k-\ell)(k-2\ell)\dots(k-(n-1)\ell)$.

Lemma 2.3. *Let $1 \le m \le n-1$ and $u(k)$ be defined on $\mathbb{N}_\ell(k_0)$. Then,*

(1) $\liminf_{k\to\infty} \Delta^m_{\alpha(\ell)} u(k) > 0$ *implies* $\lim_{k\to\infty} \Delta^i_{\alpha(\ell)} u(k) = \infty$, $0 \le i \le m-1$.
(2) $\limsup_{k\to\infty} \Delta^m_{\alpha(\ell)} u(k) < 0$ *implies* $\lim_{k\to\infty} \Delta^i_{\alpha(\ell)} u(k) = -\infty$, $0 \le i \le m-1$.

Proof. $\liminf_{k\to\infty} \Delta^m_{\alpha(\ell)} u(k) > 0$ implies that there exists a large $k_1 \in \mathbb{N}_\ell(k_0)$ such that $\Delta^m_{\alpha(\ell)} u(k) \ge c > 0$ for all $k \ge k_1$. Since

$$\Delta^{m-1}_{\alpha(\ell)} u(k) = \alpha^{\left\lceil \frac{k-k_1}{\ell} \right\rceil} \Delta^{m-1}_{\alpha(\ell)} u(k_1+j) + \sum_{r=0}^{m_1(k)} \frac{\Delta^m_{\alpha(\ell)} u(k_1+j+r\ell)}{\alpha^{\left\lceil \frac{k_1-k+j+r\ell+\ell}{\ell} \right\rceil}}$$

it follows that $\Delta^{m-1}_{\alpha(\ell)} u(k) \ge \alpha^{\left\lceil \frac{k-k_1}{\ell} \right\rceil} \Delta^{m-1}_{\alpha(\ell)} u(k_1+j) + c\left(\frac{k-k_1-j}{\ell}\right)$, and hence $\lim_{k\to\infty} \Delta^{m-1}_{\alpha(\ell)} u(k) = \infty$. The rest of the proof is by induction.

Case (2) can be treated similarly. □

Lemma 2.4. *Let $u(k)$ be defined on $\mathbb{N}_\ell(k_0)$, and $u(k) > 0$ with $\Delta^n_{\alpha(\ell)}(k)$ is of constant sign on $\mathbb{N}_\ell(k_0)$ and not zero. Then, there exists an integer m, $0 \le m \le n$ with $n+m$ odd for $\Delta^n_{\alpha(\ell)}(k) \le 0$ or $n+m$ even for $\Delta^n_{\alpha(\ell)}(k) \ge 0$ and such that $m \ge 1$ implies*

$$\Delta^i_{\alpha(\ell)} u(k) > 0 \textit{ for all large } k \in \mathbb{N}_\ell(k_0), 1 \le i \le m-1.$$

and $m \le n-1$ implies

$$(-1)^{m+i} \Delta^i_{\alpha(\ell)} u(k) > 0 \textit{ for all } k \in \mathbb{N}_\ell(k_0), m \le i \le n-1.$$

for all large $n \in \mathbb{N}_\ell(k_0)$ and $n \ge N$.

Proof. There are two possible cases.

Case 1. $\Delta^{m}_{\alpha(\ell)}u(k) \leq 0$ on $\mathbb{N}_\ell(k_0)$. First, we shall prove that $\Delta^{n-1}_{\alpha(\ell)}u(k) > 0$ on $\mathbb{N}_\ell(k_0)$. If not, then there exists some $k_1 \geq k_0$ in $\mathbb{N}_\ell(k_0)$ such that $\Delta^{n-1}_{\alpha(\ell)}u(k_1) \leq 0$. Since $\Delta^{n-1}_{\alpha(\ell)}u(k) > 0$ is decreasing and not constant on $\mathbb{N}_\ell(k_0)$, there exists $k_2 \geq k_1$ such that $\Delta^{n-1}_{\alpha(\ell)}u(k) \leq \Delta^{n-1}_{\alpha(\ell)}u(k_2) \leq \Delta^{n-1}_{\alpha(\ell)}u(k_1) \leq 0$ for all $k \geq k_2$, But, from Lemma 2.3, we find $\lim_{k\to\infty} u(k) = -\infty$ which is a contradiction to $u(k) > 0$. Thus, $\Delta^{n-1}_{\alpha(\ell)}u(k) > 0$ on $\mathbb{N}_\ell(k_0)$ and there exists a smallest integer m, $0 \leq m \leq n-1$ with $n+m$ odd and

$$(-1)^{m+i}\Delta^{i}_{\alpha(\ell)}u(k) > 0 \text{ on } \mathbb{N}_\ell(k_0), m \leq i \leq n-1. \tag{2.1}$$

Next, let $m > 1$ and

$$\Delta^{m-1}_{\alpha(\ell)}u(k) < 0 \text{ on } \mathbb{N}_\ell(k_0), \tag{2.2}$$

then once again form Lemma 2.3 it follows that

$$\Delta^{m-2}_{\alpha(\ell)}u(k) < 0 \text{ on } \mathbb{N}_\ell(k_0). \tag{2.3}$$

Inequalities (2.1)-(2.3) can be unified to

$$(-1)^{(m-2)+i}\Delta^{i}_{\alpha(\ell)}u(k) > 0 \text{ on } \mathbb{N}_\ell(k_0), m-2 \leq i \leq n-1,$$

which is a contradiction to the definition of m. So, (2.2) fails and $\Delta^{m-1}_{\alpha(\ell)}u(k) \geq 0$ for all $k \geq k_0$. From (2.1), $\Delta^{m-1}_{\alpha(\ell)}u(k)$ is non-decreasing and hence $\lim\limits_{k\to\infty} \Delta^{m-1}_{\alpha(\ell)}u(k) > 0$. If $m > 2$, we find from Lemma 2.3 that $\lim\limits_{k\to\infty} \Delta^{i}_{\alpha(\ell)}u(k) = \infty$, $1 \leq i \leq m-2$. Thus, $\Delta^{i}_{\alpha(\ell)}u(k) > 0$ for all large $k \geq k_0$, $1 \leq i \leq m-1$.

