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# SOME REMARKS ON MODIFIED PICARD OPERATORS PRESERVING SOME EXPONENTIAL FUNCTIONS

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ABSTRACT. In the present paper, we investigate the convergence properties of a class of modified Picard operators in exponential weighted Lebesgue spaces.

## 1. INTRODUCTION AND THEORETICAL BACKGROUND

Let $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{N} = \{1, 2, ...\}$. Consider the following integral operators

$$(\mathcal{P}_n f)(x) = \frac{n}{2}\int_{-\infty}^{\infty} f(x+t)\, e^{-n|t|} dt, \quad x \in \mathbb{R}, \quad n \in \mathbb{N} \tag{1.1}$$

and

$$(\mathcal{W}_n f)(x) = \frac{\sqrt{n}}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+t)\, e^{-nt^2} dt, \; x \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{1.2}$$

These operators are of type Picard and Gauss-Weierstrass, respectively. Since the kernel functions of these operators are approximate identities, they have been the most used operators in approximation theory (see, e.g., [3, 9, 12, 13, 17]). Detailed information about these operators can be found in the monograph [8]. Also, nonlinear counterparts were extensively studied in the monograph [6].

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In [14], the authors considered the Picard-type operators acting on exponential weighted Lebesgue spaces and proved some approximation theorems concerning rate of convergence of the operators and their higher order derivatives using suitable modulus of smoothness. Related Voronovskaya and quantitative Voronovskaya theorems were also presented.

In [1], the authors constructed the following modifications of the operators of type (1.1) and (1.2):

$$(\mathcal{P}_n^* f)(x) = \frac{\sqrt{n}}{2} \int_{-\infty}^{\infty} f(\beta_n(x) + t) e^{-\sqrt{n}|t|} dt, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}, \; n \geq n_a \tag{1.3}$$

and

$$(\mathcal{W}_n^* f)(x) = \frac{\sqrt{n}}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(\gamma_n(x) + t) e^{-nt^2} dt, \; x \in \mathbb{R}, \quad n \in \mathbb{N}, \tag{1.4}$$

where $\beta_n(x) = x - \frac{1}{2a} \ln\left(\frac{n}{n-(2a)^2}\right)$ with $n \geq n_a$, $\gamma_n(x) = x - \frac{a}{2n}$ and $a > 0$ is a fixed real number. Here, $n_a := [4a^2] + 1$, where $[.]$ denotes floor function. In the same work various properties of these modifications were given. After this work, in [5] and [19], some further convergence properties of the operators of type (1.3) were investigated in exponential weighted Lebesgue spaces. Recently, in [18], the author presented some results concerning the operators of type (1.4) in exponential weighted Lebesgue spaces. In particular, some characteristic features of $m-$singular generalization of the operators of type (1.4) were examined in [16].

In [4], the author constructed the following modification of Picard integral operators preserving $e^{ax}$ and $e^{2ax}$ with $a > 0$ :

(1.5)
$$(\mathcal{P}_n^{**} f)(x) = \frac{\sqrt{n}}{2} \int_{-\infty}^{\infty} e^{-a(\beta_n^{**}(x)+t)} e^{ax} f(\beta_n^{**}(x) + t) e^{-\sqrt{n}|t|} dt, \quad x \in \mathbb{R}, \; n > n_a,$$

where $\beta_n^{**}(x) = x - \beta_n$, with $\beta_n = \frac{1}{a} \ln\left(\frac{n}{n-a^2}\right)$, $n > n_a$ and $n \in \mathbb{N}$. Here, $n_a := [a^2] + 1$, where $[.]$ denotes floor function. In this work, as an extention of the work [4], we will investigate the convergence properties of the operators of type (1.5) in exponential weighted Lebesgue spaces.

