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GORENSTEIN FI-FLAT MODULES AND RELATIVE SINGULARITY CATEGORIES

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ABSTRACT. In this article we proved that the quotient triangulated category is triangle-equivalent to the stable category of the Frobenius category of all Gorenstein FI-flat and FI-cotorsion left R-modules. This result is a generalization of the result of Zhen Xing Di derived in [18] for Gorenstein flat modules. Throughout this article unless otherwise specified, R is $\mathcal{GFIF}$-closed ring.

1. INTRODUCTION

Triangle equivalences, the triangulated category and quotient triangulated category are studied for Gorenstein flat modules by many authors in [2,4,7,11]. In this article we tried to connect them to Gorenstein FI-flat modules. Gorenstein FI-Flat modules were introduced in [14] by Selvaraj et all and $\mathcal{GFIF}$-closed rings were developed in [13].

Beligiannis studied the quotient triangulated category $D^b(Rmod)/K^b(R{-}proj)$ for an arbitrary ring R, where $D^b(Rmod)$ is the bounded derived category of R-modules and $K^b(R-proj)$ is the bounded homotopy category of projective modules. Just as the singularity category, this category reflects the homological singularity of the ring R, and it treats modules which are not necessarily finitely generated. We call such a quotient triangulated category big singularity category

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of the ring R. Zhenxing Di et al in [19] extended a triangle equivalence established by Beligiannis involving Gorenstein injective left R-modules from rings with finite left Gorenstein global dimension to arbitrary rings. Also, in [20] obtained the converse of Buchweitz's triangle equivalence and a result of Beligiannis, and gave characterizations for Iwanaga–Gorenstein rings and Gorenstein algebras.

2. FI-Cotorsion Dimension of Complexes

Recall that a left R-module M is said to be Gorenstein FI-flat [13], if there exists an exact sequence of FI-flat left R-modules,

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that $M \cong Im(F_0 \to F^0)$ and such that $B \otimes_R -$ leaves the sequence exact whenever B is a FI-injective right R-module. $\mathcal{GFI}_{\mathcal{F}}$ denotes class of all Gorenstein FI-flat R-modules.

For unexplained notions and more deep understanding of the terms involed in this article, we refer the readers to [3, 5, 6, 8, 9, 12, 14, 15, 17, 18].

Definition 2.1. *[16] Let R be a ring and let M be a left R-module. Then the FI-flat dimension of M, denoted by $FIf - dim(M)$, is defined as $inf n \geq 0 : Tor^R_{n+1}(N, M) = 0$ for all FP -injective right R-modules N. If no such n exists, set $FIf - dim(M) = \infty$.*

Definition 2.2. *Let X be a complex and n an integer. The Gorenstein FI-flat dimension $GFI_F d_R(X)$ of X is defined as follows.*

- *$GFI_F d_R(X) \leq n$ if there is a quasi-isomorphism $F \longrightarrow X$ with F FI-flat such that sup $F \leq n$ and $Cj(F)$ is Gorenstein FI-flat for any integer $j \leq n$.*
- *If $GFI_F d_R(X) \leq n$ but $GFI_F d_R(X) \leq n - 1$ does not hold, then $GFI_F d_R(X) = n$.*
- *If $GFI_F d_R(X) \leq m$ for any integer m, then $GFI_F d_R(X) = -\infty$.*
- *$GFI_F d_R(X) \leq m$ does not hold for any integer m, then $GFI_F d_R(X) = \infty$.*

Definition 2.3. *Let X be a complex. A complete FI-flat resolution of X is a diagram $T \overset{\sigma_1}{\to} F \overset{\sigma_2}{\to} C \overset{\sigma_3}{\leftarrow} X$ of morphisms of complexes satisfying:*

(1) *$F \overset{\sigma_2}{\to} C \overset{\sigma_3}{\leftarrow} X$ is a FI-flat-cotorsion resolution of X.*

(2) *T is an exact complex with each entry in $\mathcal{F}\cap\mathcal{C}$ and $Z_i(T) \in \mathcal{GFI}_{\mathcal{F}}$ for each $i \in Z$.*
(3) *$T \overset{\sigma_1}{\to} F$ is a morphism such that $\sigma_1 = id_T$*

Definition 2.4. *[1] A right R-module M is called an FI-cotorsion module if $Ext_1(F, M) = 0$ for any right FI-flat R-module F.*

Definition 2.5. *Let X be a complex. The FI-cotorsion dimension of X, denoted by $dg\overline{C}-id(X)$, is defined as $dg\overline{C}-id(X) = inf sup i|C - i = 0|X \approx CwithC \in dg\overline{C}$.*

If d$dg\overline{C} - id(X) \leq n$ for all $n \leq Z$, we write $dg\overline{C} - id(X) = -\infty$. If $dg\overline{C} - id(X) \leq n$ for no $n \leq Z$, we write $dg\overline{C}-id(X) = \infty$. It is clear that $dg\overline{C}-id(X) = -\infty$ if and only if X is an exact complex .