Case 2. $\Delta^{n}_{\alpha(\ell)}u(k) \geq 0$ on $\mathbb{N}_\ell(k_0)$. Let $k_3 \geq k_2$ be such that $\Delta^{n-1}_{\alpha(\ell)}u(k_3) \geq 0$, then since $\Delta^{n-1}_{\alpha(\ell)}u(k)$ is nondecreasing and not identically constant, there exists some $k_4 \geq k_3$ such that $\Delta^{n-1}_{\alpha(\ell)}u(k) > 0$ for all $k \geq k_4$. Thus, $\lim\limits_{k\to\infty} \Delta^{n-1}_{\alpha(\ell)}u(k) > 0$ and from Lemma 2.3 $\lim_{k\to\infty} \Delta^{i}_{\alpha(\ell)}u(k) = \infty$, $1 \leq i \leq n-2$ and so $\Delta^{i}_{\alpha(\ell)}u(k) > 0$ for all large k in $\mathbb{N}_\ell(k_0)$, $1 \leq i \leq n-1$. This proves the Lemma for $m = n$. In case $\Delta^{n-1}_{\alpha(\ell)}u(k) < 0$ for all $k \in \mathbb{N}_\ell(k_0)$, we find from Lemma 2.3 that $\Delta^{n-2}_{\alpha(\ell)}u(k) > 0$ for all $k \in \mathbb{N}_\ell(k_0)$. The rest of the proof is the same as in Case 1.

□

Lemma 2.5. *Let $u(k)$ be defined on $\mathbb{N}_{\alpha(\ell)}(k_0)$, and $u(k) > 0$ with $\Delta_\ell^n u(k) \leq 0$ on $\mathbb{N}_{\alpha(\ell)}(k_0)$ and not identically equal to zero. Then, there exist a large integer k_1 in $\mathbb{N}_{\alpha(\ell)}(k_0)$ such that for all $k \in \mathbb{N}_{\alpha(\ell)}(k_1)$*

$$u(k) \geq \frac{(k-k_1-j)_\ell^{(n-1)}}{(n-1)!\ell^{(n-1)}} \alpha^{\lceil \frac{k-k_1-j+\ell}{\ell} - n \rceil} \Delta_\ell^{n-1} u(2^{n-2}k)$$

where $u_\ell^{(n)} = u(u-\ell)(u-2\ell)\cdots(u-(n-1)\ell)$. Note if further $\{u_n\}$ is increasing, then

$$u(k) \geq \frac{1}{(n-1)!\ell^{(n-1)}} \left(\frac{k-j}{2^{n-2}}\right)_\ell^{(n-1)} \alpha^{\lceil \frac{k-j+\ell}{\ell} - n \rceil} \Delta_\ell^{n-1} u(k) \; forall \; k \geq 2^{n-1}k.$$

Proof. Lemma 2.4 follow that $(-1)^{n+i}\Delta_{\alpha(\ell)}^i u(k) > 0$ and $\Delta_{\alpha(\ell)}^i u(k) > 0$ for all large k in $\mathbb{N}_\ell(k_0)$, say, for all $k \geq k_1 \geq k_0$, $1 \leq i \leq m-1$. Using these inequalities, we obtain

$$\begin{aligned}
-\Delta_{\alpha(\ell)}^{n-2} u(k) &= -\alpha^{-\infty}\Delta_{\alpha(\ell)}^{n-2} u(\infty) + \sum_{r=0}^{\infty} \frac{\Delta_{\alpha(\ell)}^{n-1} u(k+r\ell)}{\alpha^{\lceil \frac{r\ell+\ell}{\ell} \rceil}} \\
&\geq \sum_{r=0}^{\frac{k}{\ell}} \frac{\Delta_{\alpha(\ell)}^{n-1} u(k+r\ell)}{\alpha^{\lceil \frac{r\ell+\ell}{\ell} \rceil}} \geq \frac{1}{\ell\alpha} \Delta_{\alpha(\ell)}^{n-1} u(2k)(k)_\ell^{(1)} \\
\Delta_\ell^{n-3} u(k) &= \alpha^{-\infty}\Delta_\ell^{n-3} u(\infty) - \sum_{r=0}^{\infty} \frac{\Delta_{\alpha(\ell)}^{n-2} u(k+r\ell)}{\alpha^{\lceil \frac{r\ell+\ell}{\ell} \rceil}} \\
&\geq \frac{1}{\ell\alpha} \sum_{r=0}^{\frac{k}{\ell}} \frac{(k+r\ell)_\ell^{(1)} \Delta_{\alpha(\ell)}^{n-1} u(2(k+r\ell))}{\alpha^{\lceil \frac{r\ell+\ell}{\ell} \rceil}} \geq \Delta_{\alpha(\ell)}^{n-1} u(2^2k) \frac{1}{2!\ell^2\alpha^2} (k)_\ell^{(2)}
\end{aligned}$$

$$\cdots \quad \cdots \quad \cdots$$

$$\Delta_{\alpha(\ell)}^m u(k) \geq \frac{(k)_\ell^{(n-m-1)}}{(n-m-1)!\ell^{n-m-1}\alpha^{n-m-1}} \Delta_{\alpha(\ell)}^{n-1} u(2^{n-m-1}k).$$

next we get

$$\begin{aligned}\Delta_{\alpha(\ell)}^{m-1}u(k) &= \alpha^{\lceil\frac{k-k_1}{\ell}\rceil}\Delta_{\alpha(\ell)}^{m-1}u(k_1+j)+\sum_{r=0}^{m_1(k)}\frac{\Delta_{\alpha(\ell)}^{m}u(k_1+j+r\ell)}{\alpha^{\lceil\frac{k_1-k+j+r\ell+\ell}{\ell}\rceil}}\\ &\geq \sum_{r=0}^{m_1(k)}\frac{(k_1+j+r\ell)_\ell^{(n-m-1)}\Delta_{\alpha(\ell)}^{n-1}u(2^{n-m-1}(k_1+j+r\ell))}{(n-m-1)!\ell^{n-m-1}\alpha^{\lceil\frac{k_1-k+j+r\ell}{\ell}+n-m\rceil}}\\ &\geq \frac{(k-k_1-j)_\ell^{(n-m)}}{(n-m)!\ell^{(n-m)}}\alpha^{\lceil\frac{k-k_1-j}{\ell}+m-n\rceil}\Delta_\ell^{n-1}u(2^{n-m-1}k).\end{aligned}$$

Letting $m=1$ in the above inequality, we have

$$u(k)\geq \frac{(k-k_1-j)_\ell^{(n-1)}}{(n-1)!\ell^{(n-1)}}\alpha^{\lceil\frac{k-k_1-j+\ell}{\ell}-n\rceil}\Delta_\ell^{n-1}u(2^{n-2}k)$$

By replacement $k=2^{n-2}k$ the proof of the lemma is completed. □

Lemma 2.6. *Assume that*

$$\sum_{r=0}^{\infty}\frac{\alpha^{-\lceil\frac{k_1-k+j+r\ell+\ell}{\ell}\rceil}}{a(k_1+j+r\ell)}=\infty$$

and let $\{x(k)\}$ *be a positive solution of equation (1.1) . Then there exists* $k_1\geq k_0$ *such that* $z(k)>0$, $\Delta_{\alpha(\ell)}z(k)>0$, $\Delta_{\alpha(\ell)}^{m-1}z(k)>0$ *and* $\Delta_{\alpha(\ell)}^{m}z(k)\leq 0$ *for all* $k\geq k_1$.

