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# BIPOLAR VAGUE COSETS

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ABSTRACT. In this paper we introduce and study the concepts of bipolar vague cosets (BVCs) of a group , symmetric- invariant -normal bipolar vague groups (BVGs).

## 1. INTRODUCTION

The fuzzy sets was popularized first by Zadeh in 1965. Suppose $Z$ is any non-empty set.A mapping $\gamma : Z \to [0,1]$ is known as a Fuzzy subset of $Z$ . We have many extensions in the fuzzy set theory,such as intuitionistic fuzzy sets, interval - valued fuzzy sets, vague sets etc. The fuzzy set theory govern membership of an element $z$ only,it means the indication of $z$ affinity to $\gamma$.It does not take care of the indication against $z$ affinity to $\gamma$. To oppose this trouble Gau and Buehrer brought in the notion of vague set theory.According to them, a vague set $\mathcal{A}$ of a non-empty set $Z$ can be identified by functions$(t_{\mathcal{A}}, f_{\mathcal{A}})$ where $t_{\mathcal{A}}$ and $f_{\mathcal{A}}$ are functions from $Z$ to [0,1] such that $t_{\mathcal{A}}(x) + f_{\mathcal{A}}(x) \leq 1$ for all $z \in Z$ where $t_{\mathcal{A}}$ is called the truth function (or)membership function, which gives indication of how much an element $z$ belong to $\mathcal{A}$ and $f_{\mathcal{A}}$ is called the false function (or) non-membership function, which gives indication of how much an element $z$ does not belong to $\mathcal{A}$ . These approaches are being administered in various fields like decision making, fuzzy control etc. In such a way the ideology of vague sets

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is a generalization of Fuzzy set theory. Ranjit Biswas [4] proposed the theory of vague groups and authors like Eswarlal, Ramakrishna, Bhargavi, Nageswara Rao, Ragamayi introduced and studied Boolean vague sets,Vague groups, vague gamma semi rings, translate operators on Vague groups and vague gamma near rings respectively. [1, 3, 7–10] and extended the study of vague algebra and its applications. Lee [2] popularized the Bipolar - valued fuzzy sets (BVFS), which are an extension of fuzzy sets. Here the membership degree of these bipolar valued fuzzy sets (BVFS) range is extended from the interval [0,1] to [-1,1]. The degree of satisfaction to the propety corresponding to a fuzzy set and its counter property are represented by membership degrees of BVFS. This lead to a spirited field of research in distinct disciplines like algebraic structures,decision making ,graph theory, medical science, machine theory etc.Venkata kalyani U and Eswarlal T [6] have introduced and studied the homomorphism and anti-homomorphism on Bipolar Vague Normal groups (BVNGs).

In this paper we introduced bipolar vague cosets(BVC) and studied their properties. These concepts are used in the development of some important results and theorems about bipolar vague groups(BVGs) and bipolar vague normal groups (BVNGs).

## 2. Preliminaries

In this phase we recall a few number of the important standards and definitions,which are probably vital for this paper.

**Definition 2.1.** *[4] A mapping $\gamma : Z \to [0, 1]$ is referred to as a fuzzy subset of non empty set $Z$.*

**Definition 2.2.** *[4] : A vague set $\mathcal{A}$ in the universe of discourse $Z$ is a pair $(t_{\mathcal{A}}, f_{\mathcal{A}})$, where $t_{\mathcal{A}} : Z \to [0, 1], f_{\mathcal{A}} : Z \to [0, 1]$ are mappings such that $t_{\mathcal{A}}(z) + f_{\mathcal{A}}(z) \leq 1, forall z \in Z$. The functions $t_{\mathcal{A}}$ and $f_{\mathcal{A}}$ are referred as true membership function and false membership function respectively.*

**Definition 2.3.** *[4] : Let $(K, *)$ be a group.A vague set(VS) $\mathcal{A}$ of $K$ is termed as a vague group(VG) of $K$ if for any $g, h$ in $K$, if :*
$V_{\mathcal{A}}(g * h) \geq \min\{(V_{\mathcal{A}}(g), V_{\mathcal{A}}(h))\}$ *and* $V_{\mathcal{A}}(g^{-1}) \geq V_{\mathcal{A}}(g)$ *i.e*

$(i)$ $t_{\mathcal{A}}(g * h) \geq \min\{(t_{\mathcal{A}}(g), t_{\mathcal{A}}(h))\}$ *and* $f_{\mathcal{A}}(g * h) \leq \max\{f_{\mathcal{A}}(g); f_{\mathcal{A}}(h))\}$;

(ii) $t_{\mathcal{A}}(g^{-1}) \geq t_{\mathcal{A}}(g)$*and* $f_{\mathcal{A}}(g^{-1}) \leq f_{\mathcal{A}}(g)$.

**Definition 2.4.** *( [1, 4]): Consider a group* $(K.)$ *and* $\mathcal{A}$ *be a vague group(VG) of* $K$. *A vague left coset(VLC) of* $\mathcal{A}$*, denoted by* $a\mathcal{A}$*,for any* $a \in K$*,and defined by* $V_{a\mathcal{A}}(z) = V_{\mathcal{A}}(a^{-1}(z))$ *i.e* $t_{a\mathcal{A}}(z) = t_{\mathcal{A}}(a^{-1}(z))$ *and* $f_{a\mathcal{A}}(z) = f_{\mathcal{A}}(a^{-1}(z))$.

