

ON CERTAIN FRACTIONAL KINETIC EQUATIONS INVOLVING LAGUERRE POLYNOMIALS

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ABSTRACT. The purpose of the following paper is to calculate the solution of the fractional kinetic equation pertaining to Laguerre Polynomials. We obtained their solutions in expressions of the Mittag-Leffler function and interpreted their pictorial representation to discuss the nature.

1. INTRODUCTION

In recent years, we have been used distinct patterns of fractional kinetic equations in describing and solving essential questions of science. The time-dependent quantity $N(t)$ is an absolute response, then we can work out the rate $\frac{dN}{dt}$ by the following expression

$$(1.1) \quad \frac{dN}{dt} = -D + P,$$

where D denotes the destruction rate and P denotes the production rate of N . Normally, D and P depend on $N(t)$ themselves: $D = D(N)$ and $P = P(N)$. But this dependency is complex as the destruction or production at time t not only depends on $N(t)$ but also by the past research, i.e., $N(\omega), \omega < t$, of variable N . This can be explained through the mathematical expression:

$$(1.2) \quad \frac{dN}{dt} = -D(N_t) + P(N_t),$$

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where $N(t)$ is defined by $N_t(t^*) = N_t(t - t^*)$, $t^* > 0$.

Haubold and Mathai [3] further study of (1.2) and invented the following equation

$$(1.3) \quad \frac{dN_i}{dt} = -c_i N_i(t)$$

wherein $N_i = N_0$ is the initial condition and number density of species i at time $t = 0$. Equation (1.3) is known as the standard kinetic equation where Constant $C_i > 0$ and the solution of (1.3) is obtained as

$$(1.4) \quad N_i(t) = N_0 e^{-C_i t}.$$

Thereafter, Saxena and Kalla [8] invented the succeeding fractional kinetic equation:

$$(1.5) \quad N(t) - N_0 f(t) = -c_0 {}^{\nu} D_t^{-\nu} N(t), \quad \Re(\nu) > 0,$$

where $N(t)$ represent the number density of species at the time t , N_0 represent the number of densities at the time $t = 0$, c is the constant, $f(t) \in L(0, \infty)$ and ${}_0 D_t^{-\nu}$ is the Reimann-Liouville fractional operator [5], defined as

$$(1.6) \quad {}_0 D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-x)^{\nu-1} f(x) dx, \quad \Re(\nu) > 0.$$

Fractional Laguerre Polynomials.

Laguerre polynomials are the solution to the Laguerre differential equation. These polynomials are used in some physical problems, such as the explanation of the transversal profile of Laguerre-Gaussian laser beams [10]. Schrödinger's wave mechanics and Schrödinger wave equation for the hydrogen atom are the applications of Laguerre polynomials [2]. Similarly the fractional form of Laguerre polynomial, named as Fractional Laguerre polynomial [6] is also very important and useful in the propagation of electromagnetic waves along transmission lines [4].

Definition 1.1. For $n \in \mathbb{N}$ and $(n-1) < \nu < n$, $0 \leq t < \infty$ and $a > -1$, the Fractional Laguerre Polynomials are specified by the following expression

$$(1.7) \quad L_n^{\alpha}(t) = \sum_{k=0}^n \frac{(-1)^k \Gamma(n+\alpha+1) t^k}{\Gamma(k+\alpha+1) \Gamma(k+1) \Gamma(n-k+1)} \quad ; 0 \leq t < \infty.$$

Remark: For $\alpha = 0$ the Fractional Laguerre polynomial $L_n^\alpha(t)$ reduces in simple Laguerre polynomial $L_n(t)$ as

$$(1.8) \quad L_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(n+1) t^k}{[\Gamma(k+1)]^2 \Gamma(n-k+1)}, \quad n \in \mathbb{N}, 0 \leq t < \infty.$$

In the proposed work, we find the results in terms of Mittag-Leffler function [7] defined as

$$(1.9) \quad E_{\xi, \eta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\xi + \eta)}, \quad \Re(\xi) > 0, \Re(\eta) > 0, \xi, \eta \in \mathbb{C}.$$