Lemma 2.1. *Let $0 \longrightarrow X \longrightarrow X' \longrightarrow X'' \longrightarrow 0$ be a short exact sequence of complexes. If any two complexes of X, X' and X'' have finite FI-cotorsion dimension, then so does the third .*

3. TRIANGLE EQUIVALENCES

Lemma 3.1. *The subcategory $\mathcal{F} \cap \mathcal{C}$ is an FI-injective cogenerator for $\mathcal{GFI}_{\mathcal{F}}$.*

The proof is straight away as we know that a Frobenius category has projectives and injectives, and the projectives coincide with the injectives. Thus by assuming $\mathcal{B}$ an additive full subcategory of an abelian category that is closed under extensions $\mathcal{I}$ by class of all projective-injective objects of $\mathcal{B}$ we conclude that $\mathcal{B}/\mathcal{I}$ is a triangulated category

Proposition 3.1. *The subcategory $\mathcal{GFI}_{\mathcal{F}} \cap \mathcal{C}$ forms a Frobenius category whose projective-injective objects are precisely all modules in $\mathcal{F} \cap \mathcal{C}$.*

Proof. It has been proved in [13], that $\mathcal{GFI}_{\mathcal{F}}$ is closed under extensions. Hence so is $\mathcal{GFI}_{\mathcal{F}} \cap \mathcal{C}$. Then $\mathcal{GFI}_{\mathcal{F}} \cap \mathcal{C}$ becomes an exact category whose conflations are just short exact sequences with all terms in $\mathcal{GFI}_{\mathcal{F}} \cap \mathcal{C}$ .First, We show that modules in $\mathcal{F} \cap \mathcal{C}$ are projective and FI-injective in $\mathcal{GFI}_{\mathcal{F}} \cap \mathcal{C}$. According to Lemma 3.1, we see that $(\mathcal{GFI}_{\mathcal{F}}\cap\mathcal{C})\bot(\mathcal{F}\cap\mathcal{C})$. This implies that modules in $\mathcal{F}\cap\mathcal{C}$ are FI-injective. On the other hand, it is trivial that $(\mathcal{GFI}_{\mathcal{F}}\cap\mathcal{C})\bot(\mathcal{F}\cap\mathcal{C})$. Hence modules in $\mathcal{F} \cap \mathcal{C}$ are projective.

Let M be any module in $\mathcal{GFI_F} \cap \mathcal{C}$. By Lemma 3.1 again, there exists an exact sequence $0 \longrightarrow M \longrightarrow K \longrightarrow M'$ in $R - Mod$ with $K \in \mathcal{F} \cap \mathcal{C}$ and $M' \in \mathcal{GFI_F} \cap \mathcal{C}$. This shows that the exact category $\mathcal{GFI_F} \cap \mathcal{C}$ has enough FI-injective objects. On the other hand, since $M \in C$, it follows that there exists an exact sequence $0 \longrightarrow M'' \longrightarrow K' \longrightarrow M$ in $R - Mod$ with $K' \in \mathcal{F} \cap \mathcal{C}$ and $M'' \in C$. Note that M belongs to $\mathcal{GFI_F}$ as well. We conclude that M'' is also in $\mathcal{GFI_F}$ because $\mathcal{GFI_F}$ is closed under kernels of epimorphisms. Therefore, $M'' \in GFI_F \cap C$. This implies that the exact category $\mathcal{GFI_F} \cap \mathcal{C}$ has enough projective objects. From the argument above, it is direct to conclude that in the exact category$GFI_F \cap C$ the class of projective objects coincides with the class of FI-injective objects, and projective-FI-injective objects are just modules in $\mathcal{F} \cap \mathcal{C}$ □

Lemma 3.2. *Let M be a module in $\mathcal{GFI_F} \cap \mathcal{C}$.*

(1) *If F is a complex in $K^b(\mathcal{F} \cap \mathcal{C})$ such that $F_i = 0$ for $i \leq 0$, then $Hom_D(R - Mod)(M, F) = 0$.*
(2) *If F is a complex in $K^b(\mathcal{F} \cap \mathcal{C})$ such that $F_i = 0$ for $i \leq 0$, then $Hom_D(R - Mod)(F, M) = 0$.*

