ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.9, 6863–6869
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.9.45

WIENER POLYNOMIAL AND DEGREE DISTANCE POLYNOMIAL OF SOME GRAPHS

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ABSTRACT. The Wiener polynomial of a graph G is $W(G,x) = \sum_{i<j} x^{d(v_i,v_j)}$, where $d(v_i,v_j)$ is the shortest path between the vertices v_i and v_j. The Degree distance polynomial of a graph G is $DD(G,x) = \sum_{i<j} x^{[deg(v_i)+deg(v_j)]d(v_i,v_j)}$ where $d(v_i,v_j)$ denote the distance between the distinct pair of vertices and $d(v_i)$ is the degree of the vertex v_i.

1. INTRODUCTION

In mathematical chemistry, the researcher particular attention has been given to topological indices. A few properties of the graph of molecule are portrayed by a number. This number is known as the topological index. The study of topological indices is very interesting, because it has pure and applied mathematics. The topological features of the carbon atom are characterized into a number of topological indices like Degree Distance index, Wiener index have been proposed by chemists. Chemical graph is a graph whose vertices and edges denote atoms and bonds between those atoms of the underlying chemical structure. Theoretically the Wiener index introduced by H. Weiner, is the most important topological index. Wiener Index is defined as the sum of the shortest path between every pair of vertices of a graph. It is the first topological index. The Hyper Wiener index concept was given by Randic which is the generalization

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2010 *Mathematics Subject Classification.* 05C12, 05C05.

Key words and phrases. Wiener Index, Graph Degree, and Graph Distance.

of the Wiener index. Gutman, Dobrynin and Kochetova introduced the Degree Distance index as a weighted version of the Wiener index. There are three different methods to calculate Wiener Index of molecular graphs. On introducing Wiener polynomials, as a honor of H. Hosoya, the authors call this polynomial as Hosoya polynomial. Many papers have been devoted to compute the Wiener index. B. Sagan, Y-N. Yeh, P. Zhang, define a generating function, which is known as the Wiener polynomial, see [3]. For further reference see [1] and [2].

2. Preliminaries

Definition 2.1. *Let $G = (V, E)$ be a graph with the set of vertices V and set of edges E. It is denoted as V(G) and E(G). Let G be a simple connected graph with n vertices $v_1, v_2, \cdots, v_n$. Let $d(v_i, v_j)$ denote the distance between the vertices v_i and v_j. The Wiener index is defined as*

$$W(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d(v_i, v_j),$$

$$W(G) = \sum_{i<j} d(v_i, v_j), i, j = 1, 2, \cdots, n.$$

Definition 2.2. *The Wiener polynomial of a graph G is $W(G, x) = \sum\limits_{i<j} x^{d(v_i,v_j)}$, where, $d(v_i, v_j)$ is the shortest path between the vertices v_i and v_j.*

Definition 2.3. *The Wiener matrix of a graph G as a square matrix $WM(G)$ is defined as*

$$WM(G) = \begin{cases} d(v_i, v_j) & \textit{if } v_i \neq v_j \\ 0 & \textit{if } v_i = v_j \end{cases}$$

where, $d(v_i, v_j)$ is the shortest path between the vertices v_i and v_j.

Definition 2.4. *Let G be a simple connected graph with n vertices $v_1, v_2, \cdots, v_n$. Let $d(v_i, v_j)$ denote the distance between the vertices v_i and v_j, $deg(v_i)$ denote the degree of the vertex v_i.*

The Degree Distance index is defined as

$$DD(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}[(deg(v_i) + deg(v_j))d(v_i, v_j)],$$

$$DD(G) = \sum_{i<j}[(deg(v_i) + deg(v_j))d(v_i, v_j)], i, j = 1, 2, \cdots, n.$$

Definition 2.5. *The Degree Distance polynomial of a graph G is* $DD(G, x) = \sum\limits_{i<j} x^{[(deg(v_i)+deg(v_j))d(v_i,v_j)]}$*, where, $d(v_i, v_j)$ denote the distance between the distinct pair of vertices and $d(v_i)$ is the degree of the vertex v_i.*

Definition 2.6. *The Degree Distance matrix of a graph G as a square matrix $DDM(G)$ is defined as,*

$$DDM(G) = \begin{cases} [deg(v_i) + deg(v_j)]d(v_i, v_j) & \text{if } v_i \neq v_j \\ 0 & \text{if } v_i = v_j \end{cases}$$

where, $deg(v_i)$ is the degree of the vertex v_i and $d(v_i, v_j)$ is the shortest path between the vertices v_i and v_j.

