Advances in Mathematics: Scientific Journal **9** (2020), no.9, 7093–7099
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.9.59

CHARACTERIZATION OF GENERALIZED COMPLEMENTS OF A GRAPH

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ABSTRACT. For a graph $G(V,E)$, let $P = \{V_1, V_2, V_3, \dots, V_k\}$ be a partition of vertex set $V(G)$ of order $k \geq 2$. For all V_i and V_j in P, $i \neq j$, remove the edges between V_i and V_j in graph G and add the edges between V_i and V_j which are not in G. The graph G_k^P thus obtained is called the $k-$*complement* of graph G with respect to the partition P. For each set V_r in P, remove the edges of graph G inside V_r and add the edges of $\overline{G}$ (the complement of G) joining the vertices of V_r. The graph $G_{k(i)}^P$ thus obtained is called the $k(i)-$*complement of graph* G with respect to the partition P. In this paper, we characterize few properties of generalized complements of a graph.

1. INTRODUCTION

Let G be a graph on n vertices and m edges. The complement of a graph G, denoted as $\overline{G}$ has the same vertex set as that of G, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. If G is isomorphic to $\overline{G}$ then G is said to be self-complementary graph. A graph G is r regular if $\delta(G) = \Delta(G) = r$. If G is any r-regular graph then $\bar{G}$ is also $(n-r-1)$ regular. For all notations and terminologies we refer [1]. E. Sampathkumar et al. in [2, 3] introduced two types of generalized complements of a graph. For completeness we produce these here.

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2010 *Mathematics Subject Classification.* 05C30, 05C40 .

Key words and phrases. generalized complements, regular graphs, line graphs, Euler graphs.

Let $P = \{V_1, V_2, V_3, \dots, V_k\}$ be the partition of vertex set $V(G)$ of order $k \geq 2$. For all V_i and V_j in P, $i \neq j$, remove the edges between V_i and V_j in graph G and add the edges between V_i and V_j which are not in G. The graph G_k^P thus obtained is called the $k-$complement of graph G with respect to the partition P. For each set V_r in P, remove the edges of graph G inside V_r and add the edges of $\overline{G}$ (the complement of G) joining the vertices of V_r. The graph $G_{k(i)}^P$ thus obtained is called the $k(i)-$complement of graph G with respect to partition P. For more on complements of graphs we refer [5–7].

In this paper, conditions for regularity of generalized complements of a regular graph are found. The generalized complements of a graph isomorphic to its line graph are studied.

Lemma 1.1. *Let $P = \{V_1, V_2, \dots, V_k\}$ be a partition of order $k \geq 2$ of a connected graph G on n vertices. If d is degree of a vertex v in G and d_i is degree if v in $< V_i >$, then degree of v in G_k^P is $n - d + 2d_i - n_i$ where $n_i = |V_i|$.*

Proof. Lemma follows by definition of G_k^P. ☐

Proposition 1.1. *For any r-regular graph $G(V, E)$ on n vertices with partition $P = \{V_1, V_2, \dots, V_k\}$ of V of order $k \geq 2$, G_k^p is regular if (i) and either (ii) or (iii) of the following conditions hold:*

(i) *k divides n and cardinality of V_i is exactly $\frac{n}{k}$ for every $i = 1, 2, 3, \dots, k$.*
(ii) *Vertices in each partite are independent.*
(iii) *Each vertex v in $< V_i >$ has equal degree.*

Proof. Let $G(V, E)$ be any r-regular graph on n vertices. Let $P = \{V_1, V_2, \dots, V_k\}$ be partition of V of order $k \geq 2$. Suppose degree of a vertex v in $< V_i >$ is d_i for $i = 1, 2, 3, \dots, k$ then if conditions (i) and (ii) hold, the vertex v will be adjacent to exactly $n - d - (\frac{n}{k})$ vertices in G_k^p. Thus G_k^p is $n - d - (\frac{n}{k})$ regular. On the other hand, if conditions (i) and (ii) hold then the vertex v will be adjacent to exactly $n - d + d_i - (\frac{n}{k})$ vertices in G_k^p. Hence the proof. ☐

Corollary 1.1. *For any r-regular graph $G(V, E)$ on n vertices with partition $P = \{V_1, V_2, \dots, V_k\}$ of V of order $k \geq 2$,*

(1) *If G_k^p is $n - d - (\frac{n}{k})$ regular then $G_{k(i)}^p$ is $d + (\frac{n}{k}) - 1$ regular.*
(2) *If G_k^p is $n - d + 2d_i - (\frac{n}{k})$ regular then $G_{k(i)}^p$ is $d - 2d_i + (\frac{n}{k}) - 1$ regular.*

