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GENERALIZATION OF WEAKLY G-EXPANSIVE AND WEAKLY G-CONTRACTIVE MAPPINGS

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ABSTRACT. The manuscript is devoted to investigation of generalized contractive and expansive mappings in G-metric spaces. We define the (m,p)-expansive and (m,p)-contractive mappings in generalized metric spaces, which are extensions of (m,p)-expansive and (m,p)-contractive mappings in metric spaces recently introduced by the forth named author in [16] and [17]. Some of basic properties of these classes of mappings are given.

1. INTRODUCTION AND PRELIMINARIES

The concept of a generalized metric (or G-metric) space is a generalization of usual metric spaces and it is introduced by Mustafa and Sims [7], [8] and [9] in the year 2004. For more results on G-metric spaces and fixed points results, we refer the interested reader to [4, 9–12, 15, 18].

Definition 1.1. *[8] Let $\mathcal{X}$ be a non-empty set and let $G : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}_+$ be a function satisfying the following conditions:*

(1) $G(x,y,z) = 0$ *if* $x = y = z$.
(2) $0 < G(x,x,y)$ *for all* $x, y \in \mathcal{X}$ *with* $x \neq y$.
(3) $G(x,x,y) \leq G(x,y,z)$ *for all* $x, y, z \in \mathcal{X}$ *with* $y \neq z$.

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(4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ *(symmetry in all three variables)*
(5) $G(x,y,z) \leq G(x,a,a)+G(a,y,z)$ *for all* $x,y,z,a \in \mathcal{X}$, *(rectangle inequality).*

Then the function G *is called a generalized metric or a* G*-metric on* $\mathcal{X}$ *and* $(\mathcal{X},G)$ *is called a* G*-metric space.*

The study of expansive and contractive mappings in generalized metric space is a very interesting research area in fixed point theory.

Let S be a self mapping on a G-metric $(\mathcal{X},G)$ space. Then S is called G-expansive if there exists a constant $\alpha > 1$ such that for all $(x,y,z) \in \mathcal{X}^3$, we have

$$G(Sx,Sy,Sz) \geq \alpha G(x,y,z)$$

(see [11]). S is said to be weakly G-expansive mapping if for all $(x,y,z) \in \mathcal{X}^3$,

$$G(Sx,Sy,Sz) \geq G(x,y,z).$$

A self mapping S of G-metric space $(\mathcal{X},G)$ is said to be G-contractive if there exists a constant $\beta \in (0,1)$ such that for all $(x,y,z) \in \mathcal{X}^3$, we have

$$G(Sx,Sy,Sz) \leq \beta G(x,y,z).$$

S is said to be weakly G-contractive if for all $(x,y,z) \in \mathcal{X}^3$,

$$G(Sx,Sy,Sz) \leq G(x,y,z).$$

Definition 1.2. *[6] Let* $(\mathcal{X},G)$ *be a* G*-metric space. We say that* $(x_n)_n$ *is*

(i) *a* G*-Cauchy sequence if, for any* $\epsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that*

$$\forall\ (n,m,l) \in \mathbb{N}^3 : \quad n,m,l \geq n_0 \Longrightarrow G(x_n,x_m,x_l) < \epsilon.$$

(ii) *a* G*-convergent sequence to* $x \in \mathcal{X}$ *if, for an* $\epsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that*

$$\forall\ \ (n,m) \in \mathbb{N}^2 : n,m \geq n_0 \ \Longrightarrow G(x_n,x_m,x) < \epsilon.$$

(iii) $(\mathcal{X},G)$ *is said to be complete if every* G*-Cauchy sequence in* $\mathcal{X}$ *is* G*-convergent in* $\mathcal{X}$.

Definition 1.3. *[7] Let* $(\mathcal{X},G)$ *be a* G*-metric space. A mapping* $S : \mathcal{X} \to \mathcal{X}$ *is said to be* G*-continuous if* $\big(Sx_n\big)_n$ *is* G*-convergent to* Sx *whenever* $(x_n)_n$ *is* G*-convergent to* x.

In recent work T. Bermúdez et. al. introduced and studied the concept of (m,q)-isometric maps on metric spaces.

Definition 1.4. *[5] Let (E,d) be a metric space. A map $S : E \longrightarrow E$ is called an (m,q)-isometry, ($m \in \mathbb{N}$ and $q \in (0,\infty)$) if, for all $x,y \in E$*

$$\sum_{0\le k\le m}(-1)^k\binom{m}{k}d\big(S^{m-k}x,S^{m-k}y\big)^q = 0.$$

Very recently, in papers [16] and [17],the author introduced and studied a classes of mappings acting on a metric space, called (m,p)-expansive and (m,p)-hyperexpansive. Given a map S on a metric space $(\mathcal{X},d)$ into itself, set

$$\Theta_m^{(p)}(d,S;x,y) := \sum_{0\le k\le m}(-1)^k\binom{m}{k}d\big(S^kx,S^ky\big)^p\ ,\forall\ x,y\in X,$$

where $m \in \mathbb{N}$ and $p \in (0,\infty)$. The map S is said to be (m,p)-expansive if

$$\Theta_m^{(p)}(d,S;x,y) \le 0.$$

When

$$\Theta_k^{(p)}(d,S;x,y) \le 0 \quad \text{for } k \in \{1,\cdots,m\},$$

we say that S is (m,p)-hyperexpansive. Moreover if $\Theta_m^{(p)}(d,S;x,y) \ge 0$, we say that S is (m,p)-contractive and if S is (k,p)-contractive for all positive integer $k \le m$, the map S is (m,p)-hypercontractive. If $\Theta_m^{(p)}(d,S;x,y) = 0$ for all x,y, the map S is said to be an (m,p)-isometry.

2. (m,p)-EXPANSIVE AND (m,p)-CONTRACTIVE MAPPINGS IN G- METRIC SPACE

In the following, let $(\mathcal{X},G)$ be a G-metric space, $S : \mathcal{X} \longrightarrow \mathcal{X}$ be a map, $m \in \mathbb{N}$ and $p \in (0,\infty)$. We define the quantity for all $(x,y,z) \in \mathcal{X}^3$ by

$$\mathcal{P}_m^{(p)}(S;x,y,z) := \sum_{0\le k\le m}(-1)^k\binom{m}{k}G\big(S^kx,S^ky,S^kz\big)^p.$$

The concept of (m,p)-isometric mappings on G-metric spaces was introduced and studied by A.M.Ahmadi in [2].