## 2. Main Concepts and Auxiliary Results

Let $v_a(x) := e^{-a|x|}$ for $x \in \mathbb{R}$ with $a > 0$ and $1 \leq p < \infty$ be fixed real numbers. Following [7] and also [14], $L_p^a(\mathbb{R})$ denotes the space of all measurable functions $f$ for which $p-$th power of $v_a f$ is integrable in the sense of Lebesgue. The norm of $f$ satisfies $\|f\|_{L_p^a(\mathbb{R})} := \left( \int\limits_{-\infty}^{\infty} \left| e^{-a|x|} f(x) \right|^p dx \right)^{\frac{1}{p}} < \infty$.

As in [14], for $\delta \geq 0$, we use the following modulus of smoothness:

$$\omega\left(f; L_p^a; \delta\right) := \sup_{|h| \leq \delta} \left( \int\limits_{-\infty}^{\infty} e^{-ap|x|} \left| f(x+h) - f(x) \right|^p dx \right)^{\frac{1}{p}}. \tag{2.1}$$

For $\lambda, \delta \geq 0$, $\omega$ satisfies (see [10]):

i $\omega\left(f; L_p^a; \delta_1\right) \leq \omega\left(f; L_p^a; \delta_2\right)$ with $0 \leq \delta_1 < \delta_2$

ii $\omega\left(f; L_p^a; \lambda\delta\right) \leq (1+\lambda) e^{\lambda a \delta} \omega\left(f; L_p^a; \delta\right)$

iii $\lim\limits_{\delta \to 0^+} \omega\left(f; L_p^a; \delta\right) = 0.$

Now, we quote from [4] the following lemma, consisting of necessary identities, which will be used in the sequel.

**Lemma 2.1.** *[4] Let $e_0 = 1$, $e_1 = t$ and $e_2 = t^2$ for $t \in \mathbb{R}$ be test functions and $a > 0$ be fixed. For $x \in \mathbb{R}$ and sufficiently large $n$ satisfying $\sqrt{n} > a$, one has*

$\left(\mathcal{P}_n^{**} e_0\right)(x) = \frac{n^2}{(n-a^2)^2}$ tends to $1$ as $n$ tends to $+\infty$,

$\left(\mathcal{P}_n^{**} e_1\right)(x) = e^{a(x-\beta_n^{**}(x))} n \left( \frac{\beta_n^{**}(x)}{n-a^2} - \frac{2a}{(n-a^2)^2} \right)$ tends to $x$ as $n$ tends to $+\infty$,

and

$\left(\mathcal{P}_n^{**} e_2\right)(x) = \frac{e^{a\left(x-\beta_n^{**}(x)\right)}}{(n-a^2)^3} n \left( \left(\beta_n^{**}(x)\right)^2 \left(a^4 - 2a^2 n + n^2\right) + \beta_n^{**}(x) \left(4a^3 - 4an\right) + 6a^2 + 2n \right)$

tends to $x^2$ as $n$ tends to $+\infty$.

In the following lemma existence of the operators of type (1.5) in exponential weighted Lebesgue space $L_p^a(\mathbb{R})$ is established.

**Lemma 2.2.** *If $f \in L_p^a(\mathbb{R})$ with fixed $1 \leq p < \infty$ and $a > 0$, then there holds*

$$\left\| \mathcal{P}_n^{**} f \right\|_{L_p^a(\mathbb{R})} \leq e^{2a|\beta_n|} \frac{\sqrt{n}}{\sqrt{n} - 2a} \|f\|_{L_p^a(\mathbb{R})},$$

*where $\beta_n = \frac{1}{a} \ln\left(\frac{n}{n-a^2}\right)$, for sufficiently large $n \in \mathbb{N}$ satisfying $\sqrt{n} > 2a$.*

*Proof.* Using $L_p^a-$norm for $\mathcal{P}_n^{**}f$, we have

$$\begin{aligned}\left\|\mathcal{P}_n^{**}f\right\|_{L_p^a(\mathbb{R})} &= \left(\int\limits_{-\infty}^{\infty} e^{-ap|x|}\left|\left(\mathcal{P}_n^{**}f\right)(x)\right|^p dx\right)^{\frac{1}{p}} \\ &= \left(\int\limits_{-\infty}^{\infty} e^{-ap|x|}\left|\frac{\sqrt{n}}{2}\int\limits_{-\infty}^{\infty} e^{-a(\beta_n^{**}(x)+t)}e^{ax}f\left(\beta_n^{**}(x)+t\right)e^{-\sqrt{n}|t|}dt\right|^p dx\right)^{\frac{1}{p}}.\end{aligned}$$

