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THE MINIMUM MEAN HUB ENERGY OF CERTAIN GRAPHS

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ABSTRACT. For the last few decades, many researchers interested on the study of electron energy and which leads to study the application of graphs. In the present study, the minimum mean hub energy for a graph has been introduced and energies for some standard graphs with bounds has been established.

1. INTRODUCTION

In 1930, most of the researchers in chemistry were interested on the study of π - electron energy and which leads to study the application of graph energy in the area of energies of conjugated hydro carbon molecules in chemistry. Recent research on energies are based on the study of Eigen values of a variety of matrices of respective graph. In 1978, I. Gutman defined energy mathematically for all graphs and to understand the concepts refer [1] , [2].

In 2006, the hub numbers concept was introduced by M. Walsh [5]. Let G be a graph with vertex set V and x, y $\in$ K $\subseteq$ V. Any path from x to y which involves only vertices from K is called the K-path. If for every pair of vertices which are not in K, there exists a K-path in G between x and y, then K is called a hub set. Such a set K with least number of vertices is called the minimum hub set and its cardinality is known as the hub number denoted by $h(G)$.

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For a simple graph G with |V| $= n$, denote Δ(G), δ(G) denote as maximum and minimum degree of G respectively. The ordinary energy of G is defined, by I.Gutman, as the sum of the modulus values of all characteristic values of its corresponding matrix. i.e. if $h_1, h_2, ..., h_n$ be the characteristic values of corresponding matrix A(G), then energy of graph is $E(G) = \sum_{i=1}^{n} |h_i|$. Amalia Culiuc, Laura Buggy, Katelyn Mccall and Duy Nguyen [4] introduced the more general M-energy or Mean Energy of G as $E^M(G) = \sum_{i=1}^{n} |h_i - \bar{h}|$, where $\bar{h}$ is the average of $h_1, h_2, ..., h_n$.

The minimum hub energy was introduced by Sultan Senan Mahde and Veena Mathad [6]. In the present study, the minimum mean [3] hub energy, denoted by $E_H^M(G)$, has been introduced and the minimum mean hub energies of few standard graphs with bounds have been established.

2. The Minimum Mean Hub Energy

A subset K of vertex set of a graph G is called a hub set if for every pair of vertices, which are not in K, there exists a path with all internal nodes are in K. Such a set K with least number of vertices is called the minimum hub set and its cardinality is known as the hub number denoted by $h(G)$.

The minimum hub matrix $A_{HB}(G)$ of G is a $n \times n$ matrix defined as

$$A_{HB}(G) = \begin{cases} 1 & if \quad < v_i, v_j > \ \in \ E \\ 1 & if \ i = j, \ \ v_j \ \in \ H \\ 0 & otherwise. \end{cases}$$

The minimum hub energy (MHE) of graph G is defined as $E_H(G) = \sum_{i=1}^{n} |h_i|$ where $h_1, h_2, ..., h_n$ are characteristic values of $A_{HB}(G)$.

Definition 2.1. *The minimum mean hub energy (MMHE) of G is defined as* $E_H^M(G) = \sum_{i=1}^{n} |h_i - \bar{h}|$*, where* $\bar{h}$ *is the mean of characteristic values* $h_1, h_2, ..., h_n$ *of* A_H*(G).*

Remark 2.1. *[6] The MHE of a graph G varies withthe MH set and hence the MMHE of a graph G also.*

Result 2.1. *[6] For a simple graph G with* $|V|$ *= n and edge set E, if* $\bar{h}$ *is the mean of characteristic values* $h_1, h_2, ..., h_n$ *of MH matrix* $A_{HB}(G)$ *then*

(i) $\sum_{i=1}^{n} |h_i| = h(G)$,

(ii) $\sum_{i=1}^{n} h_i^2 = 2|E| + h(G)$.

Theorem 2.1. *[6]. For a simple graph G with |V|=n and |E|=m,* $\sqrt{2m + h(G)} \leq E_H(G) \leq \sqrt{n(2m + h(G))}$.

Theorem 2.2. *[6]. For a simple connected graph G with |V| = n, |E| = m,* $\sqrt{2n - m - 1} \leq E_H(G) \leq n\sqrt{n - \frac{\Delta}{n}}$.

In the following sections, the MMHE of some standard graphs and their bound has been presented.