Definition 2.1. *[2] $S : \mathcal{X} \longrightarrow \mathcal{X}$ is an (m,p)-G-isometric mapping if and only if S satisfies*

$$\mathcal{P}_m^{(p)}(S;x,y,z) = 0 \quad \forall\ (x,y,z) \in \mathcal{X}^3.$$

Remark 2.1. *Observe that if S is a self map of a G-metric space $(\mathcal{X}, G)$ then*

(i) *S is an (1, p)-G-isometric if* $G\big(Sx, Sy, Sz\big) = G\big(x, y, z\big) \quad \forall\, (x, y, z) \in \mathcal{X}^3$.

(ii) *S is an $(2, p)$-G-isometric if*

$$G\big(S^2x, S^2y, S^2z\big)^p - 2G\big(Sx, Sy, Sz\big)^p + G\big(x, y, z\big)^p = 0 \quad \forall\, (x, y, z) \in \mathcal{X}^3.$$

Some properties of $(2, 1)$-G-isometric mappings have been proved in [13].

The following definition describes the families of maps we will study in this paper.

Definition 2.2. *Let $(\mathcal{X}, G)$ be a G-metric space and let $S : \mathcal{X} \longrightarrow \mathcal{X}$ be a map. We say that:*

(i) *S is (m, p)-G-expansive if $\mathcal{P}_m^{(p)}(S; x, y, z) \leq 0 \quad \forall\, (x, y, z) \in \mathcal{X}^3$;*

(ii) *S is (m, p)-G-hyperexpansive if $\mathcal{P}_k^{(p)}(S; x, y, z) \leq 0\ \forall\ k = 1, \cdots, m$ and $(x, y, z) \in \mathcal{X}^3$;*

(iii) *S is completely p-G-hyperexpansive if $\mathcal{P}_k^{(p)}(S; x, y, z) \leq 0\ \forall\ k \in \mathbb{N}$ and $(x, y, z) \in \mathcal{X}^3$.*

Definition 2.3. *Let $(\mathcal{X}, G)$ be a G-metric space and let $S : \mathcal{X} \longrightarrow \mathcal{X}$ be a map. We say that*

(i) *S is (m, p)-G-contractive if $\mathcal{P}_m^{(p)}(S; x, y, z) \geq 0 \quad \forall\, (x, y, z) \in \mathcal{X}^3$;*

(ii) *S is (m, p)-G-hypercontractive if $\mathcal{P}_k^{(p)}(S; x, y, z) \geq 0 \quad \forall\ k = 1, 2, \cdots, m$ and $(x, y, z) \in \mathcal{X}^3$;*

(iii) *S is completely p-G-hypercontractive if S is (k, p)-G-contractive for all $k \in \mathbb{N}$.*

Remark 2.2. (i) *For any $p \in (0, \infty)$, $(1, p)$-G-expansive mappings S coincides with weakly expansive; that is,*

$$G(Sx, Sy, Sz) \geq G(x, y, z) \ \text{ for all } \ (x, y, z) \in \mathcal{X}^3.$$

(ii) *For any $p \in (0, \infty)$, $(1, p)$-G-contractive mappings coincide with weakly contractive; that is,*

$$G(Sx, Sy, Sz) \leq G(x, y, z) \ \text{ for all } \ (x, y, z) \in \mathcal{X}^3.$$

(iii) *The case of (m, p)-G-isometries is the intersection of the class of (m, p)-G-expansive maps and the class of (m, p)-G-contractive maps.*

We consider the following examples of (m, p)-G-expansive mapping and (m, p)-G-contractive mapping which are not (m, p)-G-isometric mapping.

Example 1. *Let $\mathcal{X} = [0, \infty)$ be equipped with the G-metric defined as follows:*

$$G\big(x, y, z\big) = |x - y| + |x - z| + |y - z| \quad \forall \ (x, y, z) \in \mathcal{X}^3.$$

Define $S : \mathcal{X} \longrightarrow \mathcal{X}$ by $Sx = 3x$. Then by straightforward calculation, one can show that

$$\begin{aligned}
\mathcal{P}_m^{(p)}(S; x, y, z) &= \sum_{0 \le k \le m} (-1)^{m-k} \binom{m}{k} G(S^k x, S^k y, S^k x)^p \\
&= \sum_{0 \le k \le m} (-1)^{m-k} \binom{m}{k} \Big(|S^k x - S^k y| + |S^k x - S^k z| + |S^k y - S^k z|\Big)^p \\
&= \sum_{0 \le k \le m} (-1)^{m-k} \binom{m}{k} \big(3^p\big)^k \Big(|x - y| + |x - z| + |y - z|\Big)^p \\
&= \big(1 - 3^p\big)^m \Big(|x - y| + |x - z| + |y - z|\Big)^p.
\end{aligned}$$

Hence, S is a (m, p)-G-expansive map for positive odd integer m and a (m, p)-G-contractive map for positive even integer m.

Example 2. *$\mathcal{X} = [0, 1]$ and $G(x, y, z) = |x - y| + |y - z| + |z - x|$ be a G-metric on $\mathcal{X}$. Define the map S as follows $Sx = \frac{1}{2}x + \frac{1}{4}x^2$. Clearly $S\mathcal{X} \subset \mathcal{X}$, and S is contractive on $\mathcal{X}$, since*

$$\begin{aligned}
& G(Sx, Sy, Sz) \\
= \ & |\tfrac{1}{2}x + \tfrac{1}{4}x^2 - \tfrac{1}{2}y - \tfrac{1}{4}y^2| + \tfrac{1}{2}x + \tfrac{1}{4}x^2 - \tfrac{1}{2}z - \tfrac{1}{4}z^2| + |\tfrac{1}{2}z + \tfrac{1}{4}z^2 - \tfrac{1}{2}y - \tfrac{1}{4}y^2| \\
\le \ & \tfrac{1}{2}|x - y| + \tfrac{1}{4}|x - y||x + y| + \tfrac{1}{2}|x - z| + \tfrac{1}{4}|x - z||x + z| + \tfrac{1}{2}|z - y| + \tfrac{1}{4}|z - y||z + y| \\
< \ & \tfrac{1}{2}|x - y| + \tfrac{1}{2}|x - y| + \tfrac{1}{2}|x - z| + \tfrac{1}{2}|x - z| + \tfrac{1}{2}|z - y| + \tfrac{1}{2}|z - y| \\
& = |x - y| + |x - z| + |y - z|.
\end{aligned}$$

Thus S is a weakly contractive mapping on $\mathcal{X}$.

