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INVERSION FORMULA FOR THE EIGENFUNCTION WAVELET TRANSFORM

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ABSTRACT. In this paper, we accomplished the concept of convolution of Eigenfunction transform for the study of Continuous Eigenfunction wavelet transform and discuss some of its basic properties. Finally our main goal is to find out the Plancherel and Inversion formula for the Continuous Eigenfunction Wavelet Transform (CEWT).

1. INTRODUCTION

The wavelet transform for a function$f \in L^2(\mathbf{R})$ with respect to the wavelet $\phi \in L^2(\mathbf{R})$ is defined by

$$(W_\phi f)(\sigma_2, \sigma_1) = \int_{-\infty}^{+\infty} f(t)\, \overline{\phi_{\sigma_2,\sigma_1}(t)} dt,\ \sigma_2 \in \mathbf{R},\ \sigma_1 > 0, \tag{1.1}$$

where

$$\phi_{\sigma_2,\sigma_1}(t) = \sigma_1^{-1/2} \phi\left(\frac{t-\sigma_2}{\sigma_1}\right). \tag{1.2}$$

Translation operator τ_{σ_2} is defined by

$$\tau_{\sigma_2}\phi(t) = \phi(t-\sigma_2),\ \sigma_2 \in \mathbf{R}$$

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and dilation D_{σ_1} is defined by

$$D_{\sigma_1}\phi\left(t\right)=\sigma_1^{1/2}\phi\left(\frac{t}{\sigma_1}\right),\ \sigma_1>0.$$

We can write

$$\phi_{\sigma_2,\sigma_1}\left(t\right)=\tau_{\sigma_2}D_{\sigma_1}\phi\left(t\right). \tag{1.3}$$

From (1.1), (1.2) and (1.3), it is obvious that wavelet transform of the function f on $\mathbf{R}$ is an integral transform for which the kernel is the dilated translate of ϕ.

We can also express (1.1) as the convolution:

$$\left(W_\phi f\right)\left(\sigma_2,\sigma_1\right)=\left(f*\phi\left(\sigma_1\right)\right)\left(\sigma_2\right),$$

where

$$f\left(t\right)=\overline{\phi\left(-t\right)}.$$

As for every integral transform there exists a particular type of convolution, one can define wavelet transform with respect to a integral transform using associated convolution. Integral transform including special functions as kernel play a significant role in the theory of partial differential equations. Pathak and Pandey [4] have defined Lagurree wavelet using Lagurree functions on semi- infinite interval $(0,\infty)$. Pandey and Phukan [5] studied Hermite wavelet transform and derived the many properties related to Hermite wavelet transform. Motivating from above ideas we are interested to define the wavelet transform corresponding to Eigenfunction transform and to establish inversion and Plancherel formula for Eigenfunction wavelet transform.

2. Preliminaries

Eigenfunction transform is a unification of many transforms involving infinite series representations. We first recall its definition from [7]. Let I denote any open interval $\alpha<x<\beta$ on the real line. Here $\alpha=-\infty$ and $\beta=\infty$ are permitted. Let $\Re$ denote the linear differential operator of the form:

$$\Re=\theta_0D^{n_1}\theta_1D^{n_2}\theta_2\ldots D^{n_r}\theta_r\,,$$

where $D=d/dx$, the n_r are positive integers, and ϕ_r are smooth functions on I that are never equal to zero anywhere on I . Furthermore, θ_r and n_r are such that $\Re=\overline{\theta_r}\left(-D\right)^{n_r}\ldots\left(-D\right)^{n_2}\overline{\theta_1}\left(-D\right)^{n_1}\overline{\theta_0}$, where $\overline{\theta_r}$ denotes complex conjugate of θ_r . Moreover, we assume that$\{\psi_n\}_{n=0}^{\infty}$ is a complete orthonormal sequence of eigenfunctions of $\Re$ in $L^2(I)$ and corresponding eigenvalues are $\{\lambda_n\}_{n=0}^{\infty}$such that $|\lambda_n| \to \infty$ as $n \to \infty$. Also, $\psi_n(x)$ is a smooth function in $L^2(I)$ and the zero function is not allowed as an eigenfunction.

The Eigenfunction transform [7] of $f \in L^2(I)$ is defined by

$$\wp[f](n) = \hat{f}(n) := (f, \psi_n) = \int_{\alpha}^{\beta} f(x)\overline{\psi_n(x)}dx,\ \psi_n \in L^2(I). \tag{2.1}$$

The inverse of (2.1) is given by

$$\wp^{-1}[f](x) = f(x) = \sum_{n=0}^{\infty} \hat{f}(n)\psi_n(x).$$

For various properties of the eigenfunction transform we may refer to [1,6,7]. For any $f \in L^2(I)$ the below Parseval identity holds for Eigenfunction transform

$$\sum_n \hat{f}_1(n)\hat{f}_2(n) = \int_{\alpha}^{\beta} f_1(x) f_2(x)\,dx$$

and

$$\sum_n \hat{f}_1(n)\hat{f}_2(n) = \int_{\alpha}^{\beta} \wp^{-1}\left[\hat{f}_1(n)\right]\wp^{-1}\left[\hat{f}_2(n)\right]dx. \tag{2.2}$$