By the aid of generalized Minkowski inequality [15] and using change of variables $w = x - \beta_n + t$, we get

$$\begin{aligned}\left\|\mathcal{P}_n^{**}f\right\|_{L_p^a(\mathbb{R})} &\leq e^{a|\beta_n|}\frac{\sqrt{n}}{2}\int\limits_{-\infty}^{\infty} e^{-at}e^{-\sqrt{n}|t|}\left(\int\limits_{-\infty}^{\infty}\left|f\left(\beta_n^{**}(x)+t\right)e^{-a|x|}\right|^p dx\right)^{\frac{1}{p}} dt \\ &= e^{a|\beta_n|}\frac{\sqrt{n}}{2}\int\limits_{-\infty}^{\infty} e^{-at}e^{-\sqrt{n}|t|}\left(\int\limits_{-\infty}^{\infty}\left|f(w)e^{-a|w+\beta_n-t|}\right|^p dw\right)^{\frac{1}{p}} dt \\ &\leq e^{2a|\beta_n|}\sqrt{n}\left\|f\right\|_{L_p^a(\mathbb{R})}\int\limits_{0}^{\infty} e^{(2a-\sqrt{n})t}dt \\ &= e^{2a|\beta_n|}\frac{\sqrt{n}}{\sqrt{n}-2a}\left\|f\right\|_{L_p^a(\mathbb{R})}\end{aligned}$$

provided that $\sqrt{n} > 2a$. □

## 3. Main Results

Now, we prove the following result concerning convergence rate using the modulus of smoothness defined in (2.1).

**Theorem 3.1.** *If $f \in L_p^a(\mathbb{R})$ with fixed $1 \leq p < \infty$ and $a > 0$, then for sufficiently large $n$ satisfying $\sqrt{n} > 2a$, and $|\beta_n| \leq \frac{1}{\sqrt{n}}$ there holds*

$$\begin{aligned}\left\|\mathcal{P}_n^{**}f - f\right\|_{L_p^a(\mathbb{R})} \leq &\left(\frac{2\sqrt{n}}{\sqrt{n}-2a} + \frac{n}{\left(\sqrt{n}-2a\right)^2}\right)e^{2a|\beta_n|}\omega\left(f; L_p^a; \frac{1}{\sqrt{n}}\right) \\ &+ \left|\frac{n^2}{\left(n-a^2\right)^2} - 1\right|\left\|f\right\|_{L_p^a(\mathbb{R})}.\end{aligned}$$

*Proof.* By virtue of Minkowski inequality, we may write

$$
\begin{aligned}
&\left\|\mathcal{P}_n^{**}f-f\right\|_{L_p^a(\mathbb{R})}\\
&= \left(\int_{-\infty}^{\infty} e^{-ap|x|}\left|\left(\mathcal{P}_n^{**}f\right)(x)-f(x)\right|^p dx\right)^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \left(\int_{-\infty}^{\infty} e^{-ap|x|}\left|\frac{\sqrt{n}}{2}\int_{-\infty}^{\infty} e^{-a(\beta_n^{**}(x)+t)}e^{ax}e^{-\sqrt{n}|t|}\left(f\left(\beta_n^{**}(x)+t\right)-f(x)\right)dt\right|^p dx\right)^{\frac{1}{p}}\\
&+\left(\int_{-\infty}^{\infty} e^{-ap|x|}\left|f(x)\left(\frac{\sqrt{n}}{2}\int_{-\infty}^{\infty} e^{-a(\beta_n^{**}(x)+t)}e^{ax}e^{-\sqrt{n}|t|}dt-1\right)\right|^p dx\right)^{\frac{1}{p}}\\
=\ & I_1+I_2.
\end{aligned}
$$

