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POLYNOMIAL REPRESENTATION OF ORDER-CONGRUENCE GRAPH OF $\mathbf{Z_n}$

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ABSTRACT. The order-congruence graph of the commutative ring Z_n , denoted by $C_o(Z_n)$ is defined as the graph whose vertices are the elements of the ring Z_n and any two distinct vertices x and y are adjacent whenever $x + y \equiv 0(modn)$ or $x.y \equiv 0(modn)$ with $y \neq n - x$, where n is the order of Z_n. For a simple graph G, having $\Delta(G)$ as its maximum degree, the polynomial representation of G, denoted by $P_G(x)$, is a polynomial over Z of the form $P_G(x) = \sum_{i=0}^{\Delta(G)} a_i x^i$, where a_i is the number of vertices in G with degree i for each $i = 0, 1, \ldots, \Delta(G)$. In this paper we establish the polynomial representation of order-congruence graph of Z_n for some particular values of n.

1. INTRODUCTION

A new graph structure associated to a commutative ring Z_n called order-congruence graph is introduced in [2]. The graph is denoted by $C_o(Z_n)$. In our present paper we establish the polynomial representation of Order-congruence graph of Z_n for $n = p, 2p, 3p, p^2$. We also find the polynomial representation of Line graph of Order-congruence graph of Z_p , $L(C_o(Z_p))$ and a relation between polynomial representation of $C_o(Z_p)$ and $L(C_o(Z_p))$ is also established in this paper. Besides these the relationships between $C_o(Z_n)$ and their union and join are studied. In this paper we also study some properties of the corresponding

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polynomial representation of $C_o(Z_n)$ [like whether they are monic, reducible , have constant term etc.].

For further reference see [1,2,3,4,5].

2. Preliminaries

Definition 2.1. *Let $x, y \in Z_n$ such that $x \neq y$, if $x + y \equiv 0(mod n)$ then we say x is conjugate to y and y is conjugate to x and the pair (x, y) is known as conjugate pair.*

Definition 2.2. *Line graph of order-congruence graph of Z_n, $L(C_o(Z_n))$ is the graph with all the edges of $C_o(Z_n)$ as vertices and any two distinct vertices are adjacent if and only if their corresponding edges share a common vertex in the graph $C_o(Z_n)$. We use the notation $[x, y]$ to represent a vertex in $L(C_o(Z_n))$.*

Definition 2.3. *The polynomial representation of a simple graph G , $P_G(x)$, is a polynomial over Z of the form $P_G(x) = \sum_{i=0}^{\Delta(G)} a_i x^i$, where $\Delta(G)$ is the maximum degree and a_i is the number of vertices in G with degree i for each $i = 0, 1, \ldots, \Delta(G)$.*

Definition 2.4. *A polynomial having one , two , three and four terms are respectively known as Monomial , Binomial , Trinomial and Quadrinomial.*

Definition 2.5. *A monic polynomial is a single variable polynomial in which the leading coefficient is equal to 1.*

Definition 2.6. *A polynomial is said to be reducible if it can be factored into non-trivial polynomials over the same field. Otherwise it is irreducible.*

Definition 2.7. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs , the union $G_1 \cup G_2$ is defined as $G = (V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$.*

Definition 2.8. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs , the join $G_1 + G_2$ is defined as $G_1 \cup G_2$ together with all the lines joining V_1 with V_2.*

3. Main Results

Theorem 3.1. *For the graph $G = C_o(Z_n)$,*

(i) *$P_G(x)$ is always monic for $n > 3$.*

(ii) *$P_G(x)$ never contains a constant term.*
(iii) *The degree of $P_G(x)$ is $n-1$.*
(iv) *$P_G(x)$ is monomial only if $n = 2, 3$.*

Proof. (i). For $n > 3$, apart from 0 there is no vertex in $C_o(Z_n)$ which is adjacent with all other vertices. i.e. the vertices with highest degree is unique. Thus $P_G(x)$ is always monic for $n > 3$.

(ii). As $C_o(Z_n)$ is connected $\forall\ n$, so there is no vertex in $C_o(Z_n)$ whose degree is 0. Hence there is no constant term in polynomial representation of $C_o(Z_n)$.

(iii). In $C_o(Z_n)$, the degree of vertex corresponding to 0 is $n-1$, which is the maximum degree among all vertices of $C_o(Z_n)$. Hence the degree of $P_G(x)$ is $n-1$.