3. MMHE OF FEW STANDARD GRAPHS

Theorem 3.1. *The MMHE of K_n(complete graph) is E_H^M(G) =$2n - 2 \; \forall \; n \geq 2$.*

Proof. Assume V = $\{v_1, v_2, ..., v_n\}$ is the vertex set of K_n. The minimum hub (MH) number $h(K_n) = 0$. The minimum hub(MH) matrix is

$$A_{HB}(K_n) = \begin{bmatrix} 0 & 1 & 1 & ... & 1 & 1 \\ 1 & 0 & 1 & ... & 1 & 1 \\ 1 & 1 & 0 & ... & 1 & 1 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 1 & 1 & 1 & ... & 1 & 0 \end{bmatrix}_{n \times n}$$

Then the characteristic equation $|\, A_{HB}(K_n) - hI| = 0$ is $(h+1)^{(n-1)}(h - (n-1)) = 0$. Then $h = -1; (n-1)$ times, $h = (n-1)$ a simple root. Mean $\bar{h}$ = 0.
The MMHE of $K_n = E_H^M(K_n) = \sum_{i=1}^{n-1} |-1-0| + |(n-1)-0| = (2n-2)$. ☐

Theorem 3.2. *The MMHE of $K_{1,n-1}$ (star graph) is $\dfrac{(n-2)}{n} + \sqrt{4n-3}$ for all $n \geq 2$.*

Proof. Assume V = $\{v_1, v_2, .., v_n\}$ be the vertex set of $K_{1,n-1}$.
The MH set HB = $\{v_1\}$; (v_1 is centre node). The MH matrix is

$$A_{HB}(K_{1,n-1}) = \begin{bmatrix} 1 & 1 & 1 & ... & 1 & 1 \\ 1 & 0 & 0 & ... & 0 & 0 \\ 1 & 0 & 0 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 1 & 0 & 0 & ... & 0 & 0 \end{bmatrix}_{n\times n}$$

Then the characteristic equation $|A_{HB}(K_{1,n-1})-hI| = 0$ is $h(h^2-h-(n-1)) = 0$.
By solving, characteristic values are $h = 0$; $(n-2)$ times, $\frac{1\pm\sqrt{4n-3}}{2}$.
Mean $\bar{h} = \frac{1}{n}$. The MMHE of $K_{1,n-1} = E_H^M(K_{1,n-1})$
$= \sum_{i=1}^{n-2} |0 - \frac{1}{n}| + |\frac{1\pm\sqrt{4n-3}}{2} - (\frac{1}{n})| = \frac{(n-2)}{n} + |\frac{(n-2)\pm n\sqrt{4n-3}}{2n}| = \frac{(n-2)}{n} + \sqrt{4n-3}.$ □

Theorem 3.3. *The MMHE of* $K_{n,n}$ *(Complete bipartite graph) is* $\frac{2(n-2)}{n} + \frac{(n^2+n-2)}{n} + (n-1)\sqrt{n}$, *for all* $n \geq 3$.

Proof. Assume V= (A_1, A_2) where $A_1 = \{v_1, v_2, ..., v_n\}$ and $A_2 = \{u_1, u_2, ..., u_n\}$ be the partition of $K_{n,n}$. The MH set HB= $\{u_1, v_1\}$. The MH matrix is

$$4A_{HB}(K_{n,n}) = \begin{bmatrix} 1 & 0 & 0 & .. & 0 & 1 & 1 & 1 & .. & 1 \\ 0 & 0 & 0 & .. & 0 & 1 & 1 & 1 & .. & 1 \\ 0 & 0 & 0 & .. & 0 & 1 & 1 & 1 & .. & 1 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 0 & 0 & 0 & .. & 0 & 1 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 1 & 0 & 0 & .. & 0 \\ 1 & 1 & 1 & .. & 1 & 0 & 0 & 0 & .. & 0 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 1 & 1 & 1 & .. & 1 & 1 & 0 & 0 & .. & 0 \end{bmatrix}_{2n\times 2n}$$

Then the characteristic equation $|\ A_{HB}(K_{n,n}) - hI| = 0$ is
$h^{2n-4}(h^2 + (n-1)h - (n-1))(h^2 - (n+1)h + (n-1)) = 0$. The spectrum is

$$Spec_{HB}(K_{n,n}) = \begin{pmatrix} 0 & \frac{-(n-1)\pm(n-1)\sqrt{n}}{2} & \frac{(n+1)\pm\sqrt{n^2-2n+5}}{2} \\ 2n-4 & 1(two\ \ roots) & 1(two\ \ roots) \end{pmatrix}.$$

