# SOME CONGRUENCE RELATIONS ON ODD GRACEFUL GRAPHS

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ABSTRACT. Graphs and their labeling are being applied in cryptographic models to repudiate illegal access to critical data and to deliver the required information security. Graphical passwords such as QR codes, Topsnut- graphical passwords are widely used in present world. Such passwords can be design with the help of cryptographical graphs which are developed by different types of graph labeling methods. A large scale of cryptographical graphs can be construct by using odd graceful labeling. However, identifying whether a given graph is odd graceful is a huge task which necessitates an approach for identifying odd graceful labeling by eliminating undesirable labeling. In this paper, we developed some computational techniques using congruence relations to eliminate unwanted labelings from the possible odd graceful labeling of a graph.

## 1. INTRODUCTION

The emergence of e-commerce, money transfer applications, social networks and other critical records generate huge amount of data to be managed. The security of data through real networks is one of the major challenges for the stakeholders. A technology named 'Cryptography' is widely used for ensuring the security of information or data through internet . Practitioners have used various cryptographic approaches for the protection of data from breaches and

cyber-attacks. The development of cryptographical graphs was a major break-through in cryptography and information technology mainly due to its merits over conventional techniques in terms of easiness to use for cryptographers and difficulty in deciphering for hackers [5].With this graph theoretic approach, any identified weaknesses in conventional methods of cryptography can be turned into a problem of graph theory by defining new graph labeling. Besides its application in cryptography, such studies enrich the theory of graph labeling.

In 1991, Gnanajothi introduced the concept of odd graceful graphs. She proposed Odd- Graceful Tree Conjecture and verified it for all trees of order upto 10 [4]. In [5], Hongyu Wang, Jin Xu, Bing Yao provide method to construct a series of cryptographical graphs of large size from smaller cryptographical trees, which has odd graceful labeling. Ajitha, S. Arumugan, and K. A. Germina [1] introduced a new approach to construct trees with $\alpha$-labeling from graceful trees. This approach essentially paved way for the construction of large classes of graceful or odd graceful trees. Such graphs can be used to construct cryptographical graphs and further to develop efficient algorithms [5]. Although several works have been done to develop graceful and odd graceful graphs, it is still difficult to identify whether a given labeling is odd graceful. This demands further extensive research in graph labeling to identify odd graceful graphs with advanced computational techniques.

With this background, in the present work, an approach is developed to identify odd graceful graphs using advanced computational techniques. This was achieved by introducing some necessary conditions for odd graceful graphs.

**Definition 1.1.** *An odd graceful labeling of a graph G is an injection $g : V(G) \longrightarrow \{0, 1, ...2m-1\}$ that induces a bijection $g^+ : V(G) \longrightarrow \{1, 3, ...2m-1\}$ of the edges defined by $g^+(xy) = |g(x) - g(y)|$ for all $xy \in E(G)$. The graph which admits such labeling is called an odd graceful graph.*

**Definition 1.2.** *[6] Let $g$ be any labeling of a graph $G$, then $deg_{i,n}(x)$ is the cardinality of the set $\{x \in V(G) : g^+(xy) \equiv i(mod n)$ for some y adjacent to x\}.*

Throughout this paper, we use the notation $\gamma(x)$ for $deg_{1,4}(x) - deg_{3,4}(x)$ and $\omega(xy)$ for $\frac{g(xy)+1}{2} + 1$. All graphs in this paper are finite, simple and undirected with vertex set V(G) and edge set E(G) respectively. The theoretical notations and terminologies used in the present work are based on Bondy and Murthy [2], while the adopted number theory is based on Burton [3].

## 2. MAIN RESULT

Here, we prove some interesting congruence relations on odd graceful graph.

**Theorem 2.1.** *Let G be an odd graceful graph with m edges. Then*

$$\sum_{x \in V(G)} deg(x)g(x) \equiv \begin{cases} 0 \ (mod\ 2) & \text{if m is even} \\ 1 \ (mod\ 2) & \text{if m is odd} \end{cases}$$

*Proof.* Given $G$ is an odd graceful graph with $m$ edges. Thus, the range set of $g^+$ is $\{1, 3, ...2m - 1\}$. Therefore, we have

$$\sum_{xy \in E(G)} g^+(xy) = \sum_{k=1}^{m} 2k - 1 = m^2 \text{ and } \sum_{xy \in E(G)} g^+(xy) \equiv \sum_{xy \in E(G)} (g(x) + g(y)) \ (mod\ 2).$$