**Definition 2.5.** *( [1, 4]) : Consider a group* $(K.)$ *and* $\mathcal{A}$ *be a vague group(VG) of* $K$. *A vague right coset(VRC) of* $\mathcal{A}$ *is denoted by* $\mathcal{A}a$ *and for any* $a$ *in* $K$ *defined by* $V_{\mathcal{A}a}(z) = V_{\mathcal{A}}((z)a^{-1})$ *i.e* $t_{\mathcal{A}a}(z) = t_{\mathcal{A}}((z)a^{-1})$ *and* $f_{;\mathcal{A}a}(z) = f_{\mathcal{A}}((z)a^{-1})$.

**Definition 2.6.** *( [2]): Consider a universal set* $Z$ *and* $\mathcal{A}$ *be a set over* $Z$ *that is defined by a positive membership function ,* $\mu^{+}_{\mathcal{A}} : Z \to [0, 1]$ *and a negative membership function,* $\mu^{-}_{\mathcal{A}} : Z \to [-1, 0]$. *Then A is called a bipolar-valued fuzzy set over* $Z$*,and can be written in the form* $\mathcal{A} = \{< z : \mu^{+}_{\mathcal{A}}(z), \mu^{-}_{\mathcal{A}}(z) >: z \in Z\}$

**Definition 2.7.** *( [2]): Consider a group* $K$*.A bipolar valued fuzzy subset(BVFS)* $B$ *of* $K$ *is referred as a bipolar valued fuzzy subgroup (BVFSG) of* $K$*,if for all* $g, h$ *in* $K$ *if*

(i) $B^{+}(gh) \geq \min\{B^{+}(g); B^{+}(h)\}$
(ii) $B^{+}(g^{-1}) \geq B^{+}(g)$
(iii) $B^{-}(gh) \leq \max\{B^{-}(g); B^{-}(h)\}$
(iv) $B^{-}(g^{-1}) \leq B^{-}(g)$

**Definition 2.8.** *( [2, 5]) Let* $B$ *be an object over universe of discourse* $Z$*.Then* $B$ *is called a bipolar vague set(BVS) which is of the form:*

$$B = \{< z : [t^{+}_{B}(z), 1 - f^{+}_{B}(z)], [t^{+}_{B}(z), 1 - f^{+}_{B}(z)] >: z \in Z\},$$

*where* $0 \leq t^{+}_{B}(z) + f^{+}_{B}(z) \leq 1$ *and* $-1 \leq t^{-}_{B}(z) + f^{-}_{B}(z) \leq 0$. *Here* $V^{+}_{B} = [t^{+}_{B}, 1 - f^{+}_{B}]$ *and* $V^{-}_{B} = [-1 - f^{-}_{B}, t^{-}_{B}]$ *will be used to denote a bipolar vague set.*

**Definition 2.9.** *( [2, 5]) Let* $\mathcal{A}$ *be a bipolar- vague set (BVS) in universe of discourse* $Z$. *Then* $\mathcal{A}$ *is called a bipolar valued vague group(BVG) of* $Z$ *if:*

(i) $V^{+}_{B}(gh) \geq \min\{V^{+}_{B}(g), V^{+}_{B}(h)\}$ *and* $V^{+}_{B}(g^{-1}) \geq V^{+}_{B}(g)$ *and* $V^{-}_{B}(gh) \leq \max\{V^{-}_{B}(g), V^{-}_{B}(h)\}$ *and* $V^{-}_{B}(g^{-1}) \leq V^{-}_{B}(g)$, *i.e.,* $t^{+}_{B}(gh) \geq \min\{t^{+}_{B}(g), t^{+}_{B}(h)\}$ *and* $1 - f^{+}_{B}(gh) \geq \min\{1 - f^{+}_{B}(g); 1 - f^{+}_{B}(h)\}$.
(ii) $t^{+}_{B}(g^{-1}) \geq t^{+}_{B}(g)$ *and* $1 - f^{+}_{B}(g^{-1}) \geq 1 - f^{+}_{B}(g)$.

$(iii)$ $t_B^-(gh) \leq \max\{t_B^-(g), t_B^-(h)\}$ *and* $-1 - f_B^-(gh) \leq \max\{-1 - f_B^-(g); -1 - f_B^-(h)\}$.

$(iv)$ $t_B^-(g^{-1}) \leq t_B^-(g)$ *and* $-1 - f_B^-(g^{-1}) \leq -1 - f_B^-(g)$.