Using generalized Minkowski inequality for $I_1$, we have

$$
\begin{aligned}
I_1 &= \left(\int_{-\infty}^{\infty} e^{-ap|x|}\left|\frac{\sqrt{n}}{2}\int_{-\infty}^{\infty} e^{-a(t-\beta_n)}e^{-\sqrt{n}|t|}\left[f\left(\beta_n^{**}(x)+t\right)-f(x)\right]dt\right|^p dx\right)^{\frac{1}{p}}\\
&\leq \frac{\sqrt{n}}{2}\int_{-\infty}^{\infty} e^{-a(t-\beta_n)}e^{-\sqrt{n}|t|}\left(\int_{-\infty}^{\infty} e^{-ap|x|}\left|f\left(\beta_n^{**}(x)+t\right)-f(x)\right|^p dx\right)^{\frac{1}{p}} dt\\
&\leq \frac{\sqrt{n}}{2}\omega\left(f;L_p^a;\delta\right)\int_{-\infty}^{\infty} e^{-a(t-\beta_n)}e^{-\sqrt{n}|t|}\left(1+\frac{|t-\beta_n|}{\delta}\right)e^{a|t-\beta_n|}dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\sqrt{n}}{2}e^{2a|\beta_n|}\omega\left(f;L_p^a;\delta\right)\int_{-\infty}^{\infty} e^{2a|t|}e^{-\sqrt{n}|t|}\left(1+\frac{|t|}{\delta}+\frac{|\beta_n|}{\delta}\right)dt\\
&= \frac{\sqrt{n}}{2}e^{2a|\beta_n|}\omega\left(f;L_p^a;\delta\right)\int_{-\infty}^{\infty} e^{2a|t|}e^{-\sqrt{n}|t|}\left(1+\frac{|\beta_n|}{\delta}\right)dt\\
&\quad+\frac{\sqrt{n}}{2}e^{2a|\beta_n|}\omega\left(f;L_p^a;\delta\right)\int_{-\infty}^{\infty} e^{2a|t|}e^{-\sqrt{n}|t|}\frac{|t|}{\delta}dt\\
&= I_{11}+I_{12}.
\end{aligned}
$$

Observe that

$$I_{11}=\frac{\sqrt{n}}{\sqrt{n}-2a}e^{2a|\beta_n|}\left(1+\frac{|\beta_n|}{\delta}\right)\omega\left(f;L_p^a;\delta\right)$$

and

$$I_{12}=\frac{\sqrt{n}}{\delta\left(\sqrt{n}-2a\right)^2}e^{2a|\beta_n|}\omega\left(f;L_p^a;\delta\right).$$

Choosing $\delta=\frac{1}{\sqrt{n}}$ with $\sqrt{n}>2a$ and $|\beta_n|\leq\delta$, we obtain

$$I_1\leq\left(\frac{2\sqrt{n}}{\sqrt{n}-2a}+\frac{n}{\left(\sqrt{n}-2a\right)^2}\right)e^{2a|\beta_n|}\omega\left(f;L_p^a;\frac{1}{\sqrt{n}}\right).$$

For $I_2$, we may write

$$\begin{aligned}I_2 &= \left(\int_{-\infty}^{\infty}e^{-ap|x|}\left|f(x)\left(\frac{n^2}{\left(n-a^2\right)^2}-1\right)\right|^p dx\right)^{\frac{1}{p}}\\ &\leq \left|\frac{n^2}{\left(n-a^2\right)^2}-1\right|\|f\|_{L_p^a(\mathbb{R})}.\end{aligned}$$

Hence we get the required result, that is,

$$\begin{aligned}\|\mathcal{P}_n^{**}f-f\|_{L_p^a(\mathbb{R})} &\leq \left(\frac{2\sqrt{n}}{\sqrt{n}-2a}+\frac{n}{\left(\sqrt{n}-2a\right)^2}\right)e^{2a|\beta_n|}\omega\left(f;L_p^a;\frac{1}{\sqrt{n}}\right)\\ &\quad+\left|\frac{n^2}{\left(n-a^2\right)^2}-1\right|\|f\|_{L_p^a(\mathbb{R})}.\end{aligned}$$

□

Bohman-Korovkin theorem is one of the fundamental tools in the theory of approximation (see [2]). In [11], the authors proved the Korovkin-type theorem in weighted Lebesgue spaces. The properties of the space $L_p^a(\mathbb{R})$, which we consider in this paper, fit the hypotheses of Theorem 1 in [11]. Therefore, we prove the following result by the aid of indicated theorem.