The translation Operator τ_y for $y \in [\alpha, \beta]$ associated with Eigenfunction transform is given by

$$(\tau_x f)(y) := f(x, y) = \int_{\alpha}^{\beta} f(z)\,u(x, y, z)\,dz,$$

where

$$u(x, y, z) = \sum_{n=0}^{\infty} \overline{\psi_n(x)\psi_n(y)}\psi_n(z).$$

Also,

$$\int_{\alpha}^{\beta} u(x, y, z)\,dz = 1.$$

Lemma 2.1. *Let $f \in L^2(I)$, then*

$$\|\tau_y f\|_2 \le C\,\|f\|_2,\ C > 0$$

and the map $f \to \tau_y f$*is linear and continuous in* $f \in L^2(I)$.
The convolution of $f_1, f_2 \in L^2(I)$ *is defined by*

$$\begin{aligned}(f_1 * f_2)(x) &= \int_\alpha^\beta (\tau_x f_1)(y) f_2(y)\, dy \\ &= \int_\alpha^\beta \int_\alpha^\beta f_1(z) f_2(y) u(x,y,z)\, dydz. \quad (2.3)\end{aligned}$$

Lemma 2.2. *Let* $f_1 \in L^2(I)$ *and* $f_2 \in L^2(I)$, *then the convolution* $f_1 * f_2$ *defined by Equation (2.3) satisfies the following:*

(i) $\|f_1 * f_2\|_\infty \le C \|f_1\|_2 \|f_2\|_2$.
(ii) $(f_1 * f_2)^\wedge(n) = \hat{f}_1(n) \hat{f}_2(n)$.

Convolutions involving certain orthogonal polynomials have been investigated by Gasper [2], Glaeske [3] and Gorlich and Market [4].

3. Continuous Eigenfunction Wavelet Transformation (CEWT)

Let $\phi \in L^2(I)$ be given, for $\alpha \le \sigma_2 \le \beta$ and $0 \le \sigma_1 \le \infty$ define the Legendre wavelet

$$\phi_{\sigma_2,\sigma_1}(t) = \tau_{\sigma_2} D_{\sigma_1} \phi(t) = \tau_{\sigma_2} \phi(\sigma_1 t),$$

where τ_{σ_2} translation operator associated with Legendre transform and D_{σ_1} is dilation operator.

Definition 3.1. ***Admissible Eigenfunction Wavelet***
The function $\phi \in L^2(I)$ *is called admissible Eigenfunction wavelet if* ϕ *satisfies the below admissibility condition:*

$$c_\phi = \sum_n \frac{\left|\hat{\phi}(n)\right|^2}{n} < \infty \quad (3.1)$$

Definition 3.2. ***Continuous Eigenfunction Wavelet Transform (CEWT)***
For $\phi_{\sigma_2,\sigma_1} \in L^2(I)$, $\alpha \le \sigma_2 \le \beta$, $0 \le \sigma_1 \le \infty$, *we describe the continuous*

Eigenfunction wavelet transform with help of Eigenfunction wavelet ϕ_{σ_2,σ_1} *is*

$$\begin{aligned}
(E_\phi f)(\sigma_2,\sigma_1) &= \langle f(t), \phi_{\sigma_2,\sigma_1}(t)\rangle \\
&= \int_\alpha^\beta f(t)\,\overline{\phi_{\sigma_2,\sigma_1}(t)}dt \\
&= \int_\alpha^\beta f(t)\,\overline{\tau_{\sigma_2} D_{\sigma_1}\phi(t)}dt \\
&= \int_\alpha^\beta f(t)\,\overline{\tau_{\sigma_2}\phi(\sigma_1 t)}dt \\
&= \int_\alpha^\beta \int_\alpha^\beta f(t)\,\overline{\phi(\sigma_1 z)}K(\sigma_2,t,z)\,dzdt.
\end{aligned}$$

The Eigenfunction wavelet transform can be expressed in the form of Eigenfunction transform as follows

$$E\left[(E_\phi f)(\sigma_2,\sigma_1)\right] = \hat{f}(n)\,\hat{\phi}(\sigma_1,n).$$

Also, the Eigenfunction wavelet transform can be written as

$$(E_\phi f)(\sigma_2,\sigma_1) = (f * \phi(\sigma_1,.))(\sigma_2).$$

4. Plancherel and Parsevals Formulae for Continuous Eigenfunction Wavelet Transformation (CEWT)

This section describes significant properties of CEWT, such as the Plancherel and Inversion formulae.