Also, Here, $g(x)$ occur in the summation each time if there is an edge $xy$. Thus, $g(x)$ will occur precisely $deg(x)$ times in the sum for each $x \in V(G)$. Thus,

$$\sum_{x \in V(G)} deg(x)g(x) = \sum_{xy \in E(G)} (g(u) + g(y)).$$

Therefore,

$$\sum_{x \in V(G)} deg(x) = \sum_{uv \in E(G)} (g(x) + g(y))$$

$$\equiv \sum_{xy \in E(G)} g^+(xy) \ (mod\ 2)$$

$$\equiv m^2 \ (mod\ 2) \equiv \begin{cases} 0 \ (mod\ 2) & \text{if m is even} \\ 1 \ (mod\ 2) & \text{if m is odd} \end{cases}$$

$\square$

**Corollary 2.1.** *Let G be an odd-regular graph with odd graceful labeling $g$. Then*

$$\sum_{x \in V(G)} g(x) \equiv \begin{cases} 0 \ (mod\ 2), \text{if } n \equiv 0 \ (mod\ 4) \\ 1 \ (mod\ 2), \text{if } n \equiv 2 \ (mod\ 4) \end{cases}$$

*Proof.* Let G be a k-regular graph, where $k$ is an odd number. Then number of edges in G is $\frac{nk}{2}$ which is even if $n \equiv 0 \ (mod\ 4)$ and odd if $n \equiv 2 \ (mod\ 4)$.

$$\sum_{v \in V(G)} kg(x) \equiv \begin{cases} 0 \ (mod\ 2), \text{if } n \equiv 0 \ (mod\ 4) \\ 1 \ (mod\ 2), \text{if } n \equiv 2 \ (mod\ 4) \end{cases}$$

Since $k$ is odd, we have

$$\sum_{x \in V(G)} g(x) \equiv \begin{cases} 0 \ (\text{mod } 2), \text{if } n \equiv 0 \ (\text{mod } 4) \\ 1 \ (\text{mod } 2), \text{if } n \equiv 2 \ (\text{mod } 4) \end{cases}$$

$\square$

**Corollary 2.2.** *Let $G$ be an odd graceful graph with $m$ edges. If $G$ is Eulerian, then $m$ must be even.*

*Proof.* Given G is an euler graph. Therefore, G has an eulerian circuit. Then every vertex of G has even degree. Thus,

$$\sum_{x \in V(G)} deg(x)g(x) \equiv 0 (\text{mod } 2) \text{ and by Theorem 2.1, } \sum_{x \in V(G)} deg(x)g(x) \equiv m^2 (\text{mod } 2)$$

Thus, we have $m^2 \equiv 0 (\text{mod } 2) \Rightarrow m$ is even. $\square$

**Lemma 2.1.** *Let $g$ be an odd graceful labeling of G. Then*

$$\sum_{xy \in E(G)} (g(x)^2 + g(y)^2)(-1)^{\omega(xy)} = \sum_{x \in V(G)} \gamma(x)g(x)^2.$$

*Proof.*

$$\sum_{xy \in E(G)} (g(x)^2 + g(y)^2)(-1)^{\omega(xy)} = \sum_{g^+(xy) \equiv 1(4)} (g(x)^2 + g(y)^2) - \sum_{g^+(xy) \equiv 3(4)} (g(x)^2 + g(y)^2)$$

$$= \sum_{x \in V(G)} deg_{1,4}(x)g(x)^2 - \sum_{x \in V(G)} deg_{3,4}(x)g(x)^2$$

$$= \sum_{v \in V(G)} \gamma(x)g(x)^2$$

$\square$

**Theorem 2.2.** *Let G be an odd graceful graph with m edges. Then,*

$$\sum_{x \in V(G)} (deg(x) - \gamma(x))g(x)^2 = \begin{cases} \frac{2k(16k^2 - 27k + 11)}{3} \ (mod \ 8) & \text{if m is odd} \\ \frac{2k(16k^2 + 12k - 1)}{3} \ (mod \ 8) & \text{if m is even} \end{cases}$$

*Proof.* Given graph G is odd graceful graph with m edges. Thus, we have

$$\sum_{xy \in E(G)} g^+(xy) = \sum_{xy \in E(G)} (|g(x) - g(y)|^2) = 1^2 + 3^2 + \dots + (2m-1)^2 = \frac{4m^3 - m}{3}$$

(2.1)