Proof. Since $\{x(k)\}$ is a positive solution of equation (1.1), there exists $k\geq k_0$ such that $x(k)>0$ and $x(\tau(n))>0$ for all $k\geq k_1$. Then by the definition of $z(k)$, we have $z(k)>0$ for all $k\geq k_1$. Also from the equation (1.1), we have

$$\Delta_{\alpha(\ell)}\left(a(k)\Delta_{\alpha(\ell)}^{m-1}z(n)\right)=-q(k)f(x(k-\rho\ell))<0 \text{ for all } k\geq k_0.$$

Therefore, $a(k)\Delta_{\alpha(\ell)}z(k)$ is decreasing and of one sign for all $k\geq k_1$. Since $\{a(k)\}$ is positive, we have either $\Delta_{\alpha(\ell)}^{m-1}z(k)<0$ or $\Delta_{\alpha(\ell)}^{m-1}z(k)>0$ eventually. We shall prove that $\Delta_{\alpha(\ell)}^{m-1}z(k)>0$. If not, then there exists a constant $c<0$ such that $a(k)\Delta_{\alpha(\ell)}^{m-1}z(k)\leq c<0$ for all $k\geq k_1$, which implies

$$\Delta_{\alpha(\ell)}^{m-2}z(k)-\alpha^{\lceil\frac{k-k_1}{\ell}\rceil}\Delta_{\alpha(\ell)}^{m-2}z(k_1)\leq c\sum_{r=0}^{m_1(k)}\frac{\alpha^{-\lceil\frac{k_1-k+j+r\ell+\ell}{\ell}\rceil}}{a(k_1+j+r\ell)}.$$

Letting $k\to\infty$ in the last inequality, we see that $\Delta_{\alpha(\ell)}^{m-2}z(k)\to-\infty$. That is $\Delta_{\alpha(\ell)}^{m-2}z(k)<0$ eventually. Now $\Delta_{\alpha(\ell)}^{m-2}z(k)<0$ eventually implies $\Delta_{\alpha(\ell)}^{m-3}z(k)<0$ eventually. Continuing this process, we get $z(k)<0$ eventually which is a contradiction. Hence $\Delta^{m-1}_{\alpha(\ell)}z(k) > 0$ eventually. Moreover $\{a(k)\}$ is positive and increasing and $\Delta\left(a(k)\Delta^{m-1}_{\alpha(\ell)}z(k)\right) < 0$ for all $k \geq k_1$, we have $\Delta^{m}_{\alpha(\ell)}z(k) \leq 0$ for all $k \geq k_1$. □

Lemma 2.7. *[2] The first order generalized $\alpha-$difference inequality*

$$\Delta_{\alpha(\ell)}y(k) + p(k)y(k-\rho\ell) \leq 0$$

eventually has no positive solution if

$$\liminf_{k\to\infty} \sum_{r=0}^{\frac{k-(k-\rho\ell)-j-\ell}{\ell}} p(k-\rho\ell+j+r\ell) > \frac{1}{\alpha^\rho}\left(\frac{\rho\ell}{\rho\ell+1}\right)^{\rho\ell+1}$$

or

$$\limsup_{k\to\infty} \sum_{r=0}^{\frac{k-(k-\rho\ell)-j}{\ell}} p(k-\rho\ell+j+r\ell) > \frac{1}{\alpha^\rho}.$$

3. Oscillation Results

In this section, we present a few sufficient conditions for the oscillation of all solutions of equation (1.1). Throughout this section we use the following assumptions

$$P(k) = \min\{q(k), q(\tau(k))\}, Q(k) = LP(k), \text{and } \eta(k) = \sum_{r=0}^{\infty} \frac{\alpha^{-\left\lceil\frac{k_1-k+j+r\ell+\ell}{\ell}\right\rceil}}{a(k_1+j+r\ell)}.$$

Theorem 3.1. *Assume that $\eta(k) = \infty$. If*

$$\sum_{r=0}^{\infty} \frac{P(k+j+r\ell)}{\alpha^{\left\lceil\frac{j+r\ell+\ell-k}{\ell}\right\rceil}} = \infty,$$

then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a non-oscillatory solution of equation (1.1). We may assume without loss of generality that $\{x(k)\}$ is a positive solution of equation (1.1). Then there exists a $k_1 \geq k_0$ such that $x(k) > 0$, $x(\tau(k)) > 0$ and $x(k-\rho\ell) > 0$ for all $k \geq k_1$. Then from Lemma 2.3, we have $z(k) > 0$, $\Delta_{\alpha(\ell)}z(k) > 0$, $\Delta^{m-1}_{\alpha(\ell)}z(k) > 0$ and $\Delta^{m}_{\alpha(\ell)}z(k) \leq 0$ for all $k \geq k_1$.

Now, using condition $\frac{f(x)}{x} > L$ in equation (1.1), we see that

$$\Delta_{\alpha(\ell)}\left(a(k)\Delta_{\alpha(\ell)}^{m-1}z(k)\right) = -q(k)f(x(k-\rho\ell)) \leq -Lq(k)x(k-\rho\ell) < 0 \; \forall\, k \geq k_1.$$

Therefore, $a(k)\Delta_{\alpha(\ell)}^{m-1}z(k)$ is decreasing. Also from the last inequality, we have

$$\begin{aligned}\Delta_{\alpha(\ell)}\left(a(k)\Delta_{\alpha(\ell)}^{m-1}z(k)\right) + Lq(k)x(k-\rho\ell) + p\Delta_{\alpha(\ell)}\left(a(\tau(k))\Delta_{\alpha(\ell)}^{m-1}z(\tau(k))\right)&\\ + Lq(\tau(k))px(\tau(k-\rho\ell) \leq 0,\ \forall\, k \geq k_1.&\end{aligned}$$

That is,

$$\begin{aligned}&\Delta_{\alpha(\ell)}\left(a(k)\Delta_{\alpha(\ell)}^{m-1}z(k)\right)\\ &+ LP(k)z(k-\rho\ell) + p\Delta_{\alpha(\ell)}\left(a(\tau(k))\Delta_{\alpha(\ell)}^{m-1}z(\tau(k))\right) \leq 0.\end{aligned} \tag{3.1}$$

Now summing the last inequality from k_1 to $k-\ell$, we obtain

$$\begin{aligned}&a(k)\Delta_{\alpha(\ell)}^{m-1}z(k) - \alpha^{\lceil\frac{k-k_1}{\ell}\rceil}a(k_1)\Delta_{\alpha(\ell)}^{m-1}z(k_1)\\ &+ L\sum_{r=0}^{m_1(k)}\frac{P(k_1+j+r\ell)z(k_1+j+r\ell-\rho\ell)}{\alpha^{\lceil\frac{k_1-k+j+r\ell+\ell}{\ell}\rceil}}\\ &+ pa(\tau(k))\Delta_{\alpha(\ell)}^{m-1}z(\tau(k)) - p\alpha^{\lceil\frac{k-k_1}{\ell}\rceil}a(\tau(k_1))\Delta_{\alpha(\ell)}^{m-1}z(\tau(k_1)) \leq 0.\end{aligned}$$