**Example 1.** *( [2, 5]) Let $G$={1,$\omega$,$\omega^2$} where $\omega$ is the cubic root of unity with the binary operation defined as below:*

TABLE 1. composition table of cube roots of unity

| $*$ | $1$ | $\omega$ | $\omega^2$ |
|---|---|---|---|
| $1$ | $1$ | $\omega$ | $\omega^2$ |
| $\omega$ | $\omega$ | $\omega^2$ | $1$ |
| $\omega^2$ | $\omega^2$ | $1$ | $\omega$ |

*Let $\mathcal{A} = (Z; V_\mathcal{A}^+, V_\mathcal{A}^-)$ be a bipolar vague set(BVS) in $Z$ as defined below:*

TABLE 2. Bipolar vague set in Z

| | $1$ | $\omega$ | $\omega^2$ |
|---|---|---|---|
| $V_A^+$ | [0.9,0.9] | [0.6,0.8] | [0.6,0.8] |
| $V_A^-$ | [-0.4,-0.1] | [-0.4,-0.1] | [-0.4,-0.1] |

*Then $\mathcal{A} = (Z; V_\mathcal{A}^+, V_\mathcal{A}^-)$ is a bipolar vague group(BVG) of the group $Z$.*

**Definition 2.10.** *( [2, 5]) Consider a group and $B$ be a bipolar vague set (BVS)on $Z$.Then B is known as a bipolar vague normal subgroup (BVNSG) over $Z$ if $V_B^+(ghg^{-1}) \geq V_B^+(h)$ and $V_B^-(ghg^{-1}) \leq V_B^-(h)$ for all $g, h \in Z$. The set of all bipolar vague normal subgroups on $Z$ are denoted by BVNS(Z).*

**Remark 2.1.** *( [2, 5]) Let $B$ be a bipolar vague set(BVS) on group $Z$.Then $B$ is called a bipolar vague normal subgroup over $Z$ (BVNS), if $B(ghg^{-1}) = B(h)$ for all $g, h \in Z$.*

## 3. BIPOLAR VAGUE COSETS

In this section we introduced bipolar vague cosets(BVCs) and studied their properties.

**Definition 3.1.** *Let $\mathcal{A}$ be a bipolar vague group(BVG) over group $(G, \cdot)$. For any $a \in G$, a bipolar vague left coset of $\mathcal{A}$ is denoted by $(a\mathcal{A})^L$ and $(a\mathcal{A})^L =<$*

*$a\mathcal{A}^+, a\mathcal{A}^- >$ and defined by $V^+_{a\mathcal{A}}(x) = V^+_{\mathcal{A}}(a^{-1}x)$, i.e., $t^+_{a\mathcal{A}}(x) = t^+_{\mathcal{A}}(a^{-1}x); f^+_{a\mathcal{A}}(x) = f^+_{\mathcal{A}}(a^{-1}x)$ and $V^-_{a\mathcal{A}}(x) = V^-_{\mathcal{A}}(a^{-1}x)$i.e $t^-_{a\mathcal{A}}(x) = t^-_{\mathcal{A}}(a^{-1}x) f^-_{a\mathcal{A}}(x) = f^-_{\mathcal{A}}(a^{-1}x)$ clearly bipolar vague left coset is a bipolar vague set.*

**Definition 3.2.** *Let $A$ be a bipolar vague group over group $(G, \cdot)$. For any $a \in G$, a bipolar vague right coset of $\mathcal{A}$ is denoted by $(\mathcal{A}a)^R$ and $(\mathcal{A}a)^R =< \mathcal{A}^+a, \mathcal{A}^-a >$ and defined by $V^+_{\mathcal{A}a}(x) = V^+_{\mathcal{A}}(xa^{-1})$, i.e., $t^+_{\mathcal{A}a}(x) = t^+_{\mathcal{A}}(xa^{-1}); f^+_{\mathcal{A}a}(x) = f^+_{\mathcal{A}}(xa^{-1})$ and $V^-_{\mathcal{A}a}(x) = V^-_{\mathcal{A}}(xa^{-1})$i.e $t^-_{\mathcal{A}a}(x) = t^-_{\mathcal{A}}(xa^{-1}); f^-_{\mathcal{A}a}(x) = f^-_{\mathcal{A}}(xa^{-1})$ clearly bipolar vague right coset is a bipolar vague set.*

**Example 2.** *Let $G$={1,$\omega$,$\omega^2$} where $\omega$ is the cubic root of unity with the binary operation defined as below.*

TABLE 3. composition table of cube roots of unity

| $*$ | $1$ | $\omega$ | $\omega^2$ |
|---|---|---|---|
| $1$ | $1$ | $\omega$ | $\omega^2$ |
| $\omega$ | $\omega$ | $\omega^2$ | $1$ |
| $\omega^2$ | $\omega^2$ | $1$ | $\omega$ |

*Let $\mathcal{A} = (Z; V^+_{\mathcal{A}}, V^-_{\mathcal{A}})$ be a bipolar vague set(BVS) in $Z$ as defined below:*

TABLE 4. Bipolar vague set in Z

| | $1$ | $\omega$ | $\omega^2$ |
|---|---|---|---|
| $V^+_A$ | *[0.9,0.9]* | *[0.6,0.8]* | *[0.6,0.8]* |
| $V^-_A$ | *[-0.4,-0.1]* | *[-0.4,-0.1]* | *[-0.4,-0.1]* |

$$t^+_{1\mathcal{A}}(1) = t^+_{\mathcal{A}}(1^{-1}(1)) = t^+_{\mathcal{A}}(1.1) = t^+_{\mathcal{A}}(1) = 0.9;$$

$$f^+_{1\mathcal{A}}(1) = f^+_{\mathcal{A}}(1^{-1}(1)) = f^+_{\mathcal{A}}(1.1) = f^+_{\mathcal{A}}(1) = 0.1$$