Mean $\bar{h} = \frac{1}{n}$. The MMHE can be calculated as $E_H^M(K_{n,n})$

$$= \sum_{i=1}^{2n-4} |0 - \frac{1}{n}| + |\frac{(n+1) \pm \sqrt{n^2 - 2n + 5}}{2} - \frac{1}{n}| + |\frac{-(n-1) \pm (n-1)\sqrt{n}}{2} - \frac{1}{n}|$$
$$= \frac{2(n-4)}{n} + |\frac{(n^2+n-2) \pm n\sqrt{n^2-2n+5}}{2n}| + |\frac{(n-n^2-1) \pm n(n-1)\sqrt{n}}{2n}|.$$
$$= \frac{2(n-2)}{n} + \frac{(n^2+n-2)}{n} + (n-1)\sqrt{n}.$$

□

Theorem 3.4. *The MMHE of* $S_{n,n}$ *(double star graph) is equal to* $\frac{2(n-2)}{n} + 2(\sqrt{n} + \sqrt{n-1})$*, for all* $n \geq 3$.

Proof. For $S_{n,n}$ with V = $\{v_1, v_2, ..., v_n, u_1, u_2, ..., u_n\}$ the MH set is HB= $\{v_1, u_1\}$. The MH matrix is

$$A_{HB}(S_{n,n}) = \begin{bmatrix} 1 & 1 & 1 & .. & 1 & 1 & 0 & 0 & .. & 0 \\ 1 & 1 & 0 & .. & 0 & 0 & 0 & 0 & .. & 0 \\ 1 & 0 & 1 & .. & 0 & 0 & 0 & 0 & .. & 0 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 1 & 0 & 0 & .. & 0 & 0 & 0 & 0 & .. & 0 \\ 1 & 0 & 0 & .. & 0 & 0 & 1 & 1 & .. & 1 \\ 0 & 0 & 0 & .. & 0 & 1 & 1 & 0 & .. & 0 \\ 0 & 0 & 0 & .. & 0 & 1 & 0 & 0 & .. & 0 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 0 & 0 & 0 & .. & 0 & 1 & 0 & 0 & .. & 0 \end{bmatrix}_{2n \times 2n}$$

Then the characteristic equation $|\ A_{HB}(S_{n,n}) - hI| = 0$ is $h^{2n-4}(h^2-(n-1))(h^2-2h-(n-1)) = 0$. By solving, $h = 0$ is a root of multiplicity $(2n-4)$, a pair of simple roots $1 \pm \sqrt{n}$ and $\pm\sqrt{n-1}$. Mean $\bar{h} = \frac{1}{n}$.

The MMHE

$$\begin{aligned} E_H^M(S_{n,n}) &= \sum_{i=1}^{2n-4} |0-\frac{1}{n}| + |(1\pm\sqrt{n})-\frac{1}{n}| + |\pm\sqrt{n-1}-\frac{1}{n}| \\ &= \frac{(2n-4)}{n} + |\frac{(n-1)\pm n\sqrt{n}}{n}| + |\frac{-1\pm n\sqrt{n-1}}{n}| \\ &= \frac{2(n-2)}{n} + 2(\sqrt{n}+\sqrt{n-1}). \end{aligned}$$

□

Theorem 3.5. *MMHE of $K_{n\times 2}$ (Cocktail party graph) is* $\frac{(4n^2-6n-2)}{n}+2\sqrt{2n}$ *for all $n \geq 2$.*

Proof. Let $V = \cup_{i=1}^{n}\{u_i, v_i\}$ be the vertex set of $K_{n\times 2}$. The MH set is HB = $\{u_1\}$. Then the MH matrix is

$$A_{HB}(K_{n\times 2}) = \begin{bmatrix} 1 & 0 & 1 & 1 & ..1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & ..1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & ..1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & ..1 & 1 & 1 & 1 \\ . & . & . & . & ... & . & . & . \\ . & . & . & . & ... & . & . & . \\ . & . & . & . & ... & . & . & . \\ . & . & . & . & ... & . & . & . \\ 1 & 1 & 1 & 1 & ... & . & 0 & 0 \\ 1 & 1 & 1 & 1 & ... & . & 0 & 0 \end{bmatrix}_{2n\times 2n}$$

Then the characteristic equation $|\, A_{HB}(K_{n\times 2}) - hI| = 0$ is $h^{n-1}(h-(2n-3))(h+2)^{n-2}(h^2-2n) = 0$. By solving, $h = 0; (n-1)$ times, $h = 2n-3$ (1 time), $h = -2$ ($n-2$ times), $h = \pm\sqrt{2n}$. Mean $\bar{h} = \frac{1}{2n}$. The MMHE $E_H^M(K_{n\times 2})$

$$\begin{aligned} &= \sum_{i=1}^{n-1} |0-\frac{1}{2n}| + |(2n-3)-\frac{1}{2n}| + \sum_{i=1}^{n-2} |-2-\frac{1}{2n}| + |\pm\sqrt{2n}-\frac{1}{2n}| \\ &= \frac{(n-1)}{2n} + \frac{(4n^2-6n-1)}{2n} + \frac{(n-2)(4n+1)}{2n} + |\frac{-1\pm 2n\sqrt{2n}}{2n}| \\ &= \frac{(4n^2-6n-2)}{n} + 2\sqrt{2n}. \end{aligned}$$