The following example shows that, in general, the G-expansiveness of a map S does not necessarily imply the (m, p)-G-expansiveness of S for $m \geq 2$.

Example 3. *Consider the usual G-metric $G(x, y, z) = |x - y| + |x - z| + |y - z|$ on $\mathbb{R}$. Let $S : (\mathbb{R}, G) \longrightarrow (\mathbb{R}, G)$ defined by $Sx = 3x + 2$. Then by a straightforward* *calculation, one can show that*

$$\begin{aligned} G(Sx,Sy,Sz) &= |Sx-Sy|+|Sx-Sz|+|Sy-Sz|) \\ &= 3\big(|x-y|+|x-z|+|y-z|\big) \geq G(x,y,z) \end{aligned}$$

and

$$\begin{aligned} & G(S^2x,S^2y,S^2z)^p - 2G(Sx,Sy,Sz)^p + G(x,y,z)^p \\ = \;& 3^{2p}G(x,y,z)^p - 2.3^pG(x,y,z)^p + G(x,y,z)^p = \big(3^p-1\big)^2G(x,y,z)^p \geq 0. \end{aligned}$$

Now we conclude that S is $(1,p)$-G-expansive but it fails to be an $(2,p)$-G-expansive. However, S is $(2,p)$-G-contractive but it is not $(1,p)$-G-contractive.

In same way, we have the similar example.

Example 4. *Let $(\mathcal{X},G)$ be a G-metric space. The map $d_G:\mathcal{X}\times\mathcal{X}\to\mathbb{R}_+$ defined by*

$$d_G(x,y) = \Big(G(x,y,y)^p + G(y,y,x)^p\Big)^{\frac{1}{p}} \quad \forall\ (x,y)\in\mathcal{X}^2,\ p>0$$

is a metric on $\mathcal{X}$. Let S is a self map of the G-metric space $(\mathcal{X},G)$). If S is a (m,p)-G-expansive (resp. (m,p)-G-contractive), then S is a (m,p)-expansive (resp. (m,p)-contractive in $(\mathcal{X},d_G)$.)

Remark 2.3.

$$\mathcal{P}_m^{(p)}(S;x,y,z)\leq 0 \Longleftrightarrow \mathcal{P}_m^{(p)}(S;S^nx,S^ny,S^nz)\leq 0,\ \forall\ (x,y,z)\in\mathcal{X}^3,\ \forall\ n\in\mathbb{N}_0.$$

Remark 2.4.

(i) *A self mapping S for a G-metric space $(\mathcal{X},G)$ is (m,p)-G-hyperexpansive if S is (k,p)-G-expansive for all positive integers $k\leq m$, and S is completely p-G-hyperexpansive if it is (m,p)-G-expansive for all positive integers m.*

(ii) *A self mapping S for a G-metric space $(\mathcal{X},G)$ is (m,p)-G-hypercontractive if S is (k,p)-G-contractive for all positive integers $k\leq m$, and S is completely p-G-hypercontractive if it is (m,p)-G-contractive for all positive integers m.*

We let the difference operator $\Psi:\mathbb{N}\longrightarrow\mathbb{R}$ given by the formula

$$\nabla\Psi(t) = \Psi(t) - \Psi(t+1).$$

Observe that the relations

$$\nabla^0\Psi = \Psi, \quad \nabla^n\Psi = \nabla\nabla^{n-1}\Psi$$

inductively define ∇^n for all $n \in \mathbb{N}$.

(1) A real map Ψ on $\mathbb{N}$ is said to be completely monotone if $(\nabla^n\Psi)(t) \geq 0$ for all $t \geq 0$ and $n \geq 1$.
(2) A real map Ψ on $\mathbb{N}$ is said to be completely alternating if $(\nabla^n\Psi)(t) \leq 0$ for all $t \geq 0$ and $n \geq 1$.

Proposition 2.1. *Let S be a self map of a G-metric space $(\mathcal{X}, G)$ and $p \in (0, \infty)$. The following statements hold:*

(i) *S is an completely- p-G-hyperexpansive if and only if, the map $\Psi_{x,\, y,\, z} : \mathbb{N} \to \mathbb{R}$ defined by $\Psi_{x,\, y,\, z}(n) = G\big(S^n x, S^n y, S^n z\big)^p$ for every $(x, y, z) \in \mathcal{X}^3$, is completely alternating.*
(ii) *S is completely-p-G-hypercontractive if and only if, the map $\Psi_{x,\, y,\, z} : \mathbb{N} \to \mathbb{R}$ defined by $\Psi_{x,\, y,\, z}(n) = G\big(S^n x, S^n y, S^n z\big)^p$ for every $(x, y, z) \in \mathcal{X}^3$, is completely monotone.*

Proof. By [3, Proposition 1.1, Proposition 1.2] we know that a map $\Phi : \mathbb{N} \longrightarrow \mathbb{R}$ is completely alternating if and only if $\sum_{0\leq k\leq n} (-1)^k \binom{n}{k} \Phi(m+k) \leq 0 \quad \forall\, m, n \in \mathbb{N}$ and it is completely monotone if and only if $\sum_{0\leq k\leq n} (-1)^k \binom{n}{k} \Phi(m+k) \geq 0 \quad \forall\, m, n \in \mathbb{N}$. By Choosing $\Phi = \Psi_{x,\, y,\, z}$, thus, the statements (i) and (ii) to be proved follow immediately. □

Proposition 2.2. *For a self map S of a G-metric space $(\mathcal{X}, G)$, and $(x, y, z) \in \mathcal{X}^3$, the following identity holds:*

$$\mathcal{P}_m^{(p)}(S; x, y, z) = \mathcal{P}_{m-1}^{(p)}(S; x, y, z) - \mathcal{P}_{m-1}^{(p)}(S; Sx, Sy, Sz). \tag{2.1}$$