$$t^+_{1\mathcal{A}}(\omega) = t^+_{\mathcal{A}}(1^{-1}\omega) = t^+_{\mathcal{A}}(1\omega) = t^+_{\mathcal{A}}(\omega) = 0.6;$$

$$f^+_{1\mathcal{A}}(\omega) = f^+_{\mathcal{A}}(1^{-1}\omega) = f^+_{\mathcal{A}}(1\omega) = f^+_{\mathcal{A}}(1\omega) = 0.2$$

$$t^+_{1\mathcal{A}}(\omega^2) = t^+_{\mathcal{A}}((1^{-1}\omega^2)) = t^+_{\mathcal{A}}(\omega^2) = 0.6;$$

$$f^+_{1\mathcal{A}}(1) = f^+_{\mathcal{A}}((1^{-1}\omega^2)) = f^+_{\mathcal{A}}(\omega^2) = 0.2$$

*Thus*

$$1\mathcal{A}^{+} = \{(1, 0.9, 0.1), (\omega, 0.6, 0.2), (\omega^2, 0.6, 0.2)\}$$
$$t^{-}_{1\mathcal{A}}(1) = t^{-}_{\mathcal{A}}(1^{-1}(1)) = t^{-}_{\mathcal{A}}(1.1) = t^{-}_{\mathcal{A}}(1) = -0.1;$$
$$f^{-}_{1\mathcal{A}}(1) = f^{-}_{\mathcal{A}}(1^{-1}(1)) = f^{-}_{\mathcal{A}}(1.1) = f^{-}_{\mathcal{A}}(1) = -0.6$$
$$t^{-}_{1\mathcal{A}}(\omega) = t^{-}_{\mathcal{A}}(1^{-1}\omega) = t^{-}_{\mathcal{A}}(1\omega) = 0.1;$$
$$f^{-}_{1\mathcal{A}}(\omega) = f^{-}_{\mathcal{A}}(1^{-1}\omega) = f^{-}_{\mathcal{A}}(1\omega) = 0.6$$
$$t^{-}_{1\mathcal{A}}(\omega^2) = t^{-}_{\mathcal{A}}((1^{-1}\omega^2)) = t^{-}_{\mathcal{A}}(\omega^2) = 0.1;$$
$$f^{-}_{1\mathcal{A}}(1) = f^{-}_{\mathcal{A}}((1^{-1}\omega^2)) = f^{+}_{\mathcal{A}}(\omega^2) = 0.6$$

*Thus*

$$1\mathcal{A}^{-} = \{(1, -0.1, -0.6), (\omega, -0.1, -0.6), (\omega^2, -0.1, -0.6)\}$$
$$t^{+}_{\mathcal{A}1}(1) = t^{+}_{\mathcal{A}}((1)1^{-1}) = t^{+}_{\mathcal{A}}(1.1) = t^{+}_{\mathcal{A}}(1) = 0.9;$$
$$f^{+}_{\mathcal{A}1}(1) = f^{+}_{\mathcal{A}}((1)1^{-1}) = f^{+}_{\mathcal{A}}(1.1) = f^{+}_{\mathcal{A}}(1) = 0.1$$
$$t^{+}_{\mathcal{A}1}(\omega) = t^{+}_{\mathcal{A}}(\omega 1^{-1}) = t^{+}_{\mathcal{A}}(1\omega) = t^{+}_{\mathcal{A}}(\omega) = 0.6;$$
$$f^{+}_{\mathcal{A}1}(\omega) = f^{+}_{\mathcal{A}}((1)\omega^{-1}) = f^{+}_{\mathcal{A}}(1\omega) = f^{+}_{\mathcal{A}}(\omega) = 0.2$$
$$t^{+}_{\mathcal{A}1}(\omega^2) = t^{+}_{\mathcal{A}}((\omega^2)1^{-1}) = t^{+}_{\mathcal{A}}(\omega^2) = 0.6;$$
$$f^{+}_{\mathcal{A}1}(1) = f^{+}_{\mathcal{A}}((\omega^2)1^{-1}) = f^{+}_{\mathcal{A}}(\omega^2) = 0.2$$

*Thus*

$$\mathcal{A}^{+}1 = \{(1, 0.9, 0.1), (\omega, 0.6, 0.2), (\omega^2, 0.6, 0.2)\}$$
$$t^{-}_{\mathcal{A}1}(1) = t^{-}_{\mathcal{A}}((1)1^{-1}) = t^{-}_{\mathcal{A}}(1.1) = t^{-}_{\mathcal{A}}(1) = -0.1;$$
$$f^{-}_{\mathcal{A}1}(1) = f^{-}_{\mathcal{A}}((1)1^{-1}) = f^{-}_{\mathcal{A}}(1.1) = f^{-}_{\mathcal{A}}(1) = -0.6$$
$$t^{-}_{\mathcal{A}1}(\omega) = t^{-}_{\mathcal{A}}((1)^{-1})\omega = t^{-}_{\mathcal{A}}(\omega) = -0.1;$$
$$f^{-}_{\mathcal{A}1}(\omega) = f^{-}_{\mathcal{A}}((1)^{-1}\omega) = f^{-}_{\mathcal{A}}(\omega) = -0.6$$
$$t^{-}_{\mathcal{A}1}(\omega^2) = t^{-}_{\mathcal{A}}(((1)^{-1}\omega^2)) = t^{-}_{\mathcal{A}}(\omega^2) = -0.1;$$
$$f^{-}_{\mathcal{A}1}(1) = f^{-}_{\mathcal{A}}((1)^{-1}\omega^2)) = f^{-}_{\mathcal{A}}(\omega^2) = -0.6$$

*Thus*

$\mathcal{A}^{-}1 = \{(1, -0.1, -0.6), (\omega, -0.1, -0.6), (\omega^2, -0.1, -0.6)\}$.