□

4. BOUNDS OF MMHE

Theorem 4.1. *For a simple graph G with $|V| = n$, $|E| = m$ and the hub number $h(G)$,* $\sqrt{((h(G)+2m) - 2|\bar{h}|\sqrt{n(2m+h(G))}} \leq E_H^M(G) \leq \sqrt{n((h(G)+2m) + n\bar{h}^2)}$.

Proof. By Cauchy Schwartz inequality, $(\sum_{i=1}^n |p_i q_i|)^2 = (\sum_{i=1}^n p_i^2)(\sum_{i=1}^n q_i^2)$. Take $p_i = 1$, $q_i = |h_i - \bar{h}|$. Then,

$$\begin{aligned}
[E_H^M(G)]^2 &= (\sum_{i=1}^n 1)(\sum_{i=1}^n |h_i - \bar{h}|^2) \\
&= (n)(\sum_{i=1}^n |h_i - \bar{h}|^2) \leq n(\sum_{i=1}^n h_i^2 + \sum_{i=1}^n \bar{h}^2) \\
&= n(\sum_{i=1}^n h_i^2 + n\,\bar{h}^2) \leq n((2m + h(G)) + n\bar{h}^2).
\end{aligned}$$

Then, $E_H^M(G) \leq \sqrt{n((h(G)+2m) + n\bar{h}^2)}$. Also,

$$\begin{aligned}
[E_H^M(G)]^2 &= (\sum_{i=1}^n |h_i - \bar{h}|)^2 \geq \sum_{i=1}^n |h_i - \bar{h}|^2 \geq \sum_{i=1}^n |h_i|^2 - 2|\bar{h}| \sum_{i=1}^n |h_i| \\
&\geq (h(G) + 2m) - 2|\bar{h}|\sqrt{n(h(G)+2m)}.
\end{aligned}$$

Then, $E_H^M(G) \geq \sqrt{(h(G)+2m) - 2|\bar{h}|\sqrt{n(h(G)+2m)}}$.

□

Theorem 4.2. *For a simple graph G with $|V| = n$, $|E| = m$ and the hub number $h(G)$,* $\sqrt{(h(G)+2m) - 2|\bar{h}|n\sqrt{n - \frac{\Delta}{n}}} \leq E_H^M(G) \leq \sqrt{n(n^2(n - \frac{\Delta}{n}) + n\bar{h}^2)}$.

Proof. By Cauchy Schwartz inequality, $(\sum_{i=1}^n |p_i q_i|)^2 = (\sum_{i=1}^n p_i^2)(\sum_{i=1}^n q_i^2)$. Take $p_i = 1$, $q_i = |h_i - \bar{h}|$. Then,

$$\begin{aligned}
[E_H^M(G)]^2 &= (\sum_{i=1}^n 1)(\sum_{i=1}^n |h_i - \bar{h}|^2) = (n)(\sum_{i=1}^n |h_i - \bar{h}|^2) \leq n(\sum_{i=1}^n h_i^2 + \sum_{i=1}^n \bar{h}^2) \\
&= n(\sum_{i=1}^n h_i^2 + n\,\bar{h}^2) \leq n(n^2(n - \frac{\Delta}{n}) + n\bar{h}^2)\,.
\end{aligned}$$

Thus, $E_H^M(G) \leq \sqrt{n(n^2(n-\frac{\Delta}{n}) + n\bar{h}^2)}$. Also,

$$\begin{aligned}
[E_H^M(G)]^2 &= (\sum_{i=1}^{n} |h_i - \bar{h}|)^2 \geq \sum_{i=1}^{n} |h_i - \bar{h}|^2 \geq \sum_{i=1}^{n} |h_i|^2 - 2|\bar{h}| \sum_{i=1}^{n} |h_i| \\
&\geq (h(G) + 2m) - 2|\bar{h}| n \sqrt{n - \frac{\Delta}{n}}.
\end{aligned}$$

Thus, $E_H^M(G) \geq \sqrt{(h(G)+2m) - 2|\bar{h}|n\sqrt{n - \frac{\Delta}{n}}}$. □

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