**Theorem 3.2.** *If* $f\in L_p^a(\mathbb{R})$ *with fixed* $1\leq p<\infty$ *and* $a>0$, *then there holds*

$$\lim_{n\to+\infty}\|\mathcal{P}_n^{**}f-f\|_{L_p^a(\mathbb{R})}=0.$$

*Proof.* In view of Lemma 2.2 and Theorem 1 in [11], it is sufficient to show that the following conditions hold there:

$$\lim_{n\to+\infty}\left\|\left(\mathcal{P}_n^{**}e_i\right)(x)-x^i\right\|_{L_p^a(\mathbb{R})}=0 \text{ with } i=0,1,2.$$

Let $i=0$. In view of Lemma 2.1, we have

$$\begin{aligned}\lim_{n\to+\infty}\left\|\left(\mathcal{P}_n^{**}e_0\right)(x)-1\right\|_{L_p^a(\mathbb{R})} &= \lim_{n\to+\infty}\left(\int_{-\infty}^{\infty}e^{-ap|x|}\left|\frac{n^2}{\left(n-a^2\right)^2}-1\right|^p dx\right)^{\frac{1}{p}}\\ &= \lim_{n\to+\infty}\left|\frac{n^2}{\left(n-a^2\right)^2}-1\right|\left(\frac{2}{ap}\right)^{\frac{1}{p}}\\ &= 0.\end{aligned}$$

Let $i=1$.

$$\begin{aligned}&\left\|\left(\mathcal{P}_n^{**}e_1\right)(x)-x\right\|_{L_p^a(\mathbb{R})}^p\\ =& \int_{-\infty}^{\infty}e^{-ap|x|}\left|e^{a\left(x-\beta_n^{**}(x)\right)}n\left(\frac{\beta_n^{**}(x)}{n-a^2}-\frac{2a}{\left(n-a^2\right)^2}\right)-x\right|^p dx\\ =& \int_{-\infty}^{\infty}e^{-ap|x|}\left|\frac{n^2}{n-a^2}\left(\frac{\beta_n^{**}(x)}{n-a^2}-\frac{2a}{\left(n-a^2\right)^2}\right)-x\right|^p dx\\ \leq& 2^p\int_{-\infty}^{\infty}e^{-ap|x|}\left|\frac{n^2\beta_n^{**}(x)}{\left(n-a^2\right)^2}-x\right|^p dx+2^p\left|\frac{2an^2}{\left(n-a^2\right)^3}\right|^p\frac{2}{ap}\\ \leq& 2^{2p}\left|\frac{n^2}{\left(n-a^2\right)^2}-1\right|^p\int_{-\infty}^{\infty}e^{-ap|x|}|x|^p dx+2^p\left|\frac{2an^2}{\left(n-a^2\right)^3}\right|^p\frac{2}{ap}\\ &+2^{2p}\left|\frac{\frac{1}{a}\ln\left(\frac{n}{n-a^2}\right)n^2}{\left(n-a^2\right)^2}\right|^p\frac{2}{ap}\\ =& 2^{2p+1}\left|\frac{n^2}{\left(n-a^2\right)^2}-1\right|^p(ap)^{-(p+1)}\Gamma(p+1)+2^p\left|\frac{2an^2}{\left(n-a^2\right)^3}\right|^p\frac{2}{ap}\\ &+2^{2p}\left|\frac{\frac{1}{a}\ln\left(\frac{n}{n-a^2}\right)n^2}{\left(n-a^2\right)^2}\right|^p\frac{2}{ap},\end{aligned}$$

where $\Gamma(.)$ denotes gamma function with

$$\int_{-\infty}^{\infty}e^{-ap|x|}|x|^p dx=2(ap)^{-(p+1)}\Gamma(p+1).$$

In view of above inequality, we easily see that

$$\lim_{n\to+\infty}\left\|\left(\mathcal{P}_n^{**}e_1\right)(x)-x\right\|_{L_p^a(\mathbb{R})}=0.$$