**Definition 3.3.** *A bipolar vague group $\mathcal{A}$ of group $G$ is said to be :*

(i) *bipolar vague symmetric if* $V^+_{\mathcal{A}}(x^{-1}) = V^+_{\mathcal{A}}(x)$ *and* $V^-_{\mathcal{A}}(x^{-1}) = V^-_{\mathcal{A}}(x) \forall x \in G$*, i.e.,* $t^+_{\mathcal{A}}(x^{-1}) = t^+_{\mathcal{A}}(x), f^+_{\mathcal{A}}(x^{-1}) = f^+_{\mathcal{A}}(x)$ *and* $t^-_{\mathcal{A}}(x^{-1}) = t^-_{\mathcal{A}}(x) f^-_{\mathcal{A}}(x^{-1}) = f^-_{\mathcal{A}}(x)$*.*

(ii) *bipolar vague invariant if* $V^+_{\mathcal{A}}(xy) = V^+_{\mathcal{A}}(yx)$ *and* $V^-_{\mathcal{A}}(xy) = V^-_{\mathcal{A}}(yx) \forall x, y \in G$*, i.e.,* $t^+_{\mathcal{A}}(xy) = t^+_{\mathcal{A}}(yx), f^+_{\mathcal{A}}(xy) = f^+_{\mathcal{A}}(yx)$ *and* $t^-_{\mathcal{A}}(xy) = t^-_{\mathcal{A}}(yx), f^-_{\mathcal{A}}(xy) = f^-_{\mathcal{A}}(yx) \forall x, y \in G$*.*

(iii) *Bipolar Vague normal if* $\mathcal{A}$ *is both bipolar vague symmetry and bipolar vague invariant.*

**Theorem 3.1.** *Let* $\mathcal{A}$ *be a bipolar vague group of a group* $G$*, hence* $\forall x, y \in G$*:*

$(i)$ $a\mathcal{A}^+ = b\mathcal{A}^+ \Leftrightarrow \mathcal{A}^+ a^{-1} = \mathcal{A}^+ b^{-1}$ *and* $a\mathcal{A}^- = b\mathcal{A}^- \Leftrightarrow \mathcal{A}^- a^{-1} = \mathcal{A}^- b^{-1}$ *if* $\mathcal{A}$ *is bipolar vague symmetric.*

$(ii)$ $a\mathcal{A}^+ = \mathcal{A}^+ a$ *and* $a\mathcal{A}^- = \mathcal{A}^- a \Leftrightarrow \mathcal{A}$ *is bipolar vague invariant.*

$(iii)$ $a\mathcal{A}^+ = b\mathcal{A}^+ \Leftrightarrow a^{-1}\mathcal{A}^+ = b^{-1}\mathcal{A}^+$ *and* $a\mathcal{A}^- = b\mathcal{A}^- \Leftrightarrow a^{-1}\mathcal{A}^- = b^{-1}\mathcal{A}^-$*if* $\mathcal{A}$ *is bipolar vague normal.*

*Proof.* (i) Suppose $A$ is bipolar vague symmetric and

$$\begin{aligned}
&a\mathcal{A}^+ = b\mathcal{A}^+ \text{ and } a\mathcal{A}^- = b\mathcal{A}^- \\
&\Leftrightarrow V^+_{a\mathcal{A}}(x) = V^+_{b\mathcal{A}}(x) \\
&\Leftrightarrow V^+_{\mathcal{A}}(a^{-1}x) = V^+_{\mathcal{A}}(b^{-1}x) \\
&\Leftrightarrow V^+_{\mathcal{A}}[(a^{-1}x)^{-1}] = V^+_{\mathcal{A}}[(b^{-1}x)^{-1}] \\
&\Leftrightarrow V^+_{\mathcal{A}}(x^{-1}(a^{-1})^{-1}) = V^+_{\mathcal{A}}(x^{-1}(b^{-1})^{-1}) \\
&\Leftrightarrow V^+_{\mathcal{A}a^{-1}}(x^{-1}) = V^+_{\mathcal{A}b^{-1}}(x^{-1}) \\
&\Leftrightarrow V^+_{\mathcal{A}a^{-1}}(x) = V^+_{\mathcal{A}b^{-1}}(x)
\end{aligned}$$

since $\mathcal{A}$ is bipolar vague symmetric, i.e., $\mathcal{A}^+ a^{-1} = \mathcal{A}^+ b^{-1}$. Hence $a\mathcal{A}^+ = b\mathcal{A}^+ \Leftrightarrow \mathcal{A}^+ a^{-1} = \mathcal{A}^+ b^{-1}$.