That is

$$\begin{aligned}&L\sum_{r=0}^{m_1(k)}\frac{P(k_1+j+r\ell)z(k_1+j+r\ell-\rho\ell)}{\alpha^{\lceil\frac{k_1-k+j+r\ell+\ell}{\ell}\rceil}}\\ &\leq \alpha^{\lceil\frac{k-k_1}{\ell}\rceil}a(k_1)\Delta_{\alpha(\ell)}^{m-1}z(k_1) - a(k)\Delta_{\alpha(\ell)}^{m-1}z(k) - pa(\tau(k))\Delta_{\alpha(\ell)}^{m-1}z(\tau(k))\\ &+ p\alpha^{\lceil\frac{k-k_1}{\ell}\rceil}a(\tau(k_1)\Delta_{\alpha(\ell)}^{m-1}z(\tau(k_1) \leq 0 \text{ for all } k \geq k_1.\end{aligned}$$

Since $\Delta_{\alpha(\ell)}z(k) > 0$ and $z(k) > 0$ there exists a constant $c \geq 0$ such that $z(k-\rho\ell) \geq c$ for all $k \geq k_1$ and using this and the monotonicity of $a(k)\Delta_{\alpha(\ell)}z(k)$ in the last inequity and letting $k \to \infty$, we get

$$L\sum_{r=0}^{m_1(k)}\frac{P(k_1+j+r\ell)}{\alpha^{\lceil\frac{k_1-k+j+r\ell+\ell}{\ell}\rceil}}z(k_1+j+r\ell-\rho\ell) < \infty,$$

which contradicts (3.1). Thus the proof is complete. $\square$

Theorem 3.2. *Assume that $\eta(k) = \infty$ and let $\tau(k) = k + \tau$. If either*

(3.2)
$$\liminf_{k\to\infty} \sum_{r=0}^{\frac{\rho\ell-j-\ell}{\ell}} \frac{(k-2\rho\ell+r\ell)_\ell^{(m-1)} Q((k-\rho\ell)+j+r\ell)}{\alpha^{\lceil m-\frac{k-2\rho\ell+r\ell+\ell}{\ell}\rceil} a(k+j+r\ell-2\rho\ell)} \geq \frac{\beta}{\lambda\alpha^\rho}\left(\frac{\rho\ell}{1+\rho\ell}\right)^{\rho\ell+1}$$

or

$$\limsup_{k\to\infty} \sum_{r=0}^{\frac{\rho\ell-j}{\ell}} \frac{(k+j+r\ell-2\rho\ell)_\ell^{(m-1)} Q(k+j+r\ell-\rho\ell)}{\alpha^{\lceil m-\frac{k-2\rho\ell+r\ell+\ell}{\ell}\rceil} a(k+j+r\ell-2\rho\ell)} \geq \frac{\beta}{\lambda\alpha^\rho}, \tag{3.3}$$

where $\lambda \in (0,1)$ and $\beta = (1+p)(m-1)!\ell^{m-1}$, then the solution $\{x(k)\}$ for equation (1.1) is oscillatory.

Proof. Now assume $\{x(k)\}$ is a non-oscillatory solution of equation (1.1). We can consider without loss of generality that there exists $k_1 \geq k_0$ such that $x(k) > 0$, $x(\tau(k)) > 0$ and $x(k-\rho\ell) > 0$ for all $k \geq k_1$. Now proceeding as in the previous theorem, we obtain (3.1). That is,

$$\Delta_{\alpha(\ell)}\left(a(k)\Delta_{\alpha(\ell)}^{m-1}z(k)\right) + LP(k)z(k-\rho\ell) + p\Delta_{\alpha(\ell)}\left(a(\tau(k))\Delta_{\alpha(\ell)}^{m-1}z(\tau(k))\right) \leq 0.$$

Now, since $\Delta_{\alpha(\ell)}^{m-1}z(k) > 0$, $\Delta_{\alpha(\ell)}^{m}z(k) \leq 0$, using Lemma 2.5 there exist $k_2 \geq k_1$ such that

$$\begin{aligned}
&\Delta_{\alpha(\ell)}\left(a(k)\Delta_{\alpha(\ell)}^{m-1}z(k)\right) \\
&+\frac{Q(k)}{(m-1)!\ell^{m-1}}\left(\frac{k-\rho\ell-j}{2^{m-2}}\right)_\ell^{(m-1)} \alpha^{\lceil \frac{k-\rho\ell-j+\ell}{\ell}-m\rceil}\Delta_{\alpha(\ell)}^{m-1}z(k-\rho\ell) \\
&+p\Delta_{\alpha(\ell)}\left(a(\tau(k))\Delta_{\alpha(\ell)}^{m-1}z(\tau(k))\right) \leq 0, \text{ for all } k \geq k_2 \geq 2^{m-2}.
\end{aligned}$$

Put $u(k) = a(k)\Delta_{\alpha(\ell)}^{m-1}z(k)$. Then $u(k) > 0$ and $\Delta_{\alpha(\ell)}u(k) \leq 0$ and the last inequality becomes

$$\begin{aligned}
&\Delta_{\alpha(\ell)}\left(u(k) + pu(\tau(k))\right) \\
&+\frac{\lambda Q(k)\alpha^{\lceil \frac{k-\rho\ell-j+\ell}{\ell}-m\rceil}}{(m-1)!\ell^{m-1}}\left(\frac{(k-\rho\ell-j)_\ell^{(m-1)}}{a(k-\rho\ell)}\right)u(k-\rho\ell) \leq 0,
\end{aligned} \tag{3.4}$$

for all $k \geq k_2$, for every λ, where $0 < \lambda = \left(\frac{1}{2^{m-2}}\right)^{m-1} < 1$.