Proposition 3.2. *All homology bounded complexes with both finite Gorenstein FI-flat dimension and cotorsion dimension form a triangulated full subcategory of $D^b(R - Mod)$ and is denoted by $D^b(R - Mod)$*

Proof. Let M and M' be two homology bounded complexes such that $M \cong M'$ in $D^b(R - Mod)$. Assume that M has both finite Gorenstein FI-flat dimension and FI-cotorsion dimension. Then, we see that M' has the same properties as M. This implies that $D^b(R - Mod)_{\widehat{\mathcal{GFI_F}, \mathcal{C}}}$ is closed under isomorphisms in $D^b(R - Mod)$. Moreover, it is clear that $D^b(R - Mod)_{(\widehat{\mathcal{GFI_F}, \mathcal{C}})}$ is closed under shifts. Hence it remains to show that $D^b(R - Mod)_{\widehat{\mathcal{GFI_F}, \mathcal{C}}}$ is closed under cones. To this end, let $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ be a distinguished triangle in $D^b(R - Mod)$. We may assume that it is induced by a short exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $\mathcal{C}(R - Mod)$. Now the assertion follows by ([10] , Proposition 4) and Lemma 3.2. Let T be a triangulated category and K a triangulated subcategory of T closed under summands, that is, a thick subcategory. Then one can form the triangulated quotient T/K, It is also a triangulated category. According to Proposition 3.2, we know that $D^b(R - Mod)_{\widehat{\mathcal{GFIF}, \mathcal{C}}}$ is a triangulated category. Moreover, it is clear that $K^b(\mathcal{F} \cap \mathcal{C})$ is a triangulated subcategory of $D^b(R-Mod)_{(\widehat{\mathcal{GFI}_{\mathcal{F}},\mathcal{C}})}$, which is closed under direct summands. Thus the triangulated quotient $D^b(R-Mod)_{(\widehat{\mathcal{GFI}_{\mathcal{F}},\mathcal{C}})}/K^b(\mathcal{F} \cap \mathcal{C})$ is also a triangulated category. Notice that any module in $\mathcal{GFI}_{\mathcal{F}} \cap \mathcal{C}$ as a complex has both finite Gorenstein FI-flat dimension and FI-cotorsion dimension, so there exists an embedding:

$$\mathcal{GFI}_{\mathcal{F}} \cap \mathcal{C} \hookrightarrow D^b(R-Mod)_{\widehat{\mathcal{GFI}_{\mathcal{F}},\mathcal{C}}}.$$

Let F be the composition:

$$\mathcal{GFI}_{\mathcal{F}} \cap \mathcal{C} \hookrightarrow D^b(R-Mod)_{\widehat{\mathcal{GFI}_{\mathcal{F}},\mathcal{C}}} \longrightarrow D^b(R-Mod)_{\widehat{\mathcal{GFI}_{\mathcal{F}},\mathcal{C}}}/K^b(\mathcal{F} \cap \mathcal{C}),$$

where the latter one is the natural quotient functor. It is clear that F sends modules in $\mathcal{F} \cap \mathcal{C}$ to 0 in $D^b(R-Mod)_{\widehat{\mathcal{GFI}_{\mathcal{F}},\mathcal{C}}}/K^b(\mathcal{F} \cap \mathcal{C})$, so it factors through the stable category $\underline{\mathcal{GFIF} \cap \mathcal{C}}$, see Proposition 3.1.

Consequently, there exists a functor

$$\overline{F} : \underline{\mathcal{GFIF} \cap \mathcal{C}} \longrightarrow D^b(R-Mod)_{\widehat{\mathcal{GFIF},\mathcal{C}}}/K^b(\mathcal{F} \cap \mathcal{C})\,,$$

such that $F = \overline{F}\pi$, where $\pi : \mathcal{GFI}_{\mathcal{F}} \cap \mathcal{C} \longrightarrow \underline{\mathcal{GFI}_{\mathcal{F}} \cap \mathcal{C}}$ is the natural quotient functor. □

Theorem 3.1. *The functor* $\overline{F} : \underline{\mathcal{GFI}_{\mathcal{F}} \cap \mathcal{C}} \longrightarrow D^b(R-Mod)_{(\widehat{\mathcal{GFI}_{\mathcal{F}},\mathcal{C}})}/K^b(\mathcal{F} \cap \mathcal{C})$ *is a triangle equivalence.*

Proof. We show that $\overline{F}$ is a triangle functor, and it is essentially surjective (or dense), full and faithful.