Suppose $\mathcal{A}$ is bipolar vague symmetric and

$$\begin{aligned}
&a\mathcal{A}^- = b\mathcal{A}^- \\
&\Leftrightarrow V^-_{a\mathcal{A}}(x) = V^-_{b\mathcal{A}}(x) \\
&\Leftrightarrow V^-_{\mathcal{A}}(a^{-1}x) = V^-_{\mathcal{A}}(b^{-1}x) \\
&\Leftrightarrow V^-_{\mathcal{A}}[(a^{-1}x)^{-1}] = V^-_{\mathcal{A}}[(b^{-1}x)^{-1}] \\
&\Leftrightarrow V^-_{\mathcal{A}}(x^{-1}(a^{-1})^{-1}) = V^-_{\mathcal{A}}(x^{-1}(b^{-1})^{-1}) \\
&\Leftrightarrow V^-_{\mathcal{A}a^{-1}}(x^{-1}) = V^-_{\mathcal{A}b^{-1}}(x^{-1}) \\
&\Leftrightarrow V^-_{\mathcal{A}a^{-1}}(x) = V^-_{\mathcal{A}b^{-1}}(x)
\end{aligned}$$

since $\mathcal{A}$ is bipolar vague symmetric, i.e., $\mathcal{A}^- a^{-1} = \mathcal{A}^- b^{-1}$. Hence $a\mathcal{A}^- = b\mathcal{A}^- \Leftrightarrow \mathcal{A}^- a^{-1} = \mathcal{A}^- b^{-1}$.

Thus $a\mathcal{A}^+ = b\mathcal{A}^+ \Leftrightarrow \mathcal{A}^+a^{-1} = \mathcal{A}^+b^{-1}$ and $a\mathcal{A}^- = b\mathcal{A}^- \Leftrightarrow \mathcal{A}^-a^{-1} = \mathcal{A}^-b^{-1}$ if $\mathcal{A}$ is bipolar vague symmetric.

(ii) Suppose $\mathcal{A}$ is bipolar vague invariant

$V^+_{a\mathcal{A}}(x) = V^+_{\mathcal{A}}(a^{-1}x) = V^+_{\mathcal{A}}(xa^{-1})$
$V^+_{a\mathcal{A}}(x) = V^+_{\mathcal{A}a}(x)\forall x \in G$
$\Leftrightarrow a\mathcal{A}^+ = \mathcal{A}^+a$

and

$V^-_{a\mathcal{A}}(x) = V^-_{\mathcal{A}}(a^{-1}x) = V^-_{\mathcal{A}}(xa^{-1})$
$V^-_{a\mathcal{A}}(x) = V^-_{\mathcal{A}a}(x)\forall x \in G$
$\Leftrightarrow a\mathcal{A}^- = \mathcal{A}^-a.$

Hence $a\mathcal{A}^+ = \mathcal{A}^+a$ and $a\mathcal{A}^- = \mathcal{A}^-a \Leftrightarrow \mathcal{A}$ is bipolar vague invariant.

(iii) Suppose $\mathcal{A}$ is bipolar vague normal and $a\mathcal{A}^+ = b\mathcal{A}^+$
$\Leftrightarrow \mathcal{A}^+a^{-1} = \mathcal{A}^+b^{-1}$ by (1)
$\Leftrightarrow a^{-1}\mathcal{A}^+ = b^{-1}\mathcal{A}^+$ by (2)

□

**Theorem 3.2.** *Let $\mathcal{A}$ be a bipolar vague normal group of a group $G$ and $a, b \in G$ then*

*(i)* $a\mathcal{A}^+ = b\mathcal{A}^+ \Leftrightarrow ca\mathcal{A}^+ = cb\mathcal{A}^+$, $a\mathcal{A}^- = b\mathcal{A}^- \Leftrightarrow ca\mathcal{A}^- = cb\mathcal{A}^-$
*(ii)* $a\mathcal{A}^+ = b\mathcal{A}^+ \Leftrightarrow ac\mathcal{A}^+ = bc\mathcal{A}^+$, $a\mathcal{A}^- = b\mathcal{A}^- \Leftrightarrow ac\mathcal{A}^- = bc\mathcal{A}^-$.

*Proof.* Suppose $\mathcal{A}$ is bipolar vague normal and let $a\mathcal{A}^+ = b\mathcal{A}^+$. Then $\Leftrightarrow$ $V_{a\mathcal{A}^+}[c^{-1}x] = V_{b\mathcal{A}^+}[c^{-1}x] for c^{-1}x \in G$

$\Leftrightarrow V^+_{\mathcal{A}}(a^{-1}c^{-1}x) = V^+_{\mathcal{A}}(b^{-1}c^{-1}x)$
$\Leftrightarrow V^+_{\mathcal{A}}((ca)^{-1}x) = V^+_{\mathcal{A}}(cb)^{-1}x)$
$\Leftrightarrow V_{ca}\mathcal{A}^+(x) = V_{cb}\mathcal{A}^+(x)$
$\Leftrightarrow ca\mathcal{A}^+ = cb\mathcal{A}^+.$

Now suppose $\mathcal{A}$ is bipolar vague normal and let $a\mathcal{A}^- = b\mathcal{A}^-$