(iv). $P_G(x)$ is a monomial $\Rightarrow$ degrees of all the vertices in $C_o(Z_n)$ are equal $\Rightarrow$ $C_o(Z_n)$ is a regular graph $\Rightarrow n = 2, 3$ as for $n > 3$, $C_o(Z_n)$ will never be a regular graph. Thus $P_G(x)$ is monomial only if $n = 2, 3$. □

Theorem 3.2. *The polynomial representation of $C_o(Z_p)$, $P_{C_o(Z_p)}(x) = x^{p-1} + (p-1)x^2$, where p is an odd prime.*

Proof. In $C_o(Z_p)$, the vertex corresponding to 0 is adjacent with all the remaining $p-1$ vertices. i.e. $deg(0) = p-1$. While for $x \neq 0 \in Z_p$, x is adjacent to $p-x$ and 0 in $C_o(Z_p)$. i.e. $deg(x) = 2$. Hence there are one vertex with degree $p-1$ and $p-1$ vertices with degree 2. So , $P_{C_o(Z_p)}(x) = x^{p-1} + (p-1)x^2$. □

Theorem 3.3. *The polynomial representation of $C_o(Z_{p^2})$, $P_{C_o(Z_{p^2})}(x) = x^{p^{2-1}} + (p-1)x^{p-1} + p(p-1)x^2$, where p is prime.*

Proof. In $C_o(Z_{p^2})$ by definition of Order-congruence graph , $deg(0) = p^2 - 1$ and apart from 0, no other vertex has degree $p^2 - 1$. Now the vertices corresponding to $p, 2p, 3p, ..., (p-1)p$ are adjacent with each other and 0. Hence there are $p-1$ vertices with degree $p-1$. While the remaining vertices of $C_o(Z_{p^2})$,

$1, 2, \ldots p-1$
$p+1, p+2, \ldots, P+(p-1)\ldots$
$((p-1)/2)p+1, ((p-1)/2)p+2, \ldots, ((p-1)/2)p+(p-1)$
$((p+1)/2)p+1, ((p+1)/2)p+2, \ldots, ((p+1)/2)p+(p-1)$
$((p+3)/2)p+1, ((p+3)/2)p+2, \ldots, ((p+3)/2)p+(p-1)\ldots$
$((p+(p-2))/2)p+1, ((p+(p-2))/2)p+2, \ldots, ((p+(p-2))/2)p+(p-1)$

are adjacent with 0 and their respective conjugate only.

Hence degree of all of them are 2.Thus the number of vertices with degree 2 is $((p+(p-2))/2+1)(p-1) = p(p-1)$. Therefore , $P_{C_o(Z_{p^2})}(x) = x^{p^{2-1}} + (p-1)x^{p-1} + p(p-1)x^2$, where p is prime. □

Corollary 3.1. *The polynomial representation of $C_o(Z_{p^2})$ is a Trinomial provided $p \neq 3$.*

Proof. The result follows from Theorem 3.3 immediately. □

Corollary 3.2. *Let $G = C_o(Z_p)$ and $H = C_o(Z_{p^2})$, where p is an odd-prime, then the polynomial representation of*

(i) *$G \cup H$ is $x^{p^2-1} + px^{p-1} + (p^2-1)x^2$.*
(ii) *$G + H$ is $2x^{p^2+p-1} + (p-1)x^{p^2+2} + (p-1)x^{2p-1} + p(p-1)x^{P+2}$.*

Proof. By Theorem 3.2 , $P_{C_o(Z_p)}(x) = x^{p-1} + (p-1)x^2$, where p is an odd prime. By Theorem 3.3 , $P_{C_o(Z_{p^2})}(x) = x^{p^{2-1}} + (p-1)x^{p-1} + p(p-1)x^2$, where p is prime. Therefore :

(i). $P_{G\cup H}(x) = x^{p-1} + (p-1)x^2 + x^{p^2-1} + (p-1)x^{p-1} + p(p-1)x^2$
$= x^{p^2-1} + (1+p-1)x^{p-1} + ((p-1)+p(p-1))x^2 = x^{p^2-1} + px^{p-1} + (p^2-1)x^2$.