Now put $w(k) = u(k) + pu(\tau(k))$. Then $w(k) > 0$. Since $u(k)$ is decreasing and having $\tau(k) = k + \tau \geq k$, we have

$$w(k) \leq (1+p)u(k). \tag{3.5}$$

Using (3.5) in (3.4), we notice that $w(k)$ is a positive solution of

$$\Delta_{\alpha(\ell)} w(k) + \frac{\lambda Q(k) \alpha^{\left\lceil \frac{k-\rho\ell-j+\ell}{\ell} - m \right\rceil}}{(m-1)!\ell^{m-1}} \left(\frac{(k-\rho\ell-j)_\ell^{(m-1)}}{(1+p)a(n-\rho\ell)} \right) w(k-\rho\ell) \leq 0, \tag{3.6}$$

for all $k \geq k_2$. Now there are two possibilities either (3.2) or (3.3) holds.
Case(i). If (3.2) holds, then by using the Lemma 2.7 we obtain the inequality (3.6) which has no positive solution, and that is again a contradiction.
Case(ii). If the condition (3.3) holds, by Lemma 2.7 we confirm that the inequality (3.6) has no positive solution, which intern is also a contradiction.
This completes the proof. □

Theorem 3.3. *Assume that $\eta(k) = \infty$ and $k - \rho\ell \leq \tau(k) \leq k$. If either*

$$\liminf_{k\to\infty} \sum_{r=0}^{\frac{\rho\ell-j-\ell}{\ell}} \frac{(k+r\ell-2\rho\ell)_\ell^{(m-1)} Q(k+j+r\ell-\rho\ell)}{\alpha^{\left\lceil m - \frac{k-2\rho\ell+r\ell+\ell}{\ell} \right\rceil} a(k+j+r\ell-2\rho\ell)} > \frac{\beta}{\alpha^\rho} \left(\frac{\rho\ell}{1+\rho\ell} \right)^{\rho\ell+1} \tag{3.7}$$

or

$$\limsup_{k\to\infty} \sum_{r=0}^{\frac{\rho\ell-j}{\ell}} \frac{(k+j+r\ell-2\rho\ell)_\ell^{(m-1)} Q(k+j+r\ell-\rho\ell)}{\alpha^{\left\lceil m - \frac{k-2\rho\ell+r\ell+\ell}{\ell} \right\rceil} a(k+j+r\ell-2\rho\ell)} > \frac{\beta}{\alpha^\rho}, \tag{3.8}$$

where $\beta = (1+p)(m-1)!\ell^{m-1}$, then every solution of equation (1.1) is oscillatory.

Proof. Similar to the proof of Theorem 3.2, we consider $\{x(k)\}$ is a non-oscillatory solution of equation (1.1). Then assume $\{x(k)\}$ is a positive solution of equation (1.1). It follows that there is an integer $k_1 \geq k_0$ such that $x(k) > 0$, $x(\tau(k)) > 0$ and $x(k-\rho\ell) > 0$ for all $k \geq k_1$. Now proceeding as in the previous theorem, we obtain

$$\begin{aligned} &\Delta_{\alpha(\ell)} \left(u(k) + pu(\tau(k)) \right) \\ &+ \frac{\lambda Q(k) \alpha^{\left\lceil \frac{k-\rho\ell-j+\ell}{\ell} - m \right\rceil}}{(m-1)!\ell^{m-1}} \left(\frac{(k-\rho\ell-j)_\ell^{(m-1)}}{a(k-\rho\ell)} \right) u(k-\rho\ell) \leq 0, \end{aligned} \tag{3.9}$$

Put $w(k) = u(k) + pu(\tau(k))$. Then $w(k) > 0$. Since $u(k)$ is decreasing, we have

$$w(k) = u(k) + pu(\tau(k)) \leq (1+p)u(\tau(k)) \text{ for } \tau(k) \leq k. \tag{3.10}$$

using (3.10) in (3.9), we get

(3.11)
$$\Delta_{\alpha(\ell)}w(k)+\frac{\lambda Q(k)\alpha^{\left\lceil\frac{k-\rho\ell-j+\ell}{\ell}-m\right\rceil}}{(m-1)!\ell^{m-1}}\left(\frac{(k-\rho\ell-j)_{\ell}^{(m-1)}}{(1+p)a(n-\rho\ell)}\right)w(\tau^{-1}(k-\rho\ell))\leq 0,$$

for all $k\geq k_1$.Thus $\{w(k)\}$ is a positive solution of the inequality (3.11). Now, we have to consider two cases namely:

Case(i). If (3.7) holds, then by using Lemma 2.7, we obtain the inequality (3.11), which has no positive solution, a contradiction.

Case(ii). If the condition (3.8) holds, Lemma 2.7 confirms that the inequality (3.11) has no positive solution, which is again a contradiction.

This completes the proof. □

Theorem 3.4. *Assume that $\eta(k)<\infty$ and $k-\rho\ell\leq\tau(k)\leq k$. If either (3.7) or when $\tau^{-1}(k-\rho\ell)$ is nondecreasing with (3.8) holds and for sufficiently large $k_1\geq k_0$*

$$\sum_{r=0}^{\frac{k-k_0-j-\ell}{\ell}}\left[\frac{\lambda\delta(k_2+j+r\ell)Q(k_0+j+r\ell)(k_0+j+r\ell-\rho\ell)_{\ell}^{m-2}}{(m-2)!\ell^{m-2}\alpha^{\left\lceil\frac{r\ell-\rho\ell+\ell}{\ell}-m\right\rceil}}\right.$$
$$\left.-\frac{(1+p)\alpha^{-\left\lceil\frac{r\ell+j+\ell}{\ell}\right\rceil}}{4a(k_0+j+r\ell+\ell)\delta(k_0+j+r\ell)}\right]=\infty, \tag{3.12}$$

then every solution $\{x(k)\}$ of equation (1.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a non-oscillatory and be a positive solution for equation (1.1). Then there exists an integer $k_1\geq k_0$ such that $x(k)>0$, $x(\tau(k))>0$ and $x(k-\rho\ell)>0$ for all $k\geq k_1$. From equation (1.1) we see that $\Delta_{\alpha(\ell)}\left(a(k)\Delta_{\alpha(\ell)}^{m-1}z(k)\right)\leq 0$ for all $k\geq k_1$. Since $\{a(k)\}$ is positive, $\Delta_{\alpha(\ell)}^{m-1}z(k)$ is of one sign for all $k\geq k_1$.

Case(i): Suppose $\Delta_{\alpha(\ell)}^{m-1}z(k)>0$ eventually, the proof for this case is similar to Case (i) of Theorem 3.3 and hence we omit the details.