Proof. We can use the standard binomial formula $\binom{m}{j} = \binom{m-1}{j} + \binom{m-1}{j-1}$ to prove the needed formula. Indeed

$$\begin{aligned}
&\mathcal{P}_m^{(p)}(S;x,y,z)\\
= &\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}G\big(S^kx,S^ky,S^kz\big)^p\\
= &G\big(x,y,z\big)^p+\sum_{1\leq k\leq m-1}(-1)^k\binom{m}{k}G\big(S^kx,S^ky,S^kz\big)^p+(-1)^mG\big(S^mx,S^my,S^mz\big)^p\\
= &G\big(x,y,z\big)^p+\sum_{1\leq k\leq m-1}(-1)^k\left\{\binom{m-1}{k}+\binom{m-1}{k-1}\right\}G\big(S^kx,S^ky,S^kz\big)^p+\\
&+(-1)^mG\big(S^mx,S^my,S^mz\big)^p\\
= &\mathcal{P}_{m-1}^{(p)}(S;x,y,z)-\mathcal{P}_{m-1}^{(p)}(S;Sx,Sy,Sz).
\end{aligned}$$

□

Theorem 2.1. *Let S be a self map of a G-metric space $(\mathcal{X},G)$. If S is (m,p)-G-expansive map for some $m\geq 2$ and it is $(2,p)$-G-hyperexpansive , then S is $(m-1,p)$-G-expansive.*

Proof. Since S is a $(2,p)$-G-hyperexpansive map it follows that for all $(x,y,z)\in \mathcal{X}^3$

$$G\big(x,y,z\big)^p-G\big(Sx,Sy\,Sz\big)^p\leq 0$$

and

$$G\big(x,y,z\big)^p-2G\big(Sx,Sy,Sz\big)^p+G\big(S^2x,S^2y,S^2z\big)^p\leq 0$$

or equivalently

$$G\big(S^2x,S^2y,S^2z\big)^p-G\big(Sx,Sy,Sz\big)^p\leq G\big(Sx,Sy,Sz\big)^p-G\big(Sx,Sy,Sz\big)^p.$$

Now, we prove that $\left(G\big(S^{n+1}x,S^{n+1}y,S^{n+1}z\big)^p-G\big(S^nx,S^ny,S^nz\big)^p\right)_{n\geq 0}$ is convergent. In fact, observe that this real sequence is monotonically non-increasing and bounded, so that it is convergent. Then there exists a positive constant K such that

$$G\big(S^{n+1}x,S^{n+1}y,S^{n+1}z\big)^p-G\big(S^nx,S^ny,S^nz\big)^p\to K\ \text{ as }\ n\to\infty.$$

Under the assumption $\mathcal{P}_m^{(p)}(S;x,y,z) \leq 0$ for all $(x,y,z) \in \mathcal{X}^3$ and $m \geq 2$ it follows by using (2.1) that

$$\mathcal{P}_{m-1}^{(p)}(x,y,z) \leq \mathcal{P}_{m-1}^{(p)}(S;Sx,Sy,Sz),$$

By repeating the process we get

$$\mathcal{P}_{m-1}^{(p)}(S;x,y,z) \leq \mathcal{P}_{m-1}^{(p)}(S;S^n x,S^n y,S^n z),\ n \geq 1.$$

Now to prove this desired result, it suffices to show that

$$\mathcal{P}_{m-1}^{(p)}(T;T^n x,T^n y,T^n z) \to 0 \quad \text{as } n \to \infty.$$

Here we note that

$$\mathcal{P}_{m-1}^{(p)}(S;x,y,z) = \mathcal{P}_{m-2}^{(p)}(S;x,y,z) - \mathcal{P}_{m-2}^{(p)}(S;Sx,Sy,Sz),$$

and therefore

$$\begin{aligned}&\mathcal{P}_{m-1}^{(p)}(S;S^n x,S^n y,S^n z)\\ =&\sum_{0\leq j\leq m-2}(-1)^j\binom{m-2}{j}\Big[G\big(S^{n+j}x,S^{n+j}y,S^{n+j}z\big)^p - G\big(S^{n+1+j}x,S^{n+1+j}y,S^{n+1+j}z\big)^p\Big].\end{aligned}$$

Letting $n \longrightarrow \infty$ in the preceding equality leads to

$$\mathcal{P}_{m-1}^{(p)}(S;S^n x,S^n y,S^n z) \longrightarrow \sum_{0\leq j\leq m-2}(-1)^j\binom{m-2}{j}K = 0.$$

From which we deduce that $\mathcal{P}_{m-1}^{(p)}(T;x,y,z) \leq 0$ for all $(x,y,z) \in \mathcal{X}^3$. Consequently, S is an (m,p)-G-expansive map and the proof is complete. □

The following example shows that Theorem 2.1 is not necessarily true if S is not $(2,p)$-hyperexpansive.

Example 5. *Let $\mathcal{X} = \mathbb{R}$ (the real line) and define the map $\widetilde{G} : \mathcal{X}^3 \longrightarrow \mathbb{R}_+$ as follows:*

$$\widetilde{G}(\alpha,\beta,\gamma) = \frac{1}{4}|\alpha-\beta| + \frac{1}{4}|\alpha-\gamma| + \frac{1}{4}|\beta-\gamma| \ \ (\alpha,\beta,\gamma) \in \mathcal{X}^3.$$

Define $S : \mathcal{X} \to \mathcal{X}$ by $Sx = 3+2x$. Then by a straightforward calculation, we show that S is a $(5,p)$-G-expansive but it fails to be a $(4,p)$-G-expansive.

