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VAGUE SEMI DISTRBUTIVE LATTICES

R. BULLI BABU[1], MOHAN RAO NAVULURI[2], AND B. NAGESWARA RAO[3]

ABSTRACT. In this paper we introduce vague meet semi distributive lattice, vague join semi distributive lattice and studied their properties. Further we investigate the development of some important, properties, results and theorems about vague join semi lattice, vague meet semi distributive lattices.

1. INTRODUCTION

The concept of Lattice was first defined by Dedekind in 1897 and then developed by Birkhoft.G,., imposed an operation an open problem "Is there a common abstraction which includes Boolean algebra, Boolean rings and lattice ordered group or L-group is an algebraic structure connecting lattic and group.To answer this problem many common abstractions, namely dually residuated lattice ordered semigroups,commutative lattice ordered groups. lattice ordered rings,latice ordered near rings and lattice ordered semirings are presented.Amoung them the algebraic structure lattice ordered semirings or L-semiring was introduced by Ranga Rao P.,[9]. Also the concept proposed by Zadeh L.A. [10] defining a fuzzy subset A of a given universe X characterizing the membership of an element x of X belonging to A by means of a membership function $\mu A(x)$ defined from X in to [0 1] has revolutionized the theory of Mathematical modeling. Decision making etc.,in handling the imprecise real life situations mathematically. Now several branches of fuzzy mathematics like

[1]*corresponding author*

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fuzzy algebra,fuzzy topology ,fuzzy control theory ,fuzzy measure theory etc., have emerged.But in the decision making, the fuzzy theory takes care of membership of an element x only, that is the evidence against x belonging to A. It is felt by several decision makers and researchers that in proper decision making, the evidence belongs to A and evidence not belongs to A are both necessary and how much X belongs to A or how much x does not belongs to A are necessary. Several generalizations of Zadeh's fuzzy set theory have been proposed, such as L-fuzzy sets [4]. Interval valued fuzzy sets, Intuitionistic fuzzy sets by Atanassov K.T [1] ,Vague sets [3] are mathematically equivalent. Any such set A of a given Universe X can be charactrierized by means of a pair of function $A = (t_A, f_A)$ where t_A : X $\to$ [0 1] and f_A : X $\to$ [0 1] such that $0 = t_A(x) + f_A(x) = 1$ for all x in X.The set $t_A(x)$ is called the truth function and the set $f_A(x)$ is called false function or non membership function and $t_A(x)$ gives the evidence of how much x belongs to A $f_A(x)$ gives the evidence of how much x does not belongs to A.These concepts are being applied in several areas like decision-making, fuzzy control,knowledge discovery and fault diagonsis etc. It is believed the vague sets (or equivalently instuitionistic fuzzy sets) will more useful in decision making, and other areas of Mathematical modeling. Through Atanassov's instuitionistic fuzzysets, Gau and Buehrer and some other areas of Mathematical modeling.Since then the theory fuzzy sets developed extensively and embraced almost all subjects like engineering science and technology.But the membership function $\mu_A(x)$ gives only a approximation belong to A. To avid this and obtain a better estimation and analysis of data decision making.Gau.W.L and Bueher D.J. [3] have initiated the study ofvague sets with the hope that they form a better tool to understand, interpret and solve real life problems which are in general vague, than the theory of vague sets do. Ranjit Biswas [9] initiated the study of vague groups by Ramakrishna N. [5-7] and Eswarlal T. [2] are grate extended the study of vague algebra. The objective of this paper is to contribute further to the study of vague algebra by introducing the concept of vague join semi distributive lattice and concept of vague meet semi distributive lattice with suitable examples.

2. Preliminaries

In this section we briefly present the necessary material on lattices,Boolean lattices Brouwerian lattices and illustrate with examples.

Definition 2.1. *([9]) A poset* $(L, \leq)$ *is called a lattice if* $sup\{x, y\}$ *also denoted by* $(x \vee y)$ *and* $inf\{x, y\}$ *also denoted by* $(x \wedge y)$ *exists for every pair of elements* $x, y\ in\ L$.