Following is the well-known result from Miller and Ross [5]

$$(1.10) \quad {}_0D_t^{-\lambda} t^q = \frac{\Gamma(q+1)}{\Gamma(q+\lambda+1)} t^{q+\lambda}, \quad \Re(q) > -1, 0 < \Re(\lambda) < 1, t > 0.$$

Laplace Transform [9] of a function is defined as

$$(1.11) \quad \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-pt} f(t) dt = F(p).$$

Using the definition of Laplace transform, we can have following results easily

$$(1.12) \quad \mathcal{L}\{L_n^\alpha(t)\} = \sum_{k=0}^n \frac{(-1)^k \Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1) \Gamma(n-k+1) p^{k+1}}$$

$$(1.13) \quad \mathcal{L}\{L_n^\alpha(d^\mu t^\mu)\} = \sum_{k=0}^n \frac{(-1)^k (1+\alpha)_n \Gamma(n+\alpha+1) \Gamma(k\mu+1) d^{k\mu}}{\Gamma(k+\alpha+1) \Gamma(k+1) \Gamma(n-k+1) p^{k\mu+1}}$$

$$(1.14) \quad \mathcal{L}\{{}_0D_t^{-\lambda} L_n^\alpha(t)\} = \sum_{k=0}^n \frac{(-1)^k (1+\alpha)_n}{(\alpha+1)_k \Gamma(n-k+1) p^{k+\alpha+1}},$$

where $(.)_r$ represents the Pochhammer symbol and defined as $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$. Further,

$$(1.15) \quad \mathcal{L}\{{}_0D_t^{-\lambda} L_n^\alpha(d^\mu t^\mu)\} = \sum_{k=0}^n \frac{(-1)^k (1+\alpha)_n \Gamma(k\mu+1) d^{k\mu}}{(\alpha+1)_k \Gamma(k+1) \Gamma(n-k+1) p^{k\mu+\lambda+1}}.$$

In the literature many workers has been contributed their work related to Fractional differential equations and specially with fractional kinetic equations using different functions. List of these works may include Saxena and Kalla [8], Agarwal and Bhargava [1], etc. In this sequence, the goal of this work is to discover the solution of Fractional kinetic equations pertaining to Laguerre

polynomials. We found the result in expressions of Mittag-Leffler functions and discuss the results with the pictorial presentations.

2. MAIN RESULTS

In this section we have taken new generalized forms of FKE by involving Laguerre Polynomial and its fractional derivative and find their solution using Laplace Transform technique. Further by the graphical presentation of the results for suitable parametric values, the results are interpreted.

Theorem 2.1. *Let $n \in \mathbb{N}$, $(n - 1) < \nu < n$, $0 \leq t < \infty$ and $\alpha > -1$, then the solution of the FKE*

$$(2.1) \quad N(t) - N_0 L_n^\alpha(t) = -c^\nu {}_0D_t^{-\nu} N(t)$$

is given by

$$(2.2) \quad N(t) = N_0 \sum_{k=0}^n \frac{(-1)^k \Gamma(n + \alpha + 1) t^k}{\Gamma(k + \alpha + 1) \Gamma(n - k + 1)} E_{\nu, k+1}(-c^\nu t^\nu),$$

where $E_{\nu, k+1}(\cdot)$ is the generalized Mittag-Leffler function.

Proof. Taking the Laplace transform of equation (2.1), we have

$$\begin{aligned} N(p) &= N_0 \sum_{k=0}^n \frac{(-1)^k \Gamma(n + \alpha + 1)}{\Gamma(k + \alpha + 1) \Gamma(n - k + 1) (1 + c^\nu p^{-\nu}) p^{k+1}} \\ &= N_0 \sum_{k=0}^n \frac{(-1)^k \Gamma(n + \alpha + 1)}{\Gamma(k + \alpha + 1) \Gamma(n - k + 1) (1 + c^\nu p^{-\nu}) p^{k+1}} \sum_{s=0}^{\infty} (-1)^s c^{s\nu} p^{-(s\nu + k + 1)}. \end{aligned}$$