Proposition 2.3. *Let S be a self maps of a G-metric space $(\mathcal{X}, G)$. Assume that S satisfies*

$$G\big(S^2x, S^2y, S^2z\big) = G\big(Sx, Sy, Sz\big) \text{ for all } (x, y, z) \in \mathcal{X}^3.$$

Then the following properties hold

(i) *S is (m, p)-G-expansive for positive integer m if and only if S is weakly expansive.*
(ii) *S is (m, p)-G-contractive for positive integer m if and only if,S is weakly contractive.*

Proof. Under the assumption on S, it follows that $G\big(S^kx, S^ky, S^kz\big) = G\big(Sx, Sy, Sz\big)$ for $k = 1, 2, \cdots, m$ and $(x, y, z) \in \mathcal{X}^3$. Thus, we have

$$\mathcal{P}_m^{(p)}(S; x, y, z) = G\big(x, y, z\big)^p - G\big(Sx, Sy, Sz\big)^p, \;\; \forall \;\; (x, y, z) \in \mathcal{X}^3.$$

It is clear from the foregoing that a sufficient and necessary condition for the sufficient condition for S to be (m, p)-G-expansive (resp. (m, p)-G-contractive) is that S is weakly expansive (resp. weakly contractive). □

Proposition 2.4. *Let S be a self map of a G-metric space $(\mathcal{X}, G)$. The following properties hold*

(1) *If S is weakly expansive map for which $S^2 = 0$ then, S is (m, p)-G-expansive.*
(2) *If S is (m, p)-G- contractive map for which $S^2 = 0$ then , S is weakly contractive.*

Proof. It we assume that $S^2 = 0$, we get for all $(x, y, z) \in \mathcal{X}^3$

$$\mathcal{P}_m^{(p)}(S; x, y, z) = G\big(x, y, z\big)^p - mG\big(Sx, Sy, Sz\big)^p \leq G\big(x, y, z\big)^p - G\big(Tx, Ty, Tz\big)^p.$$

(1) If S is weakly expansive, then

$$G\big(x, y, z\big) - mG\big(Sx, Sy, Sz\big) \leq 0 \Rightarrow \mathcal{P}_m^{(p)}(S; x, y, z) \leq 0.$$

Thus, we have S is (m, p)-G-expansion.

(2) If $\mathcal{P}_m^{(p)}(S; x, y, z) \geq 0$, it follows that

$$\mathcal{P}_m^{(p)}(x, y, z) = G\big(x, y, z\big)^p - mG\big(Sx, Sy, Sz\big)^p \geq 0 \Rightarrow G\big(x, y, z\big)^p \geq G(\big(Sx, Sy, Sz\big)^p.$$

Thus, we have S is weakly contractive.

□

In the following two theorems, we generalize [16, Proposition 2.8]

Theorem 2.2. *Let S be a bijective self map of G-metric space $(\mathcal{X}, G)$. If S is (m,p)-G-expansive map, then the following statements hold*

(i) *If m is even , then S^{-1} is (m,p)-G-expansive (resp. G-contractive) map.*
(ii) *If m is odd , then S^{-1} is (m,p)-G-contractive (resp. G-expansive) map.*

Proof. Under the assumption that $\mathcal{P}_m^{(p)}(S;x,y,z) \leq 0 \ \big(\text{resp. } \geq 0\big) \ \ \forall\,(x,y,z) \in \mathcal{X}^3$ we have by a computation stemming essentially from the formula

$$\binom{m}{k} = \binom{m}{m-k}; \text{for } k = 0, 1, \cdots, m,$$

$$\mathcal{P}_m^{(p)}(S^{-1};\ x,y,z) = (-1)^m \mathcal{P}_m^{(p)}(S;\ S^{-m}x, S^{-m}y, S^{-m}z).$$

Therefore for even integer m we have $\mathcal{P}_m^{(p)}(S^{-1};x,y,z) \leq 0 \quad \big(\text{resp. } \geq 0\big)$. Hence, S^{-1} is (m,p)-G-expansive $\big($resp. G-contractive $\big)$, and for odd integer m $\mathcal{P}_m^{(p)}(S^{-1};\ x,y,z) \geq 0 \ \big(\text{resp. } \geq 0\ \big)$ for all $(x,y,z) \in \mathcal{X}^3$. Hence, S^{-1} is (m,p)-G-contractive $\big($resp. G-contractive $\big)$. ☐

Theorem 2.3. *Let S be a bijective self map of G-metric space (X, G). If S is (m,p)-G-contractive map,then the following statements hold*

(i) *If m is even , then S^{-1} is (m,p)-G-contractive map.*
(ii) *If m is odd , then S^{-1} is (m,p)-G-expansive map.*

Proof. The proof is similar to the proof of the theorem above, hence we omit it here. ☐

Proposition 2.5. *Let S be a map of a G-metric space $(\mathcal{X}, G)$. If S is bijective $(2,p)$-G-expansive map, then S is $(1,p)$-G-isometric or G-isometric.*

Proof. Since S is a $(2,p)$-G-expansive, we have

$$G\big(S^2x, S^2y, S^2z\big)^p - G\big(Sx, Sy, Sz\big)^p \leq G\big(Sx, Sy, Sz\big)^p - G\big(Sx, Sy, Sz\big)^p.$$

So, it must be the case that

$$\begin{aligned} & G\big(S^{k+2}x, S^{k+2}y, S^{k+2}z\big)^p - G\big(S^{k+1}x, S^{k+1}y, S^{k+1}z\big)^p \\ \leq\ & G\big(S^{k+1}x, S^{k+1}y, S^{k+1}z\big)^p - G\big(S^kx, S^ky, S^kz\big)^p,\ k \geq 0. \end{aligned}$$

Hence

$$\begin{aligned} G\big(S^nx, S^ny, S^nz)\big)^p &\leq \sum_{1\leq k\leq n} \big(G\big(S^kx, S^ky, S^kz\big)^p - G\big(S^{k-1}x, S^{k-1}y, S^{k-1}z\big)^p\big) + G\big(x,y,z\big)^p \\ &\leq n\big(G\big(Sx,Sy,Sz\big)^p - G\big(x,y,z\big)^p\big) + G\big(x,y,z\big)^p \\ &\leq nG\big(Sx,Sy,Sz\big)^p + (1-n)G\big(x,y,z\big)^p, \end{aligned}$$

which gives,

$$G\big(Sx,Sy,Sz\big)^p \geq \frac{(1-n)}{n} G\big(x,y,z\big)^p \quad \forall\ (x,y,z) \in \mathcal{X}^3.$$