$\Leftrightarrow V_{a\mathcal{A}^-}[c^{-1}x] = V_{b\mathcal{A}^-}[c^{-1}x] for c^{-1}x \in G$
$\Leftrightarrow V^-_{\mathcal{A}}(a^{-1}c^{-1}x) = V^-_{\mathcal{A}}(b^{-1}c^{-1}x)$
$\Leftrightarrow V^-_{\mathcal{A}}((ca)^{-1}x) = V^-_{\mathcal{A}}((cb)^{-1}x)$
$\Leftrightarrow V_{ca}\mathcal{A}^-(x) = V_{cb}\mathcal{A}^-(x)$
$\Leftrightarrow ca\mathcal{A}^- = cb\mathcal{A}^-.$ □

**Theorem 3.3.** *Let $\mathcal{A}$ be a bipolar vague group of a group $G$.Then $\mathcal{A}$ is bipolar vague normal iff $\mathcal{A}$ is constant on the conjugate classes of $G$.*

*Proof.* Suppose $\mathcal{A}$ is bipolar vague normal group of $G$ and $a, b \in G$. Then $V_{\mathcal{A}}^{+}(b^{-1}ab) = V_{\mathcal{A}}^{+}(abb^{-1}) = V_{\mathcal{A}}^{+}(ae) = V_{\mathcal{A}}^{+}(a)$ and $V_{\mathcal{A}}^{-}(b^{-1}ab) = V_{\mathcal{A}}^{-}(abb^{-1}) = V_{\mathcal{A}}^{-}(ae) = V_{\mathcal{A}}^{-}(a)$. Hence $\mathcal{A}$ is constant on the conjugate classes of $G$. Consequently if $\mathcal{A}$ is constant on the conjugate classes of $G$. Let $a, b \in G$. Then

$$V_{\mathcal{A}}^{+}(ab) = V_{\mathcal{A}}^{+}(ab.aa^{-1}) = V_{\mathcal{A}}^{+}(a(ba)a^{-1}) = V_{\mathcal{A}}^{+}(ba)$$

and

$$V_{\mathcal{A}}^{-}(ab) = V_{\mathcal{A}}^{-}(ab.aa^{-1}) = V_{\mathcal{A}}^{-}(a(ba)a^{-1}) = V_{\mathcal{A}}^{-}(ba).$$

Hence $\mathcal{A}$ is bipolar vague normal group of $G$. □

**Theorem 3.4.** *Let $\mathcal{A}$ be bipolar vague group of a group $G$ then the following are equivalent.*

*(1) $a\mathcal{A}^{+} = \mathcal{A}^{+}a$ and $a\mathcal{A}^{-} = \mathcal{A}^{-}a$ foreach $a \in G$*
*(2) $a\mathcal{A}^{+}a^{-1} = \mathcal{A}^{+}, a\mathcal{A}^{-}a^{-1} = \mathcal{A}^{-}$ foreach $a \in G$.*

*Proof.* We have $\mathcal{A}$ is bipolar vague group of a group $G$ and $\mathcal{A} \in G$. Suppose $a\mathcal{A}^{+} = \mathcal{A}^{+}a$ and $a\mathcal{A}^{-} = \mathcal{A}^{-}a$. Then,

$$\Rightarrow a\mathcal{A}^{+}a^{-1} = \mathcal{A}^{+}aa^{-1} = \mathcal{A}^{+}e = \mathcal{A}^{+} \Rightarrow a\mathcal{A}^{-}a^{-1} = \mathcal{A}^{-}aa^{-1} = \mathcal{A}^{-}e = \mathcal{A}^{-}.$$

Thus $a\mathcal{A}^{+}a^{-1} = \mathcal{A}^{+}$ and $a\mathcal{A}^{-}a^{-1} = \mathcal{A}^{-}a$.

Now suppose that $a\mathcal{A}^{+}a^{-1} = \mathcal{A}$. Then

$$\Rightarrow a\mathcal{A}^{+}a^{-1}a = \mathcal{A}a \Rightarrow a\mathcal{A}^{+}e = \mathcal{A}^{+}a \Rightarrow a\mathcal{A}^{+} = \mathcal{A}^{+}a$$

and suppose that $a\mathcal{A}^{-}a^{-1} = \mathcal{A}^{-}$. Then

$$\Rightarrow a\mathcal{A}^{-}a^{-1}a = \mathcal{A}^{-}a \Rightarrow a\mathcal{A}^{-}e = \mathcal{A}^{-}a \Rightarrow a\mathcal{A}^{-} = \mathcal{A}^{-}a.$$ □

**Remark 3.1.** *Let $\mathcal{A}$ be a bipolar vague group of a group $G$. Then*

$$V_{\mathcal{A}}^{+}(xy^{-1}) = V_{\mathcal{A}}^{+}(e) \Rightarrow V_{\mathcal{A}}^{+}(x) = V_{\mathcal{A}}^{+}(y),$$

$$V_{\mathcal{A}}^{-}(xy^{-1}) = V_{\mathcal{A}}^{-}(e) \Rightarrow V_{\mathcal{A}}^{-}(x) = V_{\mathcal{A}}^{-}(y) \forall x, y \in G.$$

*The converse of the above result is not true. For example, consider the group $G = \{1, \omega, \omega^2\}$ w.r.t. the binary operation defined below, where $\omega$ is the cube root of unity. let $\mathcal{A}$ be a bipolar vague group in $G$ as in table 3 and 4. But*