Case(ii): Suppose $\Delta_{\alpha(\ell)}^{m-1}z(k)<0$ eventually, then by Lemma 2.4, we have $\Delta_{\alpha(\ell)}^{m-2}z(k)>0$ and $\Delta_{\alpha(\ell)}z(k)>0$. Now define $w(k)$ by

$$w(k)=\frac{a(k)\Delta_{\alpha(\ell)}^{m-1}z(k)}{\Delta_{\alpha(\ell)}^{m-2}z(k)}\text{ for all }k\geq k_2\geq k_1.$$

Then $w(k) < 0$ and

$$\Delta_{\alpha(\ell)} w(k) = \frac{\Delta_{\alpha(\ell)}\Big(a(k)\Delta_{\alpha(\ell)}^{m-1} z(k)\Big)}{\Delta_{\alpha(\ell)}^{m-2} z(k)} - \frac{a(k+\ell)\Delta_{\alpha(\ell)}^{m-1} z(k+\ell)}{\Delta_{\alpha(\ell)}^{m-2} z(k+\ell)\Delta_{\alpha(\ell)}^{m-2} z(k)}\Delta_{\alpha(\ell)}^{m-1} z(k)$$
$$+ (1-\alpha)\frac{a(k+\ell)\Delta_{\alpha(\ell)}^{m-1} z(k+\ell)}{\Delta_{\alpha(\ell)}^{m-2} z(k+\ell)} \text{ for all } k \geq k_2.$$

Since $a(k)\Delta_{\alpha(\ell)}^{m-1} z(k)$ is decreasing and $\Delta_{\alpha(\ell)}^{m-2} z(k)$ is increasing, we have

$$\Delta_{\alpha(\ell)} w(k) \leq \frac{\Delta_{\alpha(\ell)}\Big(a(k)\Delta_{\alpha(\ell)}^{m-1} z(k)\Big)}{\Delta_{\alpha(\ell)}^{m-2} z(k)} - \frac{w^2(k+\ell)}{a(k+\ell)} + (1-\alpha)w(k+\ell). \tag{3.13}$$

Using the decreasing nature of $a(k)\Delta_{\alpha(\ell)}^{m-1} z(k)$ we have

$$a(k_3)\Delta_{\alpha(\ell)}^{m-1} z(k_3) \leq a(k)\Delta_{\alpha(\ell)}^{m-1} z(k) \text{ for all } k_3 \geq k \geq k_2.$$

Dividing the last inequality by $a(k_3)$ and then summing it from k to $k_3 - \ell$, we obtain

$$\Delta_{\alpha(\ell)}^{m-2} z(k_3) - \alpha^{\left\lceil \frac{k_3-k}{\ell} \right\rceil}\Delta_{\alpha(\ell)}^{m-2} z(k) \leq a(k)\Delta_{\alpha(\ell)}^{m-1} z(k) \sum_{r=0}^{\frac{k_3-k-j-\ell}{\ell}} \frac{\alpha^{\left\lceil \frac{k-k_3+j+\ell+r\ell}{\ell} \right\rceil}}{a(k+j+r\ell)}.$$

Letting $l \to \infty$, we obtain

$$0 \leq \alpha^{\left\lceil \frac{k_3-k}{\ell} \right\rceil}\Delta_{\alpha(\ell)}^{m-2} z(k) + a(k)\Delta_{\alpha(\ell)}^{m-1} z(k)\delta(k)$$
$$\text{or} \quad -1 \leq \frac{a(k)\Delta_{\alpha(\ell)}^{m-1} z(k)\delta(k)}{\alpha^{\left\lceil \frac{k_3-k}{\ell} \right\rceil}\Delta_{\alpha(\ell)}^{m-2} z(k)} = \frac{w(k)\delta(k)}{\alpha^{\left\lceil \frac{k_3-k}{\ell} \right\rceil}} \leq 0 \text{ for all } k_3 \geq k \geq k_2.$$

Define $v(k)$ by

$$v(k) = \frac{a(\tau(k))\Delta_{\alpha(\ell)}^{m-1} z(\tau(k))}{\Delta_{\alpha(\ell)}^{m-2} z(k)} \text{ for all } k_3 \geq k \geq k_2. \tag{3.14}$$

We obtain $v(k) \leq 0$ and

$$-1 \leq \frac{v(k)\delta(k)}{\alpha^{\left\lceil \frac{k_3-k}{\ell} \right\rceil}} \leq 0 \text{ for all } k_3 \geq k \geq k_2.$$

From (3.14), we get

$$\Delta_{\alpha(\ell)}v(k) = \frac{\Delta_{\alpha(\ell)}\left(a(\tau(k))\Delta^{m-1}_{\alpha(\ell)}z(\tau(k))\right)}{\Delta^{m-2}_{\alpha(\ell)}z(k)} + (1-\alpha)\frac{a(\tau(k+\ell))\Delta^{m-1}_{\alpha(\ell)}z(\tau(k+\ell))}{\Delta^{m-2}_{\alpha(\ell)}z(k+\ell)}$$
$$- \frac{a(\tau(k+\ell))\Delta^{m-1}_{\alpha(\ell)}z(\tau(k+\ell))}{\Delta^{m-2}_{\alpha(\ell)}z(k+\ell)\Delta^{m-2}_{\alpha(\ell)}z(k)}\Delta^{m-1}_{\alpha(\ell)}z(k)$$

$$\leq \frac{\Delta_{\alpha(\ell)}\left(a(\tau(k))\Delta^{m-1}_{\alpha(\ell)}z(\tau(k))\right)}{\Delta^{m-2}_{\alpha(\ell)}z(k)} - \frac{v^2(k+\ell)}{a(\tau(k+\ell))} + (1-\alpha)v(k+\ell). \tag{3.15}$$

Combining (3.13) and (3.15), we obtain

$$\Delta_{\alpha(\ell)}w(k) + p\Delta_{\alpha(\ell)}v(k) \leq \frac{\Delta_{\alpha(\ell)}\left(a(k)\Delta^{m-1}_{\alpha(\ell)}z(k)\right)}{\Delta^{m-2}_{\alpha(\ell)}z(k)} - \frac{w^2(k+\ell)}{a(k+\ell)} + (1-\alpha)w(k+\ell)$$
$$+ p\frac{\Delta_{\alpha(\ell)}\left(a(\tau(k))\Delta^{m-1}_{\alpha(\ell)}z(\tau(k))\right)}{\Delta^{m-2}_{\alpha(\ell)}z(k)} - p\frac{v^2(k+\ell)}{a(\tau(k+\ell))} + p(1-\alpha)v(k+\ell).$$

Using (3.1) in the last inequality, we have

$$\Delta_{\alpha(\ell)}w(k) + p\Delta_{\alpha(\ell)}v(k) \leq \frac{-LP(k)z(k-\rho\ell)}{\Delta^{m-2}_{\alpha(\ell)}z(k)} - \frac{w^2(k+\ell)}{a(k+\ell)} - p\frac{v^2(k+\ell)}{a(\tau(k+\ell))}$$
$$+ (1-\alpha)w(k+\ell) + p(1-\alpha)v(k+\ell). \tag{3.16}$$

Now from Lemma 2.5 we obtain

$$z(k-\rho\ell) \geq \frac{\lambda\alpha^{\left\lceil \frac{k-\rho\ell-j+\ell}{\ell} - n\right\rceil}}{(m-2)!\ell^{m-2}}(k-\rho\ell-j)^{(m-2)}_{\ell}\Delta^{m-2}_{\alpha(\ell)}z(k-\rho\ell). \tag{3.17}$$