Definition 2.2. *([9]) A lattice* $(L, \leq)$ *in which every subset of L has g.l.b and l.u.b in it is called a complete lattice.*

Definition 2.3. *([7]) A lattice L is said to be distributive if it satisfies*

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ *and*

(2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ $for\ all\ x, y, z\ in\ L$.

Definition 2.4. *([9]) A lattice L is said to be bounded if L has least element and greatest element.usually least element of L is denoted by* 0_L *and greatest element is denoted by* 1_L.

Definition 2.5. *([4]) A Meet semilatice or semilattice is non-empty set S with binary operation* $" \wedge "$ *defined on it and satisfies the following.*

(1) *Idempotent:* $(a \wedge a) = a$

(2) *Commutative law :* $a \wedge b = b \wedge a$

(3) *Associative law :* $(a \wedge (b \wedge c) = (a \wedge b) \wedge c$ *,for all a,b,c* $\in S$.

(4) *Any two elements in S have a least upper bound.*

Definition 2.6. *([4]) A semilattice is non-empty set S with binary relation* $" \leq "$ *defined on it and satisfies the following.*

(1) $\leq$ *is refive :* $a \leq a$ *for all a* $\in S$

(2) $\leq$ *is antisymmetric :* $a \leq b$ *and* $b \leq a \Rightarrow a = b$.

(3) $\leq$ *is transtative law :* $a \leq b$ *and* $b \leq c \Rightarrow a \leq c$*, for all* $a, b, c \in S$.

Definition 2.7. *([5]) Let A be a meet semi lattice. A fuzy set* $\mu_A : X \to [0\ 1]$ *is called a fuzzy meet semi lattice of A if satisfies the following*

$$\mu_A(x \wedge y \geq min\{\mu_A(x), \mu_A(y)\}.$$

for all $x, y \in A$.

Example 1. *([5]) Let* $A = \{0, x_1, x_2, x_3, 1\}$ *be a semi lattice Then its a fuzzy meet semi lattice of A* μ_A *is given by* $\mu_A = \{(0, 0.8), (x_1, 0.7), (x_2, 0.5), (x_3, 0.6)(1, 0.4)\}$.

Definition 2.8. *([6]) Let* $A = (t_A, f_A),\ B = (t_B, f_B)$ *be two vague sets of set* X *then their intersection is defined as* $A \cap B = (t_{A\cap B}, f_{A\cap B})$ *where,* $t_{A\cap B} = min\{t_A, t_B\}$ *and* $f_{A\cap B} = max\{f_A, f_B\}$.

Definition 2.9. *([3]) Let $A = (t_A, f_A)$ be a vague set then it's complement is defined as $A' = (t'_A, f'_A)$, where $t'_A = 1 - f_A, f'_A = 1 - t_A$.*

Definition 2.10. *([3]) A vague set A on a set X is a pair (t_A, f_A) where $t_A : X \to [0, 1]$, $f_A : X \to [0, 1]$ such that $0 \leq t_A(x) + f_A(x) \leq 1$, for all $x \in X$.*

Definition 2.11. *([5]) Vague sets A and B are equal, written as $A = B$, iff $A \subseteq B$ and $B \subseteq A$, i.e., $t_A = t_B$, $f_A = f_B$.*

Definition 2.12. *([1]) Let $A = (t_A, f_A)$, $B = (t_B, f_B)$ be two vague sets of a set X then their union is defined as $A \cup B = (t_{A\cup B}, f_{A\cup B})$ where,*
$t_{A\cup B} = max\{t_A, t_B\}$ and $f_{A\cup B} = min\{f_A, f_B\}$.

Definition 2.13. *([10]) The interval $[t_A(x), 1 - f_A(x)]$ is called the vague value of $x \in A$, and it is denoted by $V_A(x)$. i.e.,$V_A(x) = [t_A(x), 1 - f_A(x)]$.*

3. VAGUE JOIN SEMI DISTBUTIVE LATTICE

In this section we define vague join semi distrbutive latice (VJSDL) and vague latice (VL) with suytable examples.