(1) $\overline{F}$ is a triangle functor.

Let $A \overset{u}{\to} B \longrightarrow C \longrightarrow T(A)$ be a distinguished triangle in $\underline{\mathcal{GFI}_{\mathcal{F}} \cap \mathcal{C}}$. Then it comes from a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & I(A) & \longrightarrow & T(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & T(A) & \longrightarrow & 0
\end{array}$$

in $\mathcal{GFIF} \cap \mathcal{C}$ with exact rows. This gives us a commutative diagram as follows

$$\begin{array}{ccccccc} A & \longrightarrow & I(A) & \longrightarrow & T(A) & \longrightarrow & X[1] \\ \downarrow & & \downarrow & & \| & & \downarrow \\ B & \longrightarrow & C & \longrightarrow & T(A) & \longrightarrow & Y[1] \end{array}$$

in $D^b(R-Mod)_{(\widehat{\mathcal{GFI_F}},\mathcal{C})}$. By sending this to a commutative diagram $D^b(R-Mod)_{(\widehat{\mathcal{GFIF}},\mathcal{C})}/K^b(\mathcal{F}\cap\mathcal{C})$, we get $T(A)\cong X[1]$. Thus $A\overset{u}{\to} B\longrightarrow C\longrightarrow X[1]$ is a distinguished triangle in $D^b(R-Mod)_{(\widehat{\mathcal{GFI_F}},\mathcal{C})}/K^b(\mathcal{F}\cap\mathcal{C})$ and it follows that $\overline{F}$ is a triangle functor.

(2) $\overline{F}$ is essentially surjective .

Let M be any complex in $D^b(R-Mod)_{(\widehat{\mathcal{GFI_F}},\mathcal{C})}/K^b(\mathcal{F}\cap\mathcal{C})$. Assume that $GFI_Fd_R(M)=n$ and $dg\overline{C}-id(M)=t$ for some integers n and t. Then, there exists a complex $C\in dg\overline{C}\cap C(R-mod)$ satisfying $C\cong M$ in $D^b(R-Mod)$ and C admits a complete flat resolution $T\overset{u}{\to} F\longrightarrow C\overset{id_C}{\leftarrow} C$ such that $Zi(T)\in\mathcal{GFI_F}\cap\mathcal{C}$ for each $i\in Z$, $u_i=id_{Fi}$ for all $i\geq n$ and $F\in C_\circ(R-Mod)$. Note that F is of the form

$$\cdots\longrightarrow F_{n+1}\longrightarrow F_n\longrightarrow\cdots F_{-t+1}\longrightarrow F_{-t}\longrightarrow 0\longrightarrow\cdots.$$

We have a distinguished triangle $F_{\sqsubseteq n-1}\longrightarrow F\longrightarrow F_{\sqsupseteq n}\longrightarrow F_{\sqsubseteq n-1}[1]$ in $K(R-Mod)$. Send it now to a distinguished triangle in $D^b(R-Mod)_{(\widehat{\mathcal{GFI_F}},\mathcal{C})}/K^b(\mathcal{F}\cap\mathcal{C})$.

Therefore, $F\cong F_{\sqsupseteq n}$ in $D^b(R-Mod)_{(\widehat{\mathcal{GFI_F}},\mathcal{C})}/K^b(\mathcal{F}\cap\mathcal{C})$. Moreover, it is easy to see that $F_{\sqsupseteq n}\cong C_n(F)=Z_{n-1}(T)$ in $D^b(R-Mod)$. This implies $F\cong Z_{n-1}(T)$ in $D^b(R-Mod)_{(\widehat{\mathcal{GFI_F}},\mathcal{C})}/K^b(\mathcal{F}\cap\mathcal{C})$. Note that $Z_{n-1}(T)$ belongs to $\mathcal{GFI_F}\cap\mathcal{C}$. It follows that $\overline{F}$ is essentially surjective.

(3) $\overline{F}$ is full.