Since $\Delta^{m-1}_{\alpha(\ell)}z(k) < 0$ and $k - \rho\ell \leq k$, we have

$$\Delta^{m-2}_{\alpha(\ell)}z(k) < \Delta^{m-2}_{\alpha(\ell)}z(k-\rho\ell). \tag{3.18}$$

Combining the inequalities (3.16), (3.17) and (3.18), we obtain

$$\Delta_{\alpha(\ell)}w(k) + p\Delta_{\alpha(\ell)}v(k) \leq \frac{-\lambda Q(k)\alpha^{\left\lceil \frac{k-\rho\ell-j+\ell}{\ell} - n\right\rceil}}{(m-2)!\ell^{m-2}}(k-\rho\ell-j)^{(m-2)}_{\ell}$$
$$- \frac{w^2(k+\ell)}{a(k+\ell)} - p\frac{v^2(k+\ell)}{a(k+\ell)} - (\alpha-1)w(k+\ell) - p(\alpha-1)v(k+\ell). \tag{3.19}$$

Multiplying (3.19) by $\delta(k)$ and summation is taken on the resulting inequality from k_2 to $k-\ell$, we obtain

$$\delta(k)w(k)-\alpha^{\left\lceil\frac{k-k_2}{\ell}\right\rceil}\delta(k_2)w(k_2)+\sum_{r=0}^{m_2(k)}\frac{w(k_2+j+r\ell+\ell)}{a(k_2+j+r\ell)\alpha^{\left\lceil\frac{k_2+j+\ell-k+r\ell}{\ell}\right\rceil}}$$
$$+p\delta(k)v(k)-p\alpha^{\left\lceil\frac{k-k_2}{\ell}\right\rceil}\delta(k_2)v(k_2)+p\sum_{r=0}^{m_2(k)}\frac{v(k_2+j+r\ell+\ell)}{a(k_2+j+r\ell)\alpha^{\left\lceil\frac{k_2+j+\ell-k+r\ell}{\ell}\right\rceil}}$$
$$+\sum_{r=0}^{m_2(k)}\frac{w^2(k_2+j+r\ell+\ell)}{a(k_2+j+r\ell+\ell)}\frac{\delta(k_2+j+r\ell)}{\alpha^{\left\lceil\frac{k_2+j+\ell-k+r\ell}{\ell}\right\rceil}}$$

$$+p\sum_{r=0}^{m_2(k)}\frac{v^2(k_2+j+r\ell+\ell)}{a(k_2+j+r\ell+\ell)}\frac{\delta(k_2+j+r\ell)}{\alpha^{\left\lceil\frac{k_2+j+\ell-k+r\ell}{\ell}\right\rceil}}$$
$$+(\alpha-1)\sum_{r=0}^{m_2(k)}w(k_2+j+r\ell+\ell)\delta(k_2+j+r\ell)$$
$$+(\alpha-1)\sum_{r=0}^{m_2(k)}v(k_2+j+r\ell+\ell)\delta(k_2+j+r\ell)\leq 0.$$

Since $\{a(k)\}$ is increasing, and $\{\delta(k)\}$ decreasing, and on completion of square yields

$$\sum_{r=0}^{m_2(k)}\left[\frac{\lambda Q(k_2+j+r\ell)(k_2+j+r\ell-\rho\ell)_\ell^{(m-2)}\delta(k_2+j+r\ell)}{(m-2)!\ell^{m-2}\alpha^{\left\lceil\frac{k_2-k+r\ell-\rho\ell+\ell}{\ell}-m\right\rceil}}\right.$$
$$\left.-\frac{(1+p)\alpha^{-\left\lceil\frac{k_2-k+r\ell+j+\ell}{\ell}\right\rceil}}{4a(k_2+j+r\ell+\ell)\delta(k_2+j+r\ell+\ell)}\right]$$
$$+\delta(k)w(k)+p\delta(k)v(k)\leq\alpha^{\left\lceil\frac{k-k_2}{\ell}\right\rceil}\left(\delta(k_2)w(k_2)+p\delta(k_2)v(k_2)\right).$$

When we take limit supremum as $k\to\infty$ in the last inequality, we arrive at a contradiction to (3.12). This complete the proof. □

Theorem 3.5. *let $\delta(k) < \infty$ and let $\tau(k) \geq k$. If either (3.2) holds or $\tau^{-1}(k-\rho\ell)$ is non-decreasing with (3.3) holds and for sufficiently large $k_1 \geq k_0$*

$$\sum_{r=0}^{\frac{k-k_0-j-\ell}{\ell}} \left[\frac{\lambda\delta(\tau(k_2+j+r\ell))Q(k_0+j+r\ell)(k_0+j+r\ell-\rho\ell)_\ell^{(m-2)}}{(m-2)!\ell^{m-2}\alpha^{\lceil\frac{k_2+k+r\ell-\rho\ell+\ell}{\ell}-m\rceil}} - \frac{(1+p)\alpha^{-\lceil\frac{k_2-k+r\ell+j+\ell}{\ell}\rceil}}{4a(\tau(k_0+j+r\ell+\ell))\delta(\tau(k_0+j+r\ell+\ell))}\right] = \infty,$$

where $0 < \lambda < 1$ is a constant, then every solution $\{x(k)\}$ of equation (1.1) is oscillatory.

Proof. Consider $\{x(k)\}$ is a non-oscillatory solution of equation (1.1). We shall prove the case when $\{x(k)\}$ is positive as the case for $\{x(k)\}$ negative is similar. Since $\{x(k)\}$ is positive there exists an integer $k_1 \geq k_0$ such that $x(k) > 0$, $x(\tau(k)) > 0$ and $x(k-\rho\ell) > 0$ for all $k \geq k_1$. From equation (1.1), we see that $\{a(k)\Delta_{\alpha(\ell)}^{m-1}z(k)\}$ is decreasing for all $k \geq k_1$. Then there are two cases for $\Delta_{\alpha(\ell)}^{m-1}z(k)$, namely, either $\Delta_{\alpha(\ell)}^{m-1}z(k) > 0$ eventually or $\Delta_{\alpha(\ell)}^{m-1}z(k) < 0$ eventually.
Case(i). Suppose $\Delta_{\alpha(\ell)}^{m-1}z(k) > 0$ for all $k \geq k_1$, the proof is similar to that of case (i) of Theorem 3.2 and hence the details are omitted.
Case(ii). Suppose $\Delta_{\alpha(\ell)}^{m-1}z(k) < 0$ for all $k \geq k_1$, Then by Lemma 2.4, we have $\Delta_{\alpha(\ell)}^{m-2}z(k) > 0$ and $\Delta_{\alpha(\ell)}z(k) > 0$. Now define $\gamma(k)$ by

$$\gamma(k) = \frac{a(\tau(k))\Delta_{\alpha(\ell)}^{m-1}z(\tau(k))}{\Delta_{\alpha(\ell)}^{m-2}z(k)} \text{ for all } k \geq k_2 \geq k_1.$$