Definition 3.1. *Let $A = (t_A, f_A)$ be a vague lattice and is called an vague join semi disrbutive lattice(VJSDL) if it satisfies the following properties.*

1. *$t_A(a \vee b) = t_A(a) \vee t_A(b \wedge c)$ where $t_A(a \vee b) = t_A(a \vee c)$*
2. *$f_A(a \vee b) = f_A(a) \vee f_A(b \vee c)$ where $f_A(a \vee b) = f_A(a \vee c)$*

for all $(t_A(a), f_A(a)), (t_A(b), f_A(b))$ and $(t_A(c), f_A(c))$ are $\in$ vague lattices.

Theorem 3.1. *Every vague join semi distrbutive (VJSD)is an vague lattice (VL) and the convesce need not be true.*

Proof. Given that $A = (t_A, f_A)$ vague join semi distrbutive (VJSD), i.e.,

1. $t_A(a \vee b) = t_A(a) \vee t_A(b \wedge c)$ where $t_A(a \vee b) = t_A(a \vee c)$
2. $f_A(a \vee b) = f_A(a) \vee f_A(b \vee c)$ where $f_A(a \vee b) = f_A(a \vee c)$

for all $(t_A(a), f_A(a)), (t_A(b), f_A(b))$ and $(t_A(c), f_A(c))$ are $\in$ vague lattices.

To prove that $A = (t_A, f_A)$ is a vague lattice, this is to prove that: $1.\ t_A(a \vee b) = t_A(b \vee a)$ and $2.\ f_A(a \vee b) = f_A(b \vee a)$ for all $(t_A(a), f_A(a)), (t_A(b), f_A(b))$ and $(t_A(c), f_A(c))$ are $\in$ vague lattices. Then:

1. $t_A(a \vee b) = t_A(a) \vee t_A(b \wedge c) \geq min\{t_A(a), t_A(b \wedge c)\}$

$\geq min\{t_A(a), min\{t_A(b), t_A(c)\}\} \geq min\{t_A(a), t_A(c \wedge b)\}$
$= t_A(a) \vee t_A(c \wedge b) = t_A(a \vee c).$

2. $f_A(a \vee b) = f_A(a) \vee f_A(b \wedge c) \leq max\{f_A(a), f_A(b \wedge c)\}$
$\leq max\{f_A(a), max\{f_A(b), f_A(c)\}\} \leq max\{f_A(a), f_A(c \wedge b)\}$
$= f_A(a) \vee f_A(c \wedge b) = f_A(a \vee c).$

Hence $A = (t_A, f_A)$ is an vague lattice. □

The converse of tha above theorem need not be true, that is ever vague lattice need not be vague join semi distrbutiv (VJSDL).

In the following example we will consider an vague lattice D_5 of the following figure.

FIGURE 1

FIGURE 2

Consider $(a, b) = (1, 2)$ put $c = 4, 8, 16$ equations becomes

$$t_A(a \vee b) = t_A(a) \vee t_A(b \wedge c) \quad (1)$$

$$f_A(a \vee b) = f_A(a) \vee f_A(b \wedge c) \quad (2)$$

Put $c = 4$ in equation (1):

$$t_A(1 \vee 2) \geq min\{t_A(1), t_A(2 \wedge 4)\} \geq \{t_A(1), t_A(2)\} = t_A(2).$$

Therefore $t_A(2) = t_A(2)$.

Put $c = 4$ in equation (2):

$$f_A(1 \vee 2) \leq max\{f_A(1), f_A(2 \wedge 4)\} \leq max\{f_A(1), f_A(2)\} = f_A(1).$$

Therefore $f_A(1) = 1_A(1)$.

Put $c = 8$ in equation (1):

$$t_A(1 \vee 2) \geq min\{t_A(1), t_A(2 \wedge 8)\} \geq \{t_A(1), t_A(2)\} = t_A(2).$$

Therefore $t_A(2) = t_A(2)$.