Taking the limit as $n \to \infty$, we see that

$$G\big(Sx,Sy,Sz\big)^p \geq G\big(x,y,z\big)^p, \quad \forall\ x,y,z) \in \mathcal{X}^3 \tag{2.2}$$

which implies that S is $(1,\ p)$-G-expansive or weakly expansive. Moreover, since S is a bijective $(2,p)$-G-expansive, then by application of Theorem 2.2, S^{-1} is $(2,p)$-G-expansive. Thus, we have

$$G\big(S^{-1}u, S^{-1}v, S^{-1}w\big)^p \geq G\big(u,v,w\big)^p \quad \forall\ \ (u,v,w) \in \mathcal{X}^3.$$

Thus, we deduce that,

$$G\big(S^{-1}Sx, S^{-1}Sy, S^{-1}Sz\big)^p = G\big(x,y,z\big)^p \geq G\big(Sx,Sy,Sz\big)^p, \ \ \forall\ \ (x,y,z) \in \mathcal{X}^3 \tag{2.3}$$

Since S satisfies conditions (2.2) and (2.3), then we have

$$G\big(Sx,Sy,Sz\big)^p = G\big(x,y,z\big)^p \ \ \forall\ \ (x,y,z), \in \mathcal{X}^3.$$

This means that S is a $(1,\ p)$-G-isometric or equivalently an G-isometric map. This completes the proof of the proposition. □

It is known (see [5]) that a self map S of a metric space $(\mathcal{X}, d)$ is power bounded map if

$$\sup\{\ d\big(T^nx, T^ny\big),\ n = 1,2,...\} < \infty \ \text{ for all } \ x,y \in \mathcal{X}.$$

In the following definition we extend this notion to a self map of a G-metric space as follows

Definition 2.4. *Let S be a self map for a G-metric space (X, G) . We say that S is G-power bounded if*

$$\sup\{\ G\big(S^n x, S^n y, S^n z\big),\ n = 1, 2, ...\} < \infty \ \textit{for all} \ (x, y, z) \in \mathcal{X}^3.$$

The following theorem gives a sufficient condition for an (m, p)-G-expansive map (resp.(m, p)-G-contractive) to be (m, p)-G-hyperexpansive (resp. resp. (m, p)-G-hypercontractive).

Theorem 2.4. *Let S be a self map of G metric space $(\mathcal{X}, G)$. The following statements hold.*

(1) *If S is (m, p)-G-expansive and G-power bounded, then S is (m, p)-G–hyperexpansive.*
(2) *If S is (m, p)-G-contractive and G-power bounded, then S is (m, p)-G-hypercontractive.*

Proof.

(1) It is obvious from the fact that S is (m, p)-G-expansive and (2.1) that

$$\mathcal{P}^{(p)}_{m-1}(S; x, y, z) \le \mathcal{P}^{(p)}_{m-1}(S; Sx, Sy, Sz) \le ... \le \mathcal{P}^{(p)}_{m-1}(T; T^n x, T^n y, T^n z).$$

Using (2.1),we obtain we have

$$\begin{aligned}&\mathcal{P}^{(p)}_{m-1}(S; S^n x, S^n y, S^n z)\\ = &\sum_{0\le k\le m-2} (-1)^k \binom{m-2}{k} \Big[\underbrace{G\big(S^{n+k}x, S^{n+k}y, S^{n+k}z\big)^p - G\big(S^{n+1+k}x, S^{n+1+k}y, S^{n+1+k}z\big)^p}_{=Q_n(x,y,z)}\Big].\end{aligned}$$

The condition that S is G-power bounded gives for all $(x, y, z) \in \mathcal{X}^3$ the sequence $(Q_n(x, y, z))_n \subset \mathbb{R}$ is bounded, therefore has convergent subsequence $(Q_{n_k}(x, y, z))_{k\ge 0}$ whose limit is $l \in \mathbb{R}$. We conclude that

$$\Big[G\big(S^{n_k+j}x, S^{n_k+j}y, S^{n_k+j}z\big)^p - G\big(S^{n_k+1+j}x, S^{n_k+1+j}y, S^{n_k+1+j}z\big)^p\Big] \to l \ \text{ as } \ k \to \infty.$$

So we get that

$$\mathcal{P}^{(p)}_{m-1}(S, S^{n_k}x, S^{n_k}y, S^{n_k}z) \to 0 \ \text{ as } \ k \to \infty.$$

This proves that $\mathcal{P}^{(p)}_{m-1}(S; x, y, z) \le 0$ and we can apply Definition 2.2 to obtain that S is $(m-1, p)$-G-expansive map. Regarding that S is

$(m-1,p)$-G-expansive and G-power bounded, in similar way we can get $\mathcal{P}_{m-2}^{(p)}(x,y,z) \leq 0$ for $(x,y,z) \in \mathcal{X}^3$. Analogously, we can conclude that

$$\mathcal{P}_k^{(p)}(S;x,y,z) \leq 0 \quad \text{for } \ 1 \leq k \leq m \quad \text{ and } (x,y,z) \in \mathcal{X}^3.$$

Hence, S is (m,p)-G-hyperexpansive.

(2) Regarding that S is an (m,p)-contractive together (2.1), we observe that

$$\mathcal{P}_{m-1}^{(p)}(S;x,yz) \geq \mathcal{P}_{m-1}^{(p)}(S;Sx,Sy,Sz) \geq \cdots \geq \mathcal{P}_{m-1}^{(p)}(S;S^nx,S^ny,S^nz).$$

Tanking into account (2.1),we have

$$\mathcal{P}_{m-1}^{(p)}(S;S^nx,S^ny,S^nx) = \mathcal{P}_{m-2}^{(p)}(S;S^nx,S^ny,S^nz) - \mathcal{P}_{m-2}^{(p)}(S;S^{n+1}x,S^{n+1}y,S^{n+1}z),$$

and by a routine calculation, one can verify that

$$\begin{aligned} &\mathcal{P}_{m-1}^{(p)}(S;S^nx,S^ny,S^nz) \\ = &\sum_{0\leq j\leq m-2} (-1)^j \binom{m-2}{j} \Big[G\big(S^{n+j}x,S^{n+j}y,S^{n+j}z\big)^p - G\big(S^{n+1+j}x,S^{n+1+j}y,S^{n+1+j}z\big)^p \Big]. \end{aligned}$$

Following the line of the proof of the statement (1) one can easily get

$$\mathcal{P}_{m-1}^{(p)}(S;S^{n_k}x,S^{n_k}y,S^{n_k}z) \to 0 \ \text{ as } \ k \to \infty.$$

This implies that, $\mathcal{P}_{m-1}^{(p)}(x,y,z) \geq 0$ and so that, S is $(m-1,p)$-contractive. By repeating this process, we reach the following inequalities $\mathcal{P}_k^{(p)}(S;x,y,x) \geq 0$ for $k = 1,2,\cdots,m$ and $(x,y,z) \in \mathcal{X}^3$ which shows that S is an (m,p)-G-hypercontractive.