$V_{\mathcal{A}}^{+}(1) = [0.9, 0.9]V_{\mathcal{A}}^{+}(\omega) = [0.6, 0.8]V_{\mathcal{A}}^{+}(\omega^2) = [0.6, 0.8]$. *Now* $V_{\mathcal{A}}^{+}(\omega) = V_{\mathcal{A}}^{+}(\omega^2)$. *But* $V_{\mathcal{A}}^{+}(\omega(\omega^2)^{-1}) \neq V_{\mathcal{A}}^{+}(1)$

**Theorem 3.5.** *Let $\mathcal{A}$ be a bipolar vague group of a group $G$. Then $\mathcal{A}$ is a bipolar vague normal subgroup of $G$ if and only if for all $x \in G$, $a\mathcal{A}^{+}(x) = \mathcal{A}^{+}a(x)$ and, $a\mathcal{A}^{-}(x) = \mathcal{A}^{-}a(x)$.*

*Proof.* Let us assume that $\mathcal{A}$ is a bipolar vague normal subgroup of $G$. Now we prove that $a\mathcal{A}^{+} = \mathcal{A}^{+}a$ and $a\mathcal{A}^{-} = \mathcal{A}^{-}a$.

To prove this we show that for all $y \in G$: $a\mathcal{A}^{+}(y) = \mathcal{A}^{+}a(y)$,$a\mathcal{A}^{-}(y) = \mathcal{A}^{-}a(y)$, i.e., $V_{\mathcal{A}}^{+}(a^{-1}y) = V_{\mathcal{A}}^{+}(ya^{-1}), V_{\mathcal{A}}^{-}(a^{-1}y) = V_{\mathcal{A}}^{-}(ya^{-1})$. Now

$$V_{\mathcal{A}}^{+}(a^{-1}y) = V_{\mathcal{A}}^{+}(a^{-1}ya^{-1}a) = V_{\mathcal{A}}^{+}(a^{-1}(ya^{-1})a) \geq V_{\mathcal{A}}^{+}(ya^{-1}),$$

and

$$V_{\mathcal{A}}^{+}(ya^{-1}) = V_{\mathcal{A}}^{+}(aa^{-1}ya^{-1}) = V_{\mathcal{A}}^{+}(a(a^{-1}y)a^{-1}) \geq V_{\mathcal{A}}^{+}(a^{-1}y),$$

$$V_{\mathcal{A}}^{-}(a^{-1}y) = V_{\mathcal{A}}^{-}(a^{-1}ya^{-1}a) = V_{\mathcal{A}}^{-}(a^{-1}(ya^{-1})a) \leq V_{\mathcal{A}}^{-}(ya^{-1}),$$

and

$$V_{\mathcal{A}}^{-}(ya^{-1}) = V_{\mathcal{A}}^{-}(aa^{-1}ya^{-1}) = V_{\mathcal{A}}^{-}(a(a^{-1}y)a^{-1}) \leq V_{\mathcal{A}}^{-}(a^{-1}y).$$

Hence $a\mathcal{A}^{+}(y) = \mathcal{A}^{+}a(y)$.

Similarly we can show that $a\mathcal{A}^{-}(y) = \mathcal{A}^{-}a(y)$ for all $y \in G$. Thus $a\mathcal{A}^{+} = \mathcal{A}^{+}a$ and ,$a\mathcal{A}^{-} = \mathcal{A}^{-}a$.

Conversely assume that $a\mathcal{A}^{+}(x) = \mathcal{A}^{+}a(x)$ and ,$a\mathcal{A}^{-}(x) = \mathcal{A}^{-}a(x)$ for all $x \in G$. Let prove $A$ is bipolar vague normal subgroup of $G$. Now for any $x, y \in G$, $V_{\mathcal{A}}^{+}(a^{-1}ya) = V_{\mathcal{A}}^{+}a(ya) = V_{a\mathcal{A}}^{+}(ya) = V_{\mathcal{A}}^{+}(yaa^{-1}) = V_{\mathcal{A}}^{+}(y) \geq V_{\mathcal{A}}^{+}(y)$. Thus $V_{\mathcal{A}}^{+}(a^{-1}ya) \geq V_{\mathcal{A}}^{+}(y)$ and $V_{\mathcal{A}}^{-}(a^{-1}ya) = V_{\mathcal{A}}^{-}a(ya) = V_{\mathcal{A}}^{-}(yaa^{-1}) = V_{\mathcal{A}}^{-}(y) \leq V_{\mathcal{A}}^{-}(y)$, and further, $V_{\mathcal{A}}^{-}(a^{-1}ya) \leq V_{\mathcal{A}}^{+}(y)$. Hence $\mathcal{A}$ is a bipolar vague normal subgroup(BVNSG) of $G$. □

## 4. CONCLUSION

In this paper we introduced the concept of bipolar vague cosets of a group and studied the notion of symmetric-invariant-normal bipolar vague groups. These concepts are used to further study in the development of characterizations about bipolar vague groups(BVGs) and bipolar vague normal groups(BVNGs). In order our future work is to study the notion of Bipolar vague Quotient groups (BVQGs) and investigate some of the properties based on it.

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