Theorem 4.1. ***Plancherel Formula***
Let $f_1, f_2 \in L^2(I)$*, then we have*

$$\langle (E_\phi f_1)(\sigma_1,\sigma_2), (E_\phi f_2)(\sigma_1,\sigma_2)\rangle_{L^2(I)\times L^2(I)} = c_{\phi_1,\phi_2}\langle f_1, f_2\rangle_{L^2(I)},$$

where

$$c_{\phi_1,\phi_2} = \int_0^\infty \hat{\phi_1}(\sigma_1,n)\,\hat{\phi_2}(\sigma_1,n)\,da.$$

Proof. Let $f_1, f_2 \in L^2(I)$, then we have

$$\int_0^\infty \int_\alpha^\beta (E_\phi f_1)(\sigma_1,\sigma_2)\,\overline{(E_\phi f_2)(\sigma_1,\sigma_2)}d\sigma_1 d\sigma_2$$

$$= \int_0^\infty \int_\alpha^\beta \wp^{-1}\left[\hat{f}_1(n)\,\hat{\phi}_1(\sigma_1,m)\right](\sigma_2)\,\overline{\wp^{-1}\left[\hat{f}_2(m)\,\hat{\phi}_2(\sigma_1,m)\right](\sigma_2)}d\sigma_1 d\sigma_2.$$

Now, by using (2.2) we get

$$\begin{aligned}&\int_0^\infty \int_\alpha^\beta f(x)\,\phi_1(\sigma_1,x)(\sigma_2)\,\overline{g(x)\,\phi_2(\sigma_1,x)(\sigma_2)}d\sigma_1 d\sigma_2\\ =\;& \sum_n \hat{f}_1(n)\,\overline{\hat{f}_2(n)} \int_0^\infty \hat{\phi}_1(\sigma_1,n)\,\hat{\phi}_2(\sigma_1,n)\,d\sigma_1\\ =\;& c_{\phi_1,\phi_2}\sum_n \hat{f}_1(n)\,\hat{f}_2(n).\end{aligned}$$

Hence, by using the Parseval formula for Eigenfunction transform, we get

$$\begin{aligned}&\int_0^\infty \int_\alpha^\beta (E_\phi f_1)(\sigma_1,\sigma_2)\,\overline{(E_\phi f_2)(\sigma_1,\sigma_2)}d\sigma_1 d\sigma_2\\ =\;& c_{\phi_1,\phi_2}\int_\alpha^\beta f_1(x)\,f_2(x)\,dx.\\ =\;& c_{\phi_1,\phi_2}\,\langle f_1,f_2\rangle_{L^2(I)}.\end{aligned}$$

□

Theorem 4.2. ***Inversion Formula***
Let $f \in L^2(I)$ and ϕ is an Eigenfunction wavelet, which defines CEWT, then

$$f(x) = \frac{1}{c_\phi}\int_0^\infty \int_\alpha^\beta (E_\phi f_1)(\sigma_1,\sigma_2)\,(E_\phi h)(\sigma_1,\sigma_2)\,d\sigma_1 d\sigma_2,$$

where c_ϕ is given in (3.1).

Proof. Let $h(x) \in L^2(I)$ be any function, then by applying previous theorem, we have

$$\begin{aligned}c_\phi\,\langle f,h\rangle_{L^2(I)} &= \int_0^\infty \int_\alpha^\beta (E_\phi f_1)(\sigma_1,\sigma_2)\,\overline{(E_\phi h)(\sigma_1,\sigma_2)}d\sigma_1 d\sigma_2\\ &= \int_0^\infty \int_\alpha^\beta (E_\phi f_1)(\sigma_1,\sigma_2)\int_\alpha^\beta \overline{h(t)\,\overline{\phi_{\sigma_2,\sigma_1}(t)}}dt\,d\sigma_1 d\sigma_2\\ &= \int_0^\infty \int_\alpha^\beta \int_\alpha^\beta (E_\phi f_1)(\sigma_1,\sigma_2)\,\phi_{\sigma_2,\sigma_1}(t)\,\overline{h(t)}d\sigma_1 d\sigma_2 dt\\ &= \int_\alpha^\beta g(t)\,\overline{h(t)}dt\\ &= \langle g,h\rangle,\end{aligned}$$

where

$$g(t) = \int_0^\infty \int_\alpha^\beta (E_\phi f_1)(\sigma_1, \sigma_2)\, \phi_{\sigma_2,\sigma_1}(t)\, d\sigma_1 d\sigma_2.$$

Thus, $c_\phi \langle f, h \rangle_{L^2(I)} = \langle g, h \rangle$ and

$$f = \frac{1}{c_\phi} g = \frac{1}{c_\phi} \int_0^\infty \int_\alpha^\beta (E_\phi f_1)(\sigma_1, \sigma_2)\, \phi_{\sigma_2,\sigma_1}(t)\, d\sigma_1 d\sigma_2.$$

If $f = h$, then

$$\|f\|^2_{L^2(I)} = \int_0^\infty \int_\alpha^\beta |(E_\phi f_1)(\sigma_1, \sigma_2)|^2\, d\sigma_1 d\sigma_2.$$

Moreover, the Eigenfunction wavelet transform is isometry from $L^2(I)$ to $L^2(I) \times L^2(I)$. $\square$

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