(ii). In $G + H$, every vertices of G is adjacent with every vertices of H. Hence the degree of each of the vertices of G is increased by p^2 and that of H is increased by p. Thus , $P_{G+H}(x) = x^{p-1+p^2} + (p-1)x^{2+p^2} + x^{p^2-1+p} + (p-1)x^{p-1+p} + p(p-1)x^{2+p} = 2x^{p^2+p-1} + (p-1)x^{p^2+2} + (p-1)x^{2p-1} + p(p-1)x^{P+2}$. □

Theorem 3.4. *The polynomial representation of $L(C_o(Z_p))$, $P_{L(C_o(Z_p))}(x) = (p-1)x^{p-1} + ((p-1)/2)x^2$, where p is an odd prime.*

Proof. In $L(C_o(Z_p))$, the vertices $[0,1], [0,2], ..., [0, p-1]$ induces a complete subgraph K_{p-1}. Thus the vertices of the form $[0, y]$ are adjacent with the remaining $p-2$ vertices of the complete graph K_{p-1}. Moreover $[0, y]$ is also adjacent to $[x, y]$, where (x, y) is conjugate pair in $C_o(Z_p)$. Hence $deg([0, y]) = p-1$. While the vertices of the form $[x, y]$, where (x, y) is conjugate pair in $C_o(Z_p)$ is adjacent only with $[0, x]$ and $[0, y]$. Hence $deg([x, y]) = 2$.

As there are $(p-1)/2$ conjugate pairs present in $C_o(Z_p)$, so there are $(p-1)/2$ vertices of the form $[x, y]$ present in $L(C_o(Z_p))$. Thus $P_{L(C_o(Z_p))}(x) = (p-1)x^{p-1} + ((p-1)/2)x^2$, where p is an odd prime. □

Corollary 3.3. *The degree of the polynomial representation of $C_o(Z_p)$ and $L(C_o(Z_p))$, where p is an odd prime are equal.*

Proof. By Theorem 3.2 , $P_{C_o(Z_p)}(x) = x^{p-1} + (p-1)x^2$ and by Theorem 3.4 , $P_{L(C_o(Z_p))}(x) = (p-1)x^{p-1} + ((p-1)/2)x^2$, where p is an odd prime.

$\Rightarrow$ degree of both $P_{C_o(Z_p)}(x)$ and $P_{L(C_o(Z_p))}(x)$ is $p-1$.

Hence degree of $P_{C_o(Z_p)}(x)$ = degree of $P_{L(C_o(Z_p))}(x)$. $\square$

Theorem 3.5. *The polynomial representation of $C_o(Z_p)$ and $L(C_o(Z_p))$, where p is an odd-prime are equal if and only if $p = 3$.*

Proof. For $p = 3$, $C_o(Z_p)$ becomes a cycle of length 3. As we know the line graph of cycle of length n is also a cycle of length n. So $L(C_o(Z_p))$ is also a cycle of length 3. Thus , $P_{C_o(Z_p)}(x) = P_{L(C_o(Z_p))}(x)$, when $p = 3$.

Conversely, suppose $P_{C_o(Z_p)}(x) = P_{L(C_o(Z_p))}(x) \Rightarrow$ Number of vertices of any possible degree in $C_o(Z_p)$ = Number of vertices of that degree in $L(C_o(Z_p)) \Rightarrow$ Total number of vertices in $C_o(Z_p)$ = Total number of vertices in $L(C_o(Z_p)) \Rightarrow$ $p = p - 1 + (p-1)/2 \Rightarrow p = 3$. $\square$

Theorem 3.6. *Let $G = C_o(Z_n)$, $H = C_o(Z_n) \cup C_o(Z_n) \cup ...... \cup C_o(Z_n)$ (m times) and $I = C_o(Z_n) + C_o(Z_n) + ... + C_o(Z_n)$ (m times). Then*

(i) $P_H(x) = mP_G(x)$.

(ii) $P_I(x) = mx^{(m-1)n}P_G(x)$.

Proof. (i). In $C_o(Z_n) \cup C_o(Z_n)$, the number of vertices of each possible degree of $C_o(Z_n)$ becomes $2-$times. Thus for , $H = C_o(Z_n) \cup C_o(Z_n) \cup ...... \cup C_o(Z_n)$ (m times) the number of vertices of each possible degree of $C_o(Z_n)$ becomes m-times. Hence $P_H(x) = mP_G(x)$.

(ii). In $I_1 = C_o(Z_n) + C_o(Z_n)$, the number of vertices of each possible degree of $C_o(Z_n)$ becomes double such that the degree of each vertices is increased by n. Therefore $P_{I_1}(x) = 2x^n P_G(x)$.

In $I_2 = C_o(Z_n) + C_o(Z_n) + C_o(Z_n)$, the number of vertices of each possible degree of $C_o(Z_n)$ becomes triple such that the degree of each vertices is increased by $2n$. Therefore $P_{I_2}(x) = 3x^{2n}P_G(x)$.