□

The following theorem gives a characterization of $(3,p)$-G-isometric mappings. Our inspiration cames form [14].

Theorem 2.5. *Let S be a self mapping for a G-metric space $(\mathcal{X},G)$. Then S is an $(3,p)$-G-isometric mapping if and only if S satisfies*

$$G\big(S^nx,S^ny,S^nz\big)^p = G\big(x,y,z\big)^p + nQ_1(x,y,z) + n^2Q_2(x,y,z) \tag{2.4}$$

where

$$Q_2(x,y,z) = \frac{1}{2}\bigg(G(S^2x,S^2y,S^2z)^p - 2G(Sx,Sy,Sz)^p + G\big(x,y,z\big)^p \bigg)$$

and

$$Q_1(x,y,z)=\frac{1}{2}\bigg(-G(S^2x,S^2y,S^2z)^p+4G(Sx,Sy,Sz)^p-3G\big(x,y,z\big)^p\bigg).$$

Proof. We prove the if part of the theorem. Assume that S satisfies (2.4). For $n=3$ we obtain

$$\begin{aligned}
&G\big(S^3x,S^3y,S^3z\big)^p\\
=\;&G\big(x,y,z\big)^p+3Q_1(x,y,z)+9Q_2(x,y,z)\\
=\;&G\big(x,y,z\big)^p+\frac{3}{2}\bigg(-G(S^2x,S^2y,S^2z)^p+4G(Sx,Sy,Sz)^p-3G\big(x,y,z\big)^p\bigg)\\
&+\frac{9}{2}\bigg(G(S^2x,S^2y,S^2z)^p-2G(Sx,Sy,Sz)^p+G\big(x,y,z\big)^p\bigg)\\
=\;&G\big(x,y,z\big)^p-3G(S^2x,S^2y,S^2z)^p-3G(Sx,Sy,Sz)^p.
\end{aligned}$$

Hence, we have

$$G\big(S^3x,S^3y,S^3z\big)^p-3G(S^2x,S^2y,S^2z)^p+3G(Sx,Sy,Sz)^p-G\big(x,y,z\big)^p=0,$$

and so that, S is an $(3,p)$-G-isometry.

We prove the only if part. Assume that S is an $(3,p)$-G-isometry. We prove (2.4) by mathematical induction. For $n=1$ it is true. Assume that (2.4) is true for n and prove it for $n+1.$ Indeed, for all $(x,y,z)\in\mathcal{X}^3$ we have

$$\begin{aligned}
&G\big(S^{n+1}x,S^{n+1}y,S^{n+1}z\big)^p\\
=\;&G\big(S^nSx,S^nSy,S^nSz\big)^p\\
=\;&G\big(Sx,Sy,Sz\big)^p+nQ_1(Sx,Sy,Sz)+n^2Q_2(Sx,Sy,Sz)\\
=\;&G\big(Sx,Sy,Sz\big)^p+\frac{n}{2}\bigg(-G(S^3x,S^3y,S^3z)^p+4G(S^2x,S^2y,S^2z)^p-3G\big(Sx,Sy,Sz\big)^p\bigg)\\
&+\frac{n^2}{2}\bigg(G(S^3x,S^3y,S^3z)^p-2G(S^2x,S^y,S^z)^p+G\big(Sx,Sy,Sz\big)^p\bigg)\\
=\;&\bigg(\frac{n^2-n}{2}\bigg)G\big(S^3x,S^3y,S^3z\big)^p-\big(n^2-2n\big)G\big(S^2x,S^2y,S^2z\big)^p\\
&+\bigg(\frac{n^2-3n+2}{2}\bigg)G\big(Sx,Sy,Sz\big)^p.
\end{aligned}$$

Now, using the fact that S is an $(3,p)$-G-isometry we can obtained

$$\begin{aligned}
& G\big(S^{n+1}x, S^{n+1}y, S^{n+1}z\big)^p \\
= & \left(\frac{n^2-n}{2}\right)\Big(G\big(x,y,z\big)^p + 3G(S^2x,S^2y,S^2z)^p - 3G(Sx,Sy,Sz)^p\Big) \\
& + - \big(n^2-2n\big)G\big(S^2x,S^2y,S^2z\big)^p \\
& + \left(\frac{n^2-3n+2}{2}\right)G\big(Sx,Sy,Sz\big)^p \\
= & \left(\frac{n^2+n}{2}\right)G(S^2x,S^2y,S^2z)^p + \left(\frac{-2n^2+2}{2}\right)G\big(Sx,Sy,Sz\big)^p \\
& + \left(\frac{n^2-n}{2}\right)G\big(x,y,z\big)^p \\
& = \left(\frac{n^2+n}{2}\right)\Big(G\big(x,y,z\big)^p + 2Q_1(x,y,z) + 4Q_2(x,y,z)\Big) \\
& + \left(\frac{-2n^2+2}{2}\right)\Big(G\big(x,y,z\big)^p + Q_1(x,y,z) + Q_2(x,y,z)\Big) + \left(\frac{n^2-n}{2}\right)G\big(x,y,z\big)^p \\
= & G\big(x,y,z\big)^p + (n+1)Q_1(x,y,z) + (n+1)^2Q_2(x,y,z).
\end{aligned}$$

□

Proposition 2.6. *Let $(\mathcal{X}_k, G_k)$ be a G-metric space for $k = 1, 2, \cdots n$ and let S_k be a self mapping for a the G-metric space $(\mathcal{X}_k, G_k)$,$k = 1, \cdots, n$ be . Put $\mathcal{X} = \prod_{1\le k\le n} \mathcal{X}_k$ the product space endowed with the product G-metric defined by*

$$G\big((x_k)_{1\le n}, ((y_k)_{1\le n}, z_k)_{1\le n}\big) = \Big(\sum_{1\le k\le n} G_k(x_k, y_k, z_k)^p\Big)^p, \; p > 0.$$

Define the map $S = S_1 \times S_2 \times \cdots \times S_d : (\mathcal{X}, G) \longrightarrow (\mathcal{X}, G)$ as follows

$$Sx = (S_1x_1, S_2x_2, \cdots, S_nx_n). \;\; (x_1, \cdots, x_n) \in \mathcal{X}^n.$$

the following statements hold.