Properties 2.1.

- *Degree of $W(G, x)$ is the diameter of the graph G.*
- *Wiener polynomial has no constant terms.*
- *Coefficient of x is the number of edges of the graph G.*
- $W(G, 1) = nC_2$*, where n denotes the vertices of the graph.*
- *Derivative of the Wiener polynomial is the Wiener index of the graph at $x = 1$.*

Properties 2.2.

- *Degree Distance polynomial has no constant terms.*
- *Derivative of the Degree Distance polynomial is the Degree Distance index of the graph at $x = 1$.*

3. WIENER POLYNOMIAL

Theorem 3.1. *The Wiener Polynomial of the Triangular book Graph TB_n is* $\frac{n(n-1)}{2}x^2 + (2n+1)\,x$.

Proof. The Triangular book graph has $(n+2)$ vertices and $(2n+1)$ edges.$(2n+1)$ pairs of vertices have distance one and is the total edges in triangular book graph. The diameter of the triangular book graph is two, it occurs $\left(\frac{n(n-1)}{2}\right)$ times. Therefore the Wiener Polynomial of the Triangular book Graph TB_n is $\left[\frac{n(n-1)}{2}\right]x^2 + (2n+1)\,x$. □

Theorem 3.2. *The Wiener Polynomial of the Triangular Snake Graph T_n is* $3nx + (4n-4)\,x^2 + \sum_{j=2}^{n-1} 4\,(n-j)\,x^{j+1}$.

Proof. The Triangular Snake graph has $(2n+1)$ vertices and $3n$ edges. The coefficient of x is the number of edges $3n$.The diameter of the triangular snake graph is n. The remaining $4\,(n-j)$ where $j = 2, 3, ..., n-1$ pairs of vertices contribute distance $(j+1)$. Therefore The Wiener Polynomial of the Triangular Snake Graph T_n is $3nx + (4n-4)\,x^2 + \sum_{j=2}^{n-1} 4\,(n-j)\,x^{j+1}$. □

Theorem 3.3. *The Wiener Polynomial of the Jewel Graph Jl_n is* $\left[\frac{n(n+1)}{2} + 1\right]x^2 + (2n+2)\,x.$

Proof. The Jewel graph has $(n+3)$ vertices and $(2n+2)$edges. $(2n+2)$ pairs of vertices have distance one and is the total edges in the Jewel graph. The diameter of the Jewel graph 2. It occurs $\left[\frac{n(n+1)}{2} + 1\right]$ times.The Wiener matrix of Jl_n is

$$WM(Jl_n) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{n+1} \\ v_{n+2} \\ v_{n+3} \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_3 & v_4 & \cdots & v_{n+1} & v_{n+2} & v_{n+3} \end{array} \\ \begin{pmatrix} 0 & 1 & 2 & 1 & \cdots & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 & \cdots & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & \cdots & 2 & 2 & 2 \\ 1 & 2 & 1 & 0 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 1 & 2 & 1 & \cdots & 0 & 2 & 2 \\ 2 & 1 & 2 & 1 & \cdots & 2 & 0 & 2 \\ 2 & 1 & 2 & 1 & \cdots & 2 & 2 & 0 \end{pmatrix} \end{array}$$

Hence The Wiener Polynomial of the Jewel Graph Jl_n is $\left[\frac{n(n+1)}{2} + 1\right]x^2 + (2n+2)\,x.$ □

Theorem 3.4. *The Wiener Polynomial of the Butterfly Graph Bf_n is* $\left[m^2 + n^2 + (n+3)\right]x^2 + \left[2\,(m+n)\right]x.$

Proof. The Butterfly graph has $(m+n+3)$ vertices and $(2m+n)$ edges. $(2m+n)$ pairs of vertices have distance one and is the total edges in the butterfly graph. The diameter of the Butterfly graph 2. It occurs $[m^2+n^2+(n+3)]$ times.The Wiener matrix of Bf_n is

$$WM(Bf_n) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{m+n+1} \\ v_{m+n+2} \\ v_{m+n+3} \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_3 & v_4 & \cdots & v_{m+n+1} & v_{m+n+2} & v_{m+n+3} \end{array} \\ \begin{pmatrix} 0 & 1 & 2 & 2 & \cdots & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & \cdots & 2 & 2 & 2 \\ 2 & 1 & 1 & 0 & \cdots & 2 & 2 & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 1 & 2 & 2 & \cdots & 0 & 2 & 2 \\ 2 & 1 & 2 & 2 & \cdots & 2 & 0 & 2 \\ 2 & 1 & 2 & 2 & \cdots & 2 & 2 & 0 \end{pmatrix} \end{array}$$

Hence The Wiener Polynomial of the Butterfly Graph Bf_n is
$[m^2+n^2+(n+3)]\,x^2+[2\,(m+n)]\,x$. □

Theorem 3.5. *The Wiener Polynomial of the Barbell Graph C_{K_n} is*
$(n-1)^2\,x^3+(2n-2)\,x^2+(n^2-n+1)$.