Taking inverse Laplace Transform we have

$$N(t) = N_0 \sum_{k=0}^n \frac{(-1)^k \Gamma(n + \alpha + 1) t^k}{\Gamma(k + \alpha + 1) \Gamma(n - k + 1)} \sum_{s=0}^{\infty} \frac{(-c^\nu t^\nu)^s}{\Gamma(s\nu + k + 1)}$$

Now, using the definition of Mittag-Leffler function from (1.9), we obtain the desired result. $\square$

Corollary 2.1. *By putting $\alpha = 0$ in Theorem 2.1, we get the following FKE in terms of Laguerre Polynomial*

$$(2.3) \quad N(t) - N_0 L_n(t) = -c^\nu {}_0D_t^{-\nu} N(t)$$

with the solution

$$(2.4) \quad N(t) = N_0 \sum_{k=0}^n \frac{(-1)^k \Gamma(n+1) t^k}{\Gamma(k+1) \Gamma(n-k+1)} E_{\nu, k+1}(-c^\nu t^\nu).$$

Theorem 2.2. Let $n \in \mathbb{N}$, $(n-1) < \nu < n$, $0 \leq t < \infty$, $d \neq c$, $\mu > 0$ and $\alpha > -1$, then the solution of the FKE

$$(2.5) \quad N(t) - N_0 L_n^\alpha(d^\mu t^\mu) = -c^\nu {}_0D_t^{-\nu} N(t)$$

is given by

$$(2.6) \quad N(t) = N_0 L_n^\alpha(d^\mu t^\mu) \Gamma(k\mu + 1) E_{\nu, k\mu+1}(-c^\nu t^\nu).$$

Proof. Taking the Laplace transform of equation (2.5), we have

$$\begin{aligned} N(p) &= N_0 \sum_{k=0}^n \frac{(-1)^k \Gamma(n+\alpha+1) \Gamma(k\mu+1) d^{k\mu}}{\Gamma(k+\alpha+1) \Gamma(n-k+1) \Gamma(k+1) (1+c^\nu p^{-\nu}) p^{k\mu+1}} \\ &= N_0 \sum_{k=0}^n \frac{(-1)^k \Gamma(n+\alpha+1) \Gamma(k\mu+1) d^{k\mu}}{\Gamma(k+\alpha+1) \Gamma(n-k+1) \Gamma(k+1)} \sum_{s=0}^{\infty} (-1)^s c^{s\nu} p^{-(s\nu+k\mu+1)}. \end{aligned}$$

Taking inverse Laplace Transform we have

$$N(t) = N_0 \sum_{k=0}^n \frac{(-1)^k \Gamma(n+\alpha+1) \Gamma(k\mu+1) d^{k\mu} t^{k\mu}}{\Gamma(k+\alpha+1) \Gamma(n-k+1) \Gamma(k+1)} \sum_{s=0}^{\infty} \frac{(-c^{s\nu} t^{s\nu})^s}{\Gamma(s\nu+k\mu+1)}.$$

Now, using the definition of Mittag-Leffler function from (1.9) and Fractional Laguerre Polynomial from (1.7), we obtain the desired result. $\square$

Theorem 2.3. Let $n \in \mathbb{N}$, $(n-1) < \nu < n$, $0 \leq t < \infty$, $\lambda > 0$, $\lambda \neq \nu$ and $\alpha > -1$, then the solution of the FKE

$$(2.7) \quad N(t) - N_0 [{}_0D_t^{-\lambda} (L_n^\alpha(t))] = -c^\nu {}_0D_t^{-\nu} N(t)$$

is given by

$$(2.8) \quad N(t) = N_0 \sum_{k=0}^n \frac{(-1)^k (1+\alpha)_n t^{k-\lambda}}{(1+\alpha)_k \Gamma(n-k+1)} E_{\nu, k+\lambda+1}(-c^\nu t^\nu).$$