Now, consider

$$\sum_{xy \in E(G)} (g(x) - g(y))^2 = \sum_{xy \in E(G)} (g(x) + g(y))^2 - 2 \sum_{xy \in E(G)} g(x)g(y)$$

$$= \sum_{x \in V(G)} deg(x)g(x)^2 - 2 \left( \sum_{g^+(xy) \equiv 3(4)} g(x)g(y) + \sum_{g^+(xy) \equiv 1(4)} g(x)g(y) \right)$$

$$(2.2)$$

Thus, from (2.1) and (2.2), we get

$$\frac{4m^3 - m}{3} = \sum_{x \in V(G)} deg(x)g(x)^2 - 2 \left( \sum_{g^+(xy) \equiv 3(4)} g(x)g(y) + \sum_{g^+(xy) \equiv 1(4)} g(x)g(y) \right)$$

$$(2.3)$$

If $m = 2k - 1$ is odd and by lemma 2.1, we have

$$2k^2 - 1 = \sum_{xy \in E(G)} (g(x) - g(y))^2 (-1)^{\omega(xy)}$$

$$= \sum_{xy \in E(G)} (g(x) + g(y))^2 (-1)^{\omega(xy)} - 2 \sum_{xy \in E(G)} g(x)g(y)(-1)^{\omega(xy)} \quad (2.4)$$

$$= \sum_{x \in V(G)} \gamma(x)g(x)^2 - 2 \sum_{g^+(xy) \equiv 3(4)} g(x)g(y) + 2 \sum_{g^+(xy) \equiv 1(4)} g(x)g(y)$$

Since $g(x)g(y)$ is even, subtracting (2.3) from (2.4) and applying congruence modulo $8$, we have

$$\sum_{x \in V(G)} (deg(x) - \gamma(x))g(x)^2 = \frac{2k(16k^2 - 27k + 11)}{3} \ (\text{mod } 8).$$

If $m = 2k$ is even, then, $-2(2k)^2 = \sum_{xy \in E(G)} (g(x) - g(y))^2 (-1)^{\omega(xy)}$.
  By similar procedure, we get

$$\sum_{x \in V(G)} (deg(x) - \gamma(x))g(x)^2 = \frac{2k(16k^2 + 12k - 1)}{3} \ (\text{mod } 8).$$

$$\square$$

**Example 1.** *From Table 1, we have*

$$\sum_{x \in V(G)} (deg(x) - \gamma(x))g(x)^2 \equiv 11398(mod\ 8) \equiv 6(mod\ 8)$$

FIGURE 1. An odd graceful graph with 14 edges
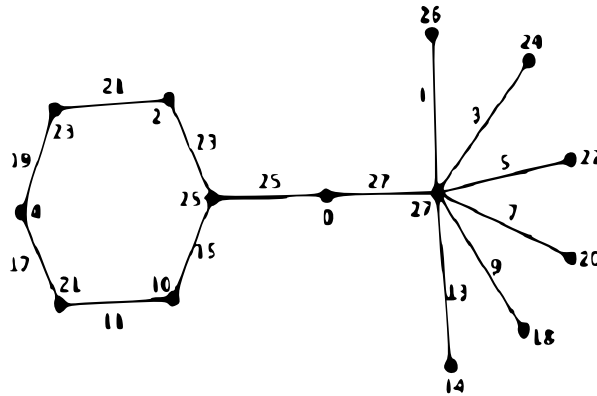
TABLE 1. Values of figure 1

| $g(x)$ | $deg(x) - \gamma(x)$ | $(deg(x) - \gamma(x))g(x)^2$ |
|---|---|---|
| 23 | 2 | 1250 |
| 2 | 2 | 8 |
| 21 | 2 | 882 |
| 4 | 2 | 32 |
| 24 | 2 | 1152 |
| 20 | 2 | 800 |
| 10 | 4 | 400 |
| 25 | 4 | 2500 |
| 27 | 6 | 4374 |
| 26,22,18,14,0 | 0 | 0 |
| Total | | 11398 |

and here $m = 14$ , therefore $k = 7$, thus

$$\frac{2k(16k^2 - 27k + 11)}{3} = 4046 \equiv 6(mod\ 8).$$

## 3. CONCLUSION

Graph labeling has got wide application in cryptography and construction of odd graceful graphs has been an area of continuous research in Mathematics. Identifying whether a graph is odd graceful is one of the most challenging task in

the theory of graph labeling. The complexity of problem generally increases with the size of graphs which necessitates an approach for identifying odd graceful labeling by eliminating undesirable labeling. An approach is presented here introduced some necessary conditions on odd graceful graphs using congruence relations. The approach was demonstrated successfully to eliminate undesirable labeling and thereby identify odd graceful graphs. While the application could be useful for cryptographers to identify odd graceful labeling, the approach is expected to enrich the theory of graph labeling.

## References

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