Proof. The Barbell graph has $(2n)$ vertices and (n^2-n+1)edges. (n^2-n+1) pairs of vertices have distance one and is the total edges in the Barbell graph. The diameter of the Barbell graph 3. It occurs $(n-1)^2$ times.The Wiener matrix of C_{K_n} is

$$WM(C_{K_n}) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{2n-2} \\ v_{2n-1} \\ v_{2n} \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_3 & v_4 & \cdots & v_{2n-2} & v_{2n-1} & v_{2n} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 3 & 3 & 3 \\ 1 & 0 & 1 & 1 & \cdots & 3 & 3 & 3 \\ 1 & 1 & 0 & 1 & \cdots & 3 & 3 & 3 \\ 1 & 1 & 1 & 0 & \cdots & 3 & 3 & 3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 3 & 3 & 3 & 3 & \cdots & 0 & 1 & 1 \\ 3 & 3 & 3 & 3 & \cdots & 1 & 0 & 1 \\ 3 & 3 & 3 & 3 & \cdots & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Hence The Wiener Polynomial of the Barbell Graph C_{K_n} is
$(n-1)^2 x^3 + (2n-2) x^2 + (n^2 - n + 1)$. □

4. Degree Distance Polynomial

Theorem 4.1. *The Degree Distance polynomial of the Cycle graph C_n is*

$$\sum_{j=0}^{n-2} 2nx^{4(1+j)} + nx^{4n}, \; if \; n \; is \; even.$$

$$\sum_{j=0}^{n-1} (2n+1)x^{4(1+j)}, \; if \; n \; is \; odd.$$

Proof. The Cycle graph has n vertices and n edges.

Case 1. n is even.

Suppose the Cycle graph C_n has even number of vertices. Then $2n$ pairs of vertices have degree of x as $4(1+j)$, where $j = 0, 1, \cdots, n-2$. The remaining n pairs of vertices have degree of x as $4n$.
Hence, the Degree Distance polynomial of the Cycle C_n is

$$\sum_{j=0}^{n-2} 2nx^{4(1+j)} + nx^{4n}.$$

Case 2. n is odd.

Consider the Cycle graph C_n having odd number of vertices. $2n+1$ pairs of vertices become the coefficient of x with degree of x as $4(1+j)$, where $j = 0, 1, \cdots, n-1$.
Hence, the Degree Distance polynomial of the Cycle C_n is

$$\sum_{j=0}^{n-1} (2n+1)x^{4(1+j)}.$$

□

Theorem 4.2. *The Degree Distance polynomial of the Fan graph f_n, $n > 2$ is*

$$2x^5 + (n-3)x^6 + x^8 + 2(n-3)x^{10} + \left(\frac{(n-3)(n-4)}{2}\right) x^{12} + 2x^{n+2} + (n-2)x^{n+3}.$$

Proof. The Fan graph f_n, $(n > 2)$ has $n+1$ vertices and $(2n-1)$ edges. Let $v_1, v_2, v_3, \cdots, v_n$ be n vertices of the Path and v_{n+1} be the common vertex. $deg(v_{n+1}) = n$ and $deg(v_i) = 2$,$1 \leq i \leq n$. Two pairs of vertices have degree of x as five and $(n+2)$. $(n-3)$, $2(n-3)$, $\left(\frac{(n-3)(n-4)}{2}\right)$ and $(n-2)$ pairs of vertices have the power of x as six, ten, twelve and $(n+3)$ respectively. Only one pair of vertices has degree of x as eight.Therefore,the Degree Distance polynomial of the Fan graph f_n is

$$2x^5 + (n-3)x^6 + x^8 + 2(n-3)x^{10} + \left(\frac{(n-3)(n-4)}{2}\right)x^{12} + 2x^{n+2} + (n-2)x^{n+3}.$$

□

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