Proof. Taking the Laplace transform of equation (2.7), we have

$$\begin{aligned} N(p) &= N_0 \sum_{k=0}^n \frac{(-1)^k (1+\alpha)_n}{(1+\alpha)_k \Gamma(n-k+1) (1+c^\nu p^{-\nu}) p^{k+\lambda+1}} \\ &= N_0 \sum_{k=0}^n \frac{(-1)^k (1+\alpha)_n}{(1+\alpha)_k \Gamma(n-k+1)} \sum_{s=0}^{\infty} (-1)^s c^{s\nu} p^{-(s\nu+k+\lambda+1)}. \end{aligned}$$

Now, taking inverse Laplace Transform and using the definition of Mittag-Leffler function from (1.9), we achieve the required result. $\square$

Theorem 2.4. *Let $n \in \mathbb{N}$, $(n-1) < \nu < n$, $0 \leq t < \infty$, $d \neq c$, $\mu > 0$, $\lambda > 0$, $\lambda \neq \nu$ and $\alpha > -1$, then the solution of the FKE*

$$(2.9) \quad N(t) - N_0[{}_0D_t^{-\lambda}(L_n^\alpha(d^\mu t^\mu))] = -c^\nu {}_0D_t^{-\nu} N(t)$$

is given by

$$(2.10) \quad N(t) = N_0 \sum_{k=0}^n \frac{(-1)^k (1+\alpha)_n \Gamma(k\mu+1) d^{k\mu} t^{k\mu+\lambda}}{(1+\alpha)_k \Gamma(k+1) \Gamma(n-k+1)} E_{\nu, k\mu+\lambda+1}(-c^\nu t^\nu).$$

Proof. Proceeding on the similar lines of Theorem 2.1, Theorem 2.2 and Theorem 2.3 we can easily prove the result (2.10). $\square$

3. SPECIAL CASES

(i) If we take $d = c$ in Theorem 2.2, then the equation (2.3) reduces in

$$(3.1) \quad N(t) - N_0 L_n^\alpha(c^\mu t^\mu) = -c^\nu {}_0D_t^{-\nu} N(t)$$

with the solution

$$(3.2) \quad N(t) = N_0 L_n^\alpha(c^\mu t^\mu) \Gamma(k\mu+1) E_{\nu, k\mu+1}(-c^\nu t^\nu).$$

Other conditions are same as with (2.2).

(ii) If we put $\lambda = \nu$ in Theorem 2.3, then the equation (2.7) reduces in

$$(3.3) \quad N(t) - N_0[{}_0D_t^{-\nu}(L_n^\alpha(t))] = -c^\nu {}_0D_t^{-\nu} N(t)$$

with the solution

$$(3.4) \quad N(t) = N_0 t^\nu L_n^\alpha(t) \Gamma(k+1) E_{\nu, k+\nu+1}(-c^\nu t^\nu).$$

Other conditions are same as with (2.3).

(iii) If we put $d = c$ and $\lambda = \nu$ in Theorem 2.4, then the equation (2.9) reduces in

$$(3.5) \quad N(t) - N_0[{}_0D_t^{-\nu}(L_n^\alpha(c^\mu t^\mu))] = -c^\nu {}_0D_t^{-\nu} N(t)$$

with the solution

$$(3.6) \quad N(t) = N_0 t^\nu L_n^\alpha(c^\mu t^\mu) \Gamma(k\mu+1) E_{\nu, k\mu+\nu+1}(-c^\nu t^\nu).$$

4. CONCLUSION

Here, we propose the solution of generalized fractional kinetic equations involving Laguerre Polynomial in the form of four theorems by using the approach of Laplace transform. The results obtained here are believed to be new and have wide applications in science and technology. Further, the behaviour of these results are interpreted by graphs, taking distinct values of the parameters.

Graphical Interpretation of Results.

We draw the graphs of the solutions of FKE mentioned in Theorem 2.1 and corollary 2.1. Also, the graphs for the solutions of FKE mentioned in Theorem 2.2, 2.3 and Theorem 2.4 are there. For specific parametric values, we can observe that $N(t) > 0$ for $t > 0$.