Let $i=2$.

$$\begin{aligned}
&\left\|\left(\mathcal{P}_n^{**}e_2\right)(x)-x^2\right\|_{L_p^a(\mathbb{R})}^p\\
=&\int_{-\infty}^{\infty}\left|\frac{n^2}{(n-a^2)^4}\left(\left(\beta_n^{**}(x)\right)^2\left(a^4-2a^2n+n^2\right)+\beta_n^{**}(x)\left(4a^3-4an\right)+6a^2+2n\right)-x^2\right|^p\\
&\times e^{-ap|x|}dx\\
\leq\ & 2^p\int_{-\infty}^{\infty}\left|\frac{n^2}{(n-a^2)^4}\left(\beta_n^{**}(x)\right)^2\left(a^4-2a^2n+n^2\right)-x^2\right|^p e^{-ap|x|}dx\\
&+2^p\int_{-\infty}^{\infty}\left|\frac{n^2}{(n-a^2)^4}\left(\beta_n^{**}(x)\left(4a^3-4an\right)+6a^2+2n\right)\right|^p\\
=&:A_1+A_2.
\end{aligned}$$

In view of the same considerations as in the case $i=1$, it is not difficult to see that $A_2\to0$ as $n\to\infty$. Now, we will deal with $A_1$. If we write $\beta_n^{**}(x)$ in explicit form, we have

$$\begin{aligned}
A_1 &= 2^p\int_{-\infty}^{\infty}\left|\frac{n^2}{(n-a^2)^4}\left(x-\frac{1}{a}\ln\left(\frac{n}{n-a^2}\right)\right)^2\left(a^4-2a^2n+n^2\right)-x^2\right|^p e^{-ap|x|}dx\\
&\leq 2^{2p}\left|\frac{n^2\left(a^4-2a^2n+n^2\right)}{(n-a^2)^4}-1\right|^p\int_{-\infty}^{\infty}|x|^{2p}e^{-ap|x|}dx\\
&\quad+2^{2p}\left|\frac{n^2\left(a^4-2a^2n+n^2\right)}{(n-a^2)^4}\right|^p\int_{-\infty}^{\infty}\left|\frac{1}{a^2}\ln^2\left(\frac{n}{n-a^2}\right)-\frac{2x}{a}\ln\left(\frac{n}{n-a^2}\right)\right|^p e^{-ap|x|}dx\\
&=:A_{11}+A_{12}.
\end{aligned}$$

By the same argument used in the case $i=1$, we have $A_{12}\to0$ as $n\to\infty$. Now, it remains to show that $A_{11}\to0$ as $n\to\infty$. Since

$$\int_{-\infty}^{\infty}|x|^{2p}e^{-ap|x|}dx=2\left(ap\right)^{-(1+2p)}\Gamma\left(1+2p\right),$$

we have

$$\begin{aligned} A_{11} &= 2^{2p} \left| \frac{n^2 \left( a^4 - 2a^2 n + n^2 \right)}{\left( n - a^2 \right)^4} - 1 \right|^p \int\limits_{-\infty}^{\infty} |x|^{2p} \, e^{-ap|x|} dx \\ &= 2^{2p+1} \left| \frac{n^2 \left( a^4 - 2a^2 n + n^2 \right)}{\left( n - a^2 \right)^4} - 1 \right|^p (ap)^{-(1+2p)} \, \Gamma \left( 1 + 2p \right). \end{aligned}$$

The result follows, that is,

$$\lim_{n \to +\infty} \left\| \left( \mathcal{P}_n^{**} e_2 \right) (x) - x^2 \right\|_{L_p^a(\mathbb{R})} = 0.$$

Thus the proof is completed. □

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