Then $\gamma(k) < 0$ for all $k \geq k_2$. Since $a(k)\Delta_{\alpha(\ell)}^{m-1}z(k)$ is decreasing we have

$$a(\tau(k_3))\Delta_{\alpha(\ell)}^{m-1}z(\tau(k_3)) \leq a(\tau(k))\Delta_{\alpha(\ell)}^{m-1}z(\tau(k)) \text{ for all } k3 \geq k \geq k_2.$$

Divide the last inequality by $a(\tau(k_3))$ and sum if from k to $k_3-\ell$, we obtain

$$\begin{aligned}&\Delta_{\alpha(\ell)}^{m-2}z(\tau(k_3)) - \alpha^{\lceil\frac{k_3-k}{\ell}\rceil}\Delta_{\alpha(\ell)}^{m-2}z(\tau(k)) \\ &\leq a(\tau(k))\Delta_{\alpha(\ell)}^{m-1}z(\tau(k)) \sum_{r=0}^{\frac{k_3-k-j-\ell}{\ell}} \frac{\alpha^{\lceil\tau(\frac{k-k_3+j+\ell+r\ell}{\ell})\rceil}}{a(\tau(k+j+r\ell))}.\end{aligned}$$

Letting $k_3 \to \infty$, we obtain

$$0 \leq \alpha^{\lceil\frac{k_3-k}{\ell}\rceil}\Delta_{\alpha(\ell)}^{m-2}z(\tau(k)) \leq a(\tau(k))\Delta_{\alpha(\ell)}^{m-2}z(\tau(k))\delta(\tau(k)). \tag{3.20}$$

Since $\Delta^{m-1}_{\alpha(\ell)} z(k) < 0$ and $\Delta^{m-1}_{\alpha(\ell)} z(k)$ is decreasing and for $\tau(k) \geq k$, we have

$$\Delta^{m-2}_{\alpha(\ell)} z(\tau(k)) \leq \Delta^{m-2}_{\alpha(\ell)} z(k). \tag{3.21}$$

Combining the inequalities (3.20), (3.21) and (3.18), we have

$$-1 \leq \frac{\gamma(k)\delta(\tau(k))}{\alpha^{\lceil \frac{k_3-k}{\ell} \rceil}} \leq 0 \text{ for all } k \geq k_2.$$

Similarly defining $w(k)$ by

$$w(k) = \frac{a(k)\Delta^{m-1}_{\alpha(\ell)} z(k)}{\Delta^{m-2}_{\alpha(\ell)} z(k)} \text{ for all } k \geq k_2,$$

we get

$$-1 \leq \frac{w(k)\delta(\tau(k))}{\alpha^{\lceil \frac{k_3-k}{\ell} \rceil}} \leq 0 \text{ for all } k \geq k_2.$$

Based on the proof of Theorem 3.4 we obtain (3.19). Multiplying (3.19) by $\delta(\tau(k))$ and then sum it form k_2 to $k-\ell$, we get

$$\begin{aligned}
&\delta(\tau(k))w(k) - \alpha^{\lceil \frac{k-k_2}{\ell} \rceil}\delta(\tau(k_2))w(k_2) + \sum_{r=0}^{m_2(k)} \frac{w(k_2+j+r\ell+\ell)}{a(\tau(k_2+j+r\ell))\alpha^{\lceil \frac{k_2+j+\ell-k+r\ell}{\ell} \rceil}} \\
&+ p\delta(\tau(k))v(\tau(k)) - p\alpha^{\lceil \frac{k-k_2}{\ell} \rceil}\delta(\tau(k_2))v(k_2) \\
&+ p\sum_{r=0}^{m_2(k)} \frac{v(k_2+j+r\ell+\ell)}{a(\tau(k_2+j+r\ell))\alpha^{\lceil \frac{k_2+j+\ell-k+r\ell}{\ell} \rceil}} \\
&+ \sum_{r=0}^{m_2(k)} \frac{w^2(k_2+j+r\ell+\ell)}{a(\tau(k_2+j+r\ell+\ell))} \frac{\delta(\tau(k_2+j+r\ell))}{\alpha^{\lceil \frac{k_2+j+\ell-k+r\ell}{\ell} \rceil}} \\
&+ \frac{\lambda}{(m-2)!\ell^{m-2}} \sum_{r=0}^{m_2(k)} \frac{Q(k_2+j+r\ell)(k_2+r\ell-\rho\ell)^{(m-2)}_{\ell}\delta(\tau(k_2+j+r\ell))}{\alpha^{\lceil \frac{k_2-\rho\ell+\ell-k+r\ell}{\ell} - m \rceil}} \\
&+ p\sum_{r=0}^{m_2(k)} \frac{v^2(k_2+j+r\ell+\ell)}{a(\tau(k_2+j+r\ell+\ell))} \frac{\delta(\tau(k_2+j+r\ell))}{\alpha^{\lceil \frac{k_2+j+\ell-k+r\ell}{\ell} \rceil}} \\
&+ (\alpha-1)\sum_{r=0}^{m_2(k)} w(k_2+j+r\ell+\ell)\delta(\tau(k_2+j+r\ell)) \\
&\qquad + (\alpha-1)\sum_{r=0}^{m_2(k)} v(k_2+j+r\ell+\ell)\delta(\tau(k_2+j+r\ell)) \leq 0.
\end{aligned}$$

Since $\{a(k)\}$ increasing, and $\{\delta(k)\}$ decreasing and by the completion of square, we arrive at

$$\begin{aligned}
&\delta(k)w(k)+p\delta(k)v(k)\\
&+\sum_{r=0}^{m_2(k)}\left[\frac{\lambda Q(k_2+j+r\ell)(k_2+j+r\ell-\rho\ell)_\ell^{(m-2)}\delta(\tau(k_2+j+r\ell))}{(m-2)!\ell^{m-2}\alpha^{\left\lceil\frac{k_2-k+r\ell-\rho\ell+\ell}{\ell}-m\right\rceil}}\right.\\
&\qquad\left.-\frac{(1+p)\alpha^{-\left\lceil\frac{k_2-k+r\ell+j+\ell}{\ell}\right\rceil}}{4a(\tau(k_2+j+r\ell+\ell))\delta(\tau(k_2+j+r\ell+\ell))}\right]\\
&\qquad\le \alpha^{\left\lceil\frac{k-k_2}{\ell}\right\rceil}\left(\delta(\tau(k_2))w(k_2)+p\delta(\tau(k_2))v(k_2)\right).
\end{aligned}$$

By taking limit as $k\to\infty$ in the last inequality and arrive at a result which is contrary to (3.12). That completes the proof. □

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