In this way , for $I = C_o(Z_n) + C_o(Z_n) + ... + C_o(Z_n)$ (m times) , the number of vertices of each possible degree of $C_o(Z_n)$ becomes m times such that the degree of each vertices is increased by $(m-1)n$. Hence $P_I(x) = mx^{(m-1)n}P_G(x)$. $\square$

Theorem 3.7. *The polynomial representation of $C_o(Z_{2p})$, $P_{C_o(Z_{2p})}(x) = x^{2p-1} + x^p + (p-1)x^3 + (p-1)x^2$, where p is an odd prime.*

Proof. In $C_o(Z_{2p})$, by definition of order-congruence graph , $deg(0) = 2p - 1$. The vertex corresponding to p is adjacent with the vertices corresponding to $2, 4, \ldots, p-1$ and their conjugate pairs $2p-2, 2p-4, \ldots, p+1$. Moreover it is adjacent to 0 also. Hence $deg(p) = 2((p-1)/2) + 1 = p$.

It follows that the vertices corresponding to $2, 4, \ldots, p-1, 2p-2, 2p-4, \ldots, p+1$ are adjacent with $0, p$ and their respective conjugate only. Hence degree of all of them are 3. Thus the number of vertices having degree 3 equal to $p-1$.

Also the vertices corresponding to $1, 3, \ldots, p-2, 2p-1, 2p-3, \ldots, p+2$ are adjacent with their respective conjugate and 0 only. Hence degree of all of them are 2. Thus the number of vertices having degree 2 equal to $p-1$.

Therefore $P_{C_o(Z_{2p})}(x) = x^{2p-1} + x^p + (p-1)x^3 + (p-1)x^2$, where p is an odd prime. □

Theorem 3.8. *The polynomial representation of* $C_o(Z_{3p})$*,* $P_{C_o(Z_{3p})}(x) = x^{3p-1} + 2x^{p+1} + (p-1)x^4 + 2(p-1)x^2$*, where* $p \neq 3$ *is an odd prime.*

Proof. In $C_o(Z_{3p})$, by definition of order-congruence graph , $deg(0) = 3p - 1$.

The vertices corresponding to p and $2p$ are adjacent with $3, 6, \ldots, 3((p-1)/2)$ and their conjugate $3p-3, 3p-6, \ldots, 3p-3((p-1)/2) = 3((p+1)/2)$. Moreover the vertices corresponding to p and $2p$ are also adjacent with each other and 0. Hence $deg(p) = deg(2p) = 2((p-1)/2) + 2 = p + 1$.

It follows that the vertices corresponding to $3, 6, \ldots, 3((p-1)/2), 3p-3, 3p-6, \ldots, 3p-3((p-1)/2) = 3((p+1)/2)$ are adjacent with $0, p, 2p$ and their respective conjugate only. Hence the degree of these $p-1$ vertices is 4.

While the remaining $3p-(p-1+3) = 2(p-1)$ vertices [excluding $0, p, 2p$ and multiple of 3] are adjacent with 0 and their respective conjugate only. Hence the degree of these $2(p-1)$ vertices is 2.

Therefore $P_{C_o(Z_{3p})}(x) = x^{3p-1} + 2x^{p+1} + (p-1)x^4 + 2(p-1)x^2$, where $p \neq 3$ is an odd prime. □

Corollary 3.4. *The polynomial representation of* $C_o(Z_{2p})$ *and* $C_o(Z_{3p})$*, where* $p \neq 3$ *is an odd prime are both Quadrinomial.*

Proof. The result follows from Theorem 3.7 and Theorem 3.8 immediately. □

4. Conclusion

From the above discussions it has been observed that the polynomial representation of $C_o(Z_n)$ is monic for $n > 3$ and has no constant term with degree $n-1$. For $n = p, p^2, 2p, 3p$ the polynomial representation of $C_o(Z_n)$ is reducible over Z. From Theorem 3.2 and 3.4, it follows that the polynomial representation of $C_o(Z_p)$ and $L(C_o(Z_p))$ are both binomial provided $p \neq 3$ is an odd-prime. Also for an odd-prime p the degree of the polynomial representation of $C_o(Z_p)$ and its line graph are equal. Moreover $P_{C_o(Z_p)}(x) = P_{L(C_o(Z_p))}(x)$, if and only if $p = 3$.

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