Put $c = 8$ in equation (2):

$$f_A(1 \vee 2) \leq max\{f_A(1), f_A(2 \wedge 8)\} \leq max\{f_A(1), f_A(1)\} = f_A(1).$$

Therefore $f_A(1) = 1_A(1)$.

Put $c = 16$ in equation (1):

$t_A(1 \vee 2) \geq min\{t_A(1), t_A(2 \wedge 16)\} \geq \{t_A(1), t_A(2)\} = t_A(2).$

Therefore $t_A(2) = t_A(2)$.

Put $c = 16$ in equation (2):

$f_A(1 \vee 2) \leq max\{f_A(1), f_A(2 \wedge 16)\} \leq max\{f_A(1), f_A(1)\} = f_A(1).$

Therefore $f_A(1) = 1_A(1)$.

Therefore $(t_A(1)$, $f_A(1))$, $(t_A(2)$, $f_A(2))$ are vague join semi distrbutive lattices (VSJDL).

Consider $(a, b) = (2, 4)$ put $c = 1, 8, 16$ equations become:

Put $c = 1$ in equation (1):

$t_A(2 \vee 4) \geq min\{t_A(2), t_A(4 \wedge 1)\} \geq \{t_A(1), t_A(2)\} = t_A(2).$

Therefore $t_A(4) \neq\geq t_A(2)$.

Put $c = 1$ in equation (2):

$f_A(2 \vee 4) \leq max\{f_A(2), f_A(4 \wedge 1)\} \leq max\{f_A(2), f_A(1)\} = f_A(1).$

Therefore $f_A(2) \neq 1_A(1)$.

Put $c = 8$ in equation (1):

$t_A(2 \vee 4) \geq min\{t_A(2), t_A(4 \wedge 8)\} \geq \{t_A(2), t_A(4)\} = t_A(4).$

Therefore $t_A(4) = t_A(4)$.

Put $c = 8$ in equation (2):

$f_A(2 \vee 4) \leq max\{f_A(2), f_A(4 \wedge 8)\} \leq max\{f_A(2), f_A(4)\} = f_A(2).$

Therefore $f_A(2) = f_A(2)$.

Put $c = 16$ in equation (1):

$t_A(2 \vee 4) \geq min\{t_A(2), t_A(4 \wedge 16)\} \geq \{t_A(2), t_A(4)\} = t_A(4).$

Therefore $t_A(4) = t_A(4)$.

Put $c = 16$ in equation (2):
$f_A(2 \vee 4) \leq max\{f_A(2), f_A(4 \wedge 8)\} \leq max\{f_A(2), f_A(4)\} = f_A(2)$.
Therefore $f_A(2) = f_A(2)$.

Hence $(t_A(2), f_A(2)), (t_A(4), f_A(4))$ is not vague join semi distrbutive lattices (VSJDL).

4. VAGUE MEET SEMI DISTBUTIVE LATTICE

In this section we define vague join semi distrbutive latice (VJSDL) and vague latice (VL) with suytable examples.

Definition 4.1. *Let $A = (t_A, f_A)$ be a vague lattice and is called an vague meet semi disrbutive lattice(VMSDL) if it satisfies the following properties:*
1. $t_A(a \wedge b) = t_A(a) \wedge t_A(b \vee c)$
2. $f_A(a \wedge b) = f_A(a) \wedge f_A(b \wedge c)$

for all $(t_A(a), f_A(a)), (t_A(b), f_A(b))$ and $(t_A(c), f_A(c))$ are $\in$ vague lattices.

Theorem 4.1. *Every vague meet semi distrbutive lattice(VMSDL)is an vague lattice (VL) and the convesce need not be true.*

Proof. Given that $A = (t_A, f_A)$ vague meet semi distrbutive lattice (VMSDL), i.e. $1.\ t_A(a \wedge b) = t_A(a) \wedge t_A(b \vee c)$ and $2.\ f_A(a \wedge b) = f_A(a) \wedge f_A(b \wedge c)$ for all $(t_A(a), f_A(a)), (t_A(b), f_A(b))$ and $(t_A(c), f_A(c))$ are $\in$ vague lattices.