Since we have $F=\overline{F}\pi$, it suffices to show that F is full. Let

$$X\overset{f}{\leftarrow} Z\overset{g}{\to} Y$$

be a morphism in $D^b(R-Mod)_{(\widehat{\mathcal{GFI_F}},\mathcal{C})}/K^b(\mathcal{F}\cap\mathcal{C})$ with $X,Y\in\mathcal{GFI_F}\cap\mathcal{C}$ and f lies in the compatible saturated multiplicative system corresponding to $K^b(\mathcal{F}\cap\mathcal{C})$. Complete f to a distinguished triangle

$$X[-1]\overset{w}{\to} Q\longrightarrow Z\overset{f}{\to} X$$

with $Q \in K^b(\mathcal{F} \cap \mathcal{C})$. Consider the distinguished triangle

$$Q_{\sqsubseteq -1} \longrightarrow Q \longrightarrow Q_{\sqsupseteq 0} \longrightarrow Q_{\sqsubseteq -1}[1] \text{ in } K^b(\mathcal{F} \cap \mathcal{C})\,.$$

Consider now the following commutative of distinguished triangles

$$\begin{array}{ccccccc} X[-1] & \longrightarrow & Q_{\sqsubseteq -1} & \longrightarrow & Z' & \longrightarrow & X \\ \| & & \downarrow & & \downarrow & & \| \\ X[-1] & \longrightarrow & Q & \longrightarrow & Z & \longrightarrow & X \end{array}$$

where $s, l, f$are all in the compatible saturated multiplicative system corresponding to $K^b(\mathcal{F} \cap \mathcal{C})$. Since by Lemma 3.2, there exists some $k : X \longrightarrow Y$ such that $gl = ks = kfl$. So we have $k = gf^{-1}$. Thus, F is proved to be full.

(4) F is faithful.

Suppose that there exists a morphism $f : X \longrightarrow Y$ in $\mathcal{GFI}_{\mathcal{F}} \cap \mathcal{C}$ such that $F(f) = 0$. We want to show $f = 0$. To this end, complete f to a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow X[1]$$

in $\underline{\mathcal{GFI}_{\mathcal{F}} \cap \mathcal{C}}$. Since $\overline{F}(f) = 0$, $\overline{F}(g)$ is a section. According to (3), we know that F is full. So there exists some morphism $\alpha : Z \longrightarrow Y$ such that $1_{\overline{F}(Y)} = \overline{F}(\alpha g)$. Let $\beta = \alpha g$ and complete β to a distinguished triangle

$$Y \xrightarrow{\beta} Y \longrightarrow C(\beta) \longrightarrow Y[1]$$

in $\underline{\mathcal{GFI}_{\mathcal{F}} \cap \mathcal{C}}$. We have $F(C(\beta)) \in K^b(\mathcal{F} \cap \mathcal{C})$. We know that any Gorenstein FI-flat module with finite flat dimension is FI-flat. It follows that $C(\beta) \in \mathcal{F} \cap \mathcal{C}$. Hence β is an isomorphism in $\underline{\mathcal{GFI}_{\mathcal{F}} \cap \mathcal{C}}$. This implies that g is a section, and hence $f = 0$. This completes the proof.

□

Lemma 3.3. *Let R be a Gorenstein ring. Then we have* $D^b(R - Mod)_{(\mathcal{GFI}_{\mathcal{F}} \cap \mathcal{C})} = D^b(R - Mod)$.

Lemma 3.4. *Let R be a* $\mathcal{GFI}_{\mathcal{F}}$*-closed ring. Then there exists a triangle equivalence* $\underline{\mathcal{GFI}_{\mathcal{F}} \cap \mathcal{C}} = D^b(R - Mod)/K^b(\mathcal{F} \cap \mathcal{C})$.

Let $\mathcal{X}$ be a subcategory of R-Mod. Then an exact complex of modules in $\mathcal{X}$ is called totally $\mathcal{X}$-acyclic if it is $Hom_R(\mathcal{X}, -)$-exact and $Hom_R(-, \mathcal{X})$-exact. Denote by $G(\mathcal{X})$ the subcategory of R-Mod whose modules are of the form $M \cong Z_{-1}(\mathcal{X})$ for some totally $\mathcal{X}$-acyclic complex $\mathcal{X}$.

Lemma 3.5. *Let R be a Gorenstein ring. Then we have $G(\mathcal{F} \cap \mathcal{C}) = \mathcal{GFI}_{\mathcal{F}} \cap \mathcal{C}$.*

Lemma 3.6. *Let R be a FI-Gorenstein ring. Then there exists a triangle equivalence $\underline{G(\mathcal{F} \cap \mathcal{C})} = D^b(R - Mod)/K^b(\mathcal{F} \cap \mathcal{C})$.*

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