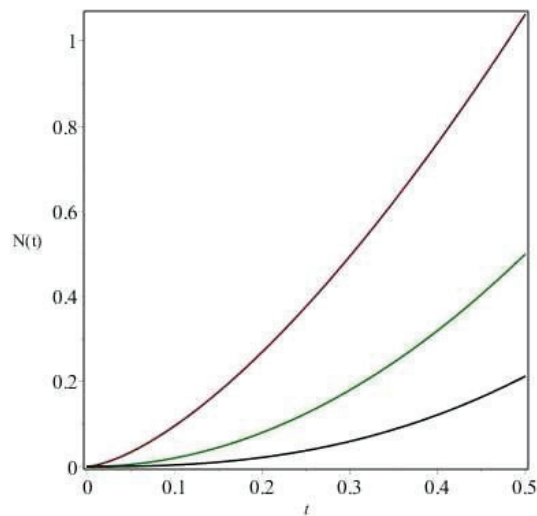


FIGURE 1. for $k = 1$,
 $s = 1$ and $\nu =$
 $0.5(0.5)1.5$

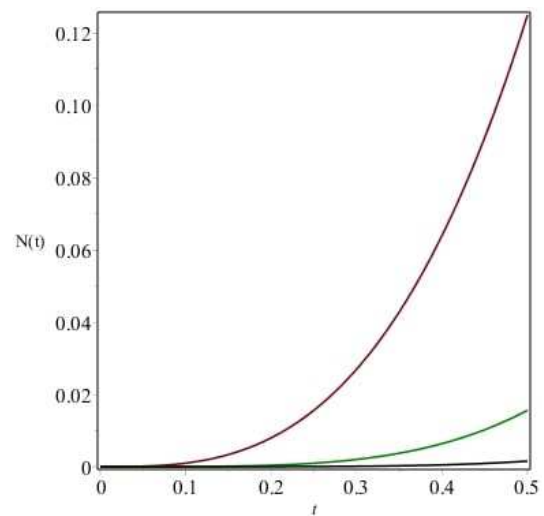


FIGURE 2. for $k = 2$,
 $s = 2$ and $\nu =$
 $0.5(0.5)1.5$

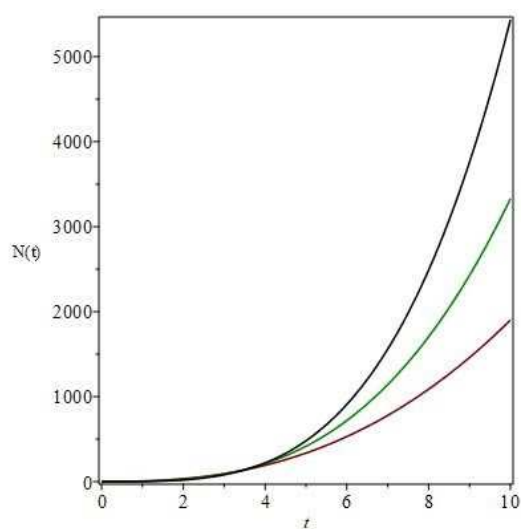


FIGURE 3. for $k = 1$,
 $s = 1$, $a = 1/2$ and $d = 2$

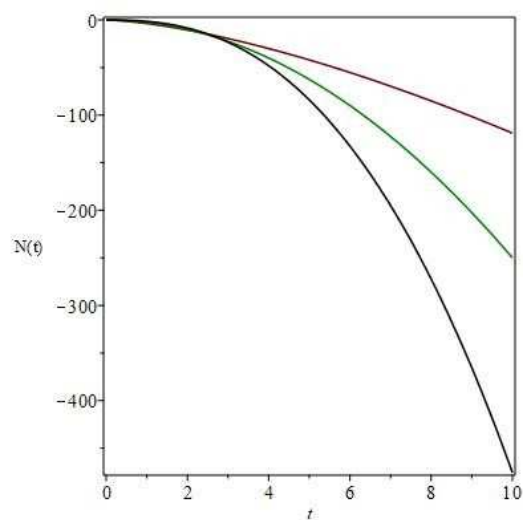


FIGURE 4. for $k = 1$,
 $s = 1$, $a = 1/2$ and $d = -2$

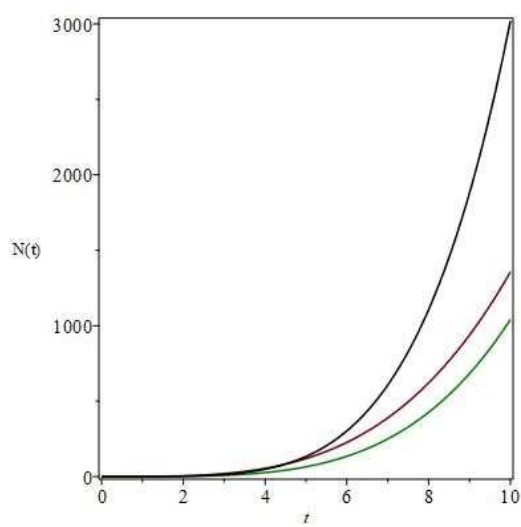