To prove that $A = (t_A, f_A)$ is a vague lattice, this is to prove that $1.\ t_A(a \vee b) = t_A(b \vee a)$ and $2.\ f_A(a \vee b) = f_A(b \vee a)$ and for all $(t_A(a), f_A(a)), (t_A(b), f_A(b))$ and $(t_A(c), f_A(c))$ are $\in$ vague lattices. Then:

1. $t_A(a \vee b) = t_A(a) \vee t_A(b \wedge c) \geq min\{t_A(a), t_A(b \wedge c)\}$
$\geq min\{t_A(a), min\{t_A(b), t_A(c)\}\} \geq min\{t_A(a), t_A(c \wedge b)\}$
$= t_A(a) \vee t_A(c \wedge b) = t_A(a \vee c)$.
2. $f_A(a \vee b) = f_A(a) \vee f_A(b \wedge c) \leq max\{f_A(a), f_A(b \wedge c)\}$
$\leq max\{f_A(a), max\{f_A(b), f_A(c)\}\} \leq max\{f_A(a), f_A(c \wedge b)\}$
$= f_A(a) \vee f_A(c \wedge b) = f_A(a \vee c)$.

Hence $A = (t_A, f_A)$ is an vague lattice. □

The converse of tha above theorem need not be true, that is ever vague lattice need not be vague meet semi distrbutiv (VMSDL).

We shall the following example consider an vague lattice S_8 of the following figure.

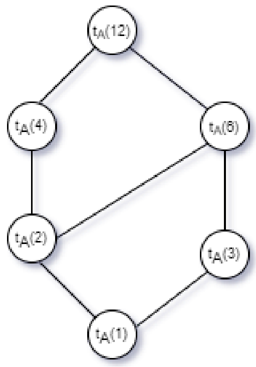

FIGURE 3

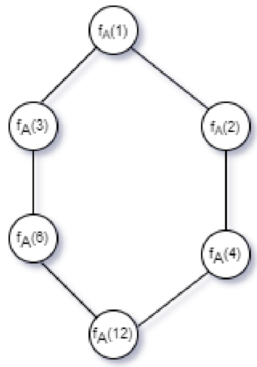

FIGURE 4

Consider $(a, b) = (3, 2)$ put $c = 4, 6, 8, 12$ equations becomes

1. $t_A(a \wedge b) = t_A(a) \wedge t_A(b \vee c)$ (3)

2. $f_A(a \wedge b) = f_A(a) \wedge f_A(b \wedge c)$ (4)

Put $c = 4$ in equation (3):

$t_A(3 \wedge 2) \geq min\{t_A(3), t_A(2 \vee 4)\} \geq \{t_A(3), t_A(4)\} = t_A(1)$.

Therefore $t_A(1) = t_A(1)$.

Put $c = 4$ in equation (4):

$f_A(3 \wedge 2) \leq max\{f_A(3), f_A(2 \wedge 4)\} \leq max\{f_A(3), f_A(4)\} = f_A(12)$.

Therefore $f_A(6) \neq f_A(12)$.

Put $c = 6$ in equation (3):

$t_A(3 \wedge 2) \geq min\{t_A(3), t_A(2 \vee 6)\} \geq \{t_A(3), t_A(12)\} = t_A(3)$.

Therefore $t_A(1) = t_A(3)$.

Put $c = 6$ in equation (4):

$f_A(3 \wedge 2) \leq max\{f_A(3), f_A(2 \wedge 6)\} \leq max\{f_A(3), f_A(6)\} = f_A(6)$.

Therefore $f_A(6) = f_A(6)$.

Put $c = 8$ in equation (3):

$t_A(3 \wedge 2) \geq min\{t_A(3), t_A(2 \vee 8)\} \geq \{t_A(3), t_A(8)\} = t_A(1)$.

Therefore $t_A(1) = t_A(1)$.