(i) *If each S_k is an (m,p)-G-isometric mapping, then S is an (m,p)-G-isometric mapping.*
(ii) *If each S_k is an (m,p)-G-expansive mapping, then S is an (m,p)-G-expansive mapping.*
(iii) *If each S_k is an (m,p)-G-contractive mapping, then S is an (m,p)-G-contractive.*

(iv) *If S_k is (m_k, p)-G-hyperexpansive mapping for $1 \leq k \leq n$, then S is an (m,p)-G-expansive where $m = \min(m_1, \cdots, m_n$.*

(v) *If T_k is (m_k, p)-G-hypercontractive mapping for $1 \leq k \leq n$, then S is an (m,p)-G-contractive where $m = \min(m_1, \cdots, m_n$.*

Proof. Let $x = (x_k)_{1\leq k\leq d}, y = (y_k)_{1\leq k\leq d}$ and $z = (z_k)_{1\leq k\leq d} \in \mathcal{X}$. We have that

$$\mathcal{P}_m^{(p)}(S; x, y, z) = \sum_{0\leq j\leq m} (-1)^j \binom{m}{j} G(S^j x, S^j y, S^j z)^p$$

$$= \sum_{0\leq j\leq m} (-1)^j \binom{m}{j} \Big(\sum_{1\leq k\leq d} \Big(G_k(S_k^j x_k, S_k^j y_k, S_k^j z_k)^p \Big) \Big)$$

$$= \sum_{1\leq k\leq n} \Big(\sum_{0\leq j\leq m} (-1)^j \binom{m}{j} G(T_k^j x_k, T_k^j y_k, T_k^j z_k)^p \Big)$$

$$= \sum_{1\leq k\leq n} \mathcal{P}_m^{(p)}(S_k, x_k, y_k, z_k)..$$

The statements (i),(ii) and (iii) follows immediately.

(iv) If S_k is (m_k, p)-G-hyperexpansive for $k = 1, \cdots, n$, it follows that S_k is (m,p)-G-expansive mapping and hence S is (m,p)-G-expansive by statement (ii).

(v) If S_k is (m_k, p)-G-hypercontractive for $k = 1, \cdots, n$, it follows that S_k is (m,p)-G-contractive mapping and hence S is (m,p)-G-contractive by statement (iii). □

Recall that an bounded operator $S : \mathcal{H} \longrightarrow \mathcal{H}$ ($\mathcal{H}$ is a Hilbert space) is called an m-isometric if S satisfies

$$\sum_{0\leq k\leq m} (-1)^{m-k} \binom{m}{k} S^{*k} S^k = 0.$$

In [1], it was proved that if S is an m-isometric operator , then S is injective ant its range is closed.

In the following theorem, we generalize the above-mentioned results according to (m,p)-G-isometric mapping in compete G-metric space.

Theorem 2.6. *Let S be a self mapping of a complete G-metric space $(\mathcal{X}, G)$. If S is G-continuous (m,p)-G-isometric mapping. Then S is injective and $\mathcal{R}(S)$ (the range of S) is G-closed in $\mathcal{X}$.*

Proof. Firstly, we prove that S is injective. Let $x, y \in \mathcal{X}$ such that $Sx = Sy$. Assume that $x \neq y$. Since S is an (m,p)-G-isometric mapping and $S^k x = S^k y$ for $k = 1, \cdots, m$ it follows that

$$\mathcal{P}_m^{(p)}(S,x,y,z) = 0 \Leftrightarrow G\big(x,y,z\big)^p + \sum_{1\leq k\leq m}(-1)^k\binom{m}{k}G\big(S^kx, S^kx, S^kz\big)^p = 0.$$

By taking $x = z$ we obtain

$$G\big(x,y,x\big)^p + \sum_{1\leq k\leq m}(-1)^k\binom{m}{k}\underbrace{G\big(S^kx,S^kx,S^kx\big)^p}_{=0} = 0.$$

So, it must be the case that $G\big(x,y,x\big) = 0$. By the second condition of G-metric, we get a contradiction. Hence, $x = y$ and S is injective map.

We prove that $\mathcal{R}(S)$ is G-closed. Let $(x_n)_n$ be a sequence in $\mathcal{X}$ such that $Tx_n \longrightarrow y$ in $(\mathcal{X}, G)$. Since S is G-continuous we have $S^k x_n \longrightarrow S^k y$ in $(\mathcal{X}, G)$ for $k = 1, \cdots, m.$. Under the assumption that S is an (m,p)-G-isometric, we get

$$G\big(x_n,x_m,x_l\big)^p = -\sum_{1\leq k\leq m}(-1)^k\binom{m}{k}G\big(S^kx_n,S^kx_m,S^kx_l\big) \;\text{ for all }\; n,m,l>0.$$

It is well know that $(S^k x_n)_n$ are Cauchy sequences in $(\mathcal{X}, G)$ for $k = 1, 2, \cdots, m$. We obtain that $(x_n)_n$ is a Cauchy sequence in $(\mathcal{X}, G)$. Due to the completeness of $(\mathcal{X}, G)$, there exists $x \in \mathcal{X}$ such that $(x_n)_n$ is G-convergent to $x.$ On the other hand, using the fact that S is G-continuous, which yields that $Sx_n \longrightarrow Sx$ as $n \to \infty$. This implies that $y = Sx$, which yields that $\mathcal{R}(S)$ is G-closed. □

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