FIGURE 5. for $k = 1$,
 $s = 1$, $a = 1/2$ and $\alpha = 2$

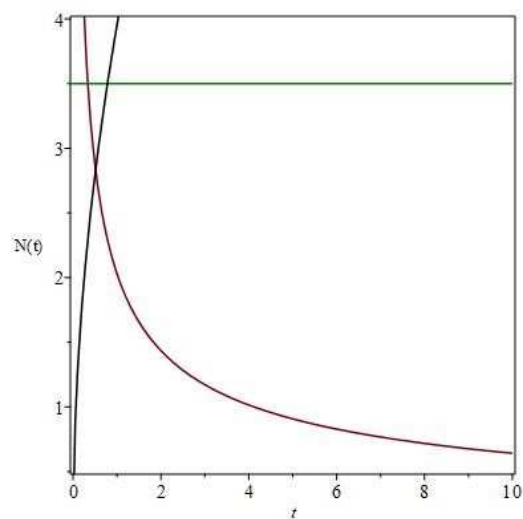


FIGURE 6. for $k = 2$,
 $s = 2$, $a = 1/2$ and $\alpha = -2$

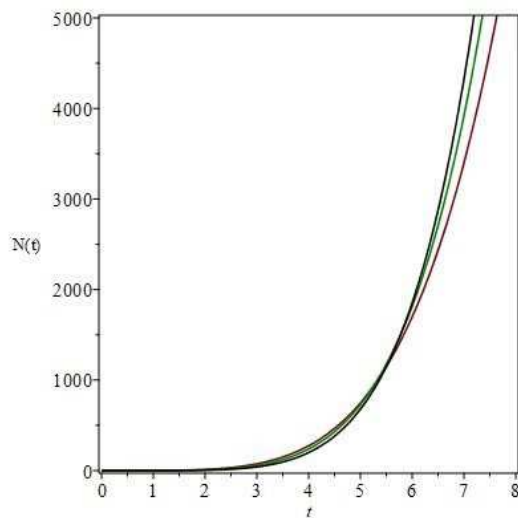


FIGURE 7. for $k = 1$,
 $s = 1$, $a = 1/2$ and $\alpha = 2$

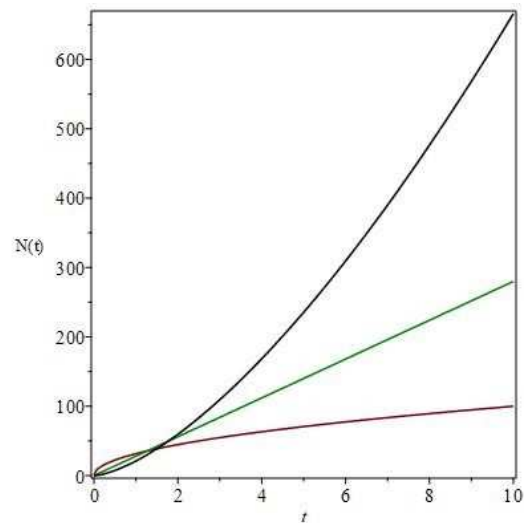


FIGURE 8. for $k = 2$,
 $s = 2$, $a = 1/2$ and $\alpha = -2$

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