Put $c = 8$ in equation (4):

$f_A(3 \wedge 8) \leq max\{f_A(3), f_A(2 \wedge 6)\} \leq max\{f_A(3), f_A(8)\} = f_A(8)$.

Therefore $f_A(6) \neq f_A(8)$.

Put $c = 12$ in equation (3):

$t_A(3 \wedge 2) \geq min\{t_A(3), t_A(2 \vee 12)\} \geq \{t_A(3), t_A(12)\} = t_A(12)$.

Therefore $t_A(1) \neq t_A(3)$.

Put $c = 12$ in equation (4):

$$f_A(3 \wedge 2) \leq max\{f_A(3), f_A(2 \wedge 12)\} \leq max\{f_A(3), f_A(12)\} = f_A(12).$$

Therefore $f_A(6) \neq f_A(12)$.

Therefore $\{(t_A(3), f_A(3)), (t_A(2), f_A(2))\}$ is not vague meet semi distrbutive lattices (VSMDL).

Theorem 4.2. *Vague dual of vague join semi distrbutive lattice (VJSD)L is an vague modular semi distrbutive lattice(VMSDL).*

Proof. Given that $A = (t_A, f_A)$ vague join semi distrbutive lattice (VJSDL), i.e., 1. $t_A(a \vee b) = t_A(a) \vee t_A(b \wedge c)$ where $t_A(a \vee b) = t_A(a \vee c)$, 2. $f_A(a \vee b) = f_A(a) \vee f_A(b \vee c)$ where $f_A(a \vee b) = f_A(a \vee c)$ for all $(t_A(a), f_A(a)), (t_A(b), f_A(b))$ and $(t_A(c), f_A(c))$ are $\in$ vague lattices vague dual of $t_A(a \vee b) = t_A(a \vee c)$ and $f_A(a \wedge b) = f_A(a \wedge c)$ this implies that $t_A(a \vee b) = t_A(a) \vee t_A(b \wedge c)$ and $1.f_A(a \wedge b) = f_A(a) \wedge f_A(b \wedge c)$ for all for all $(t_A(a), f_A(a)), (t_A(b), f_A(b))$ and $(t_A(c), f_A(c))$ are in vague lattices.

Therefore Vague dual of vague join semi distrbutive lattice (VJSD)L is an vague modular semi distrbutive lattice(VMSDL). □

Theorem 4.3. *Every vague modular lattice need not be vague meet semi distributive lattice.*

Proof. Let given $A = (t_A, f_A)$ vague modular lattice then $A = (t_A, f_A)$ contains vague modular lattice which is isomorphic to M_4.

An vague lattice $A = (t_A, f_A)$ is an vague modular lattice if and only if it doesnot contain an vague sub lattice isomorphic to N_5. Which implies $A = (t_A, f_A)$ is not an vague modular lattice.this is contradiction. hence $A = (t_A, f_A)$ doesnot contain an vague sub lattice isomorphic to N_5.

Conversely assume that vague lattice $A = (t_A, f_A)$ does not contains an vague sub lattice isomorphic to N_5. Suppose $A = (t_A, f_A)$ is not an vague modular lattice but $A = (t_A, f_A)$ contains an vague sub lattice isomorphic to N_5. This is a contradiction to our assumption. Which implies $A = (t_A, f_A)$ is a vague modular lattice. Therefore $A = (t_A, f_A)$ is not an vague meet semi distributive lattice hence every vague meet semi distributive lattice is a vague meet lattice and the converse need not be true. □

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[1,3]DEPARTMENT OF S&H, LENDI INSTITUTE OF ENGINEERING AND TECHNOLOGY, JONNADA VIZIAYANAGARAM, ANDHRA PRADESH, INDIA
E-mail address: [1]Bullibabu123@gmail.com

DEPARTMENT OF MATHEMATICS, GOVT. COLLEGE OF ENGINEERING, BODINAYAKKANUR, TMILANADU-625582, INDIA.
E-mail address: [2]mohanaraonavuluri@gmail.com

E-mail address: [3]bendi151977@gmail.com