Proof. Follows from noting that G_k^p and $G_{k(i)}^p$ are complements of each other. ☐

Example 1. *For $2-$regular graph on 4 vertices and $k=2$, let us realize Proposition 1.1 and Corollary 1.1*

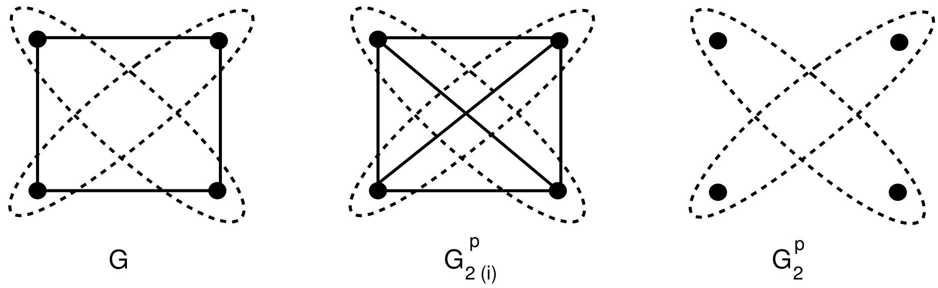

FIGURE 1

Here $n=4, d=2, d_i=0, \frac{n}{k}=2, k=2$

1.1. $k(i)-$ Complement of a graph isomorphic to its line graph.

In this section the $k(i)-$complement of a graph G isomorphic to its line graph are studied. Here for any graph G , $L(G)$ denotes line graph, $G^p_{k(i)}$ denotes $k(i)-$ complement of G, $|V(G)|=n$ the number of vertices in G and $|E(G)|=e$ number of edges in G. C_n the cycle of length n, $K_{1,n}$ the star graph and P_n path of length $n-1$.

Theorem 1.1. *[1] A connected graph G is isomorphic to its line graph $L(G)$, if and only if G is a cycle.*

Martin Aigner [4] showed that there are only two graphs namely C_5 and C_3 with one pendant edge emanating at each of its vertices have the property that their complement is isomorphic to line graph

Observation.

(1) Let G be any graph, for an automorphism between $V(G)$ and $V(G^p_{k(i)})$ one must have $|E(G)|=|V(G)|=|V(G^p_{k(i)})|$. This implies that G is connected unicyclic graph or G consists of c components each of which are unicyclic.

(2) Let $P=\{V_1, V_2, \ldots, V_k\}$, $k\geq 2$, be a partition of V . Then none of $< Vi >$ $i=1,2,\ldots,k$ must be complement of any of nine forbidden graphs [1] for line graphs.

Theorem 1.2. *For any graph G of order n and size q, $G^p_{k(i)}$ is isomorphic to $L(G)$ if any one of the following conditions hold.*

(1) *G is any cycle C_n, and $P = \{V_1, V_2, \ldots, V_n\}$ partition of $V(G)$, with $V_i = 1$ for all i.*
(2) *G is unicyclic with at least one pendant edge attached to a cycle C_n and $P = \{V_1, V_2, \ldots, V_n\}$ partition of V such that each V_i for $i = 1, 2, \ldots, n$, has exactly one vertex v_i of C_n and all the pendant vertices at v_{i+1}, all the pendant vertices at v_q are in the partite that contains v_1.*
(3) *G is unicyclic with at least one path attached to a cycle C_n and $P = \{V_1, V_2, \ldots, V_{n-x}\}$ partition of V where x is number of vertices of the path at distance one from vertices of C_n such that each V_i for $i = 1, 2, \ldots, n$, has exactly one vertex v_i of C_n and all the vertices at distance one from v_{i+1} of C_n, all the vertices at distance one from v_q belong to the partite that contains v_1.*

Proof. Let G be any graph satisfying condition 1, then by definition of $G^p_{k(i)}$ is C_n. Thus in this case $G^p_{k(i)}$ isomorphic to $L(G)$.

Let G be any graph satisfying condition 2. Let $e_1, e_2, \ldots, e_j$, be the pendant edges at a vertex v_j of C_n. Then these edges form a complete subgraph in $L(G)$. This is true at every vertex of C_n. Thus $L(G)$ is C_n with complete subgraphs attached at the vertices of C_n for the partition $P = \{V_1, V_2, \ldots, V_n\}$ of V such that each V_i for $i = 1, 2, \ldots, n$ has exactly one vertex v_i of C_n and all the pendant vertices at v_{i+1}, all the pendant vertices at v_q are in the partite that contains v_1. Then $G^P_{k(i)}$ is C_n with complete subgraphs attached at the vertices of C_n and isomorphic to $L(G)$.

If G is any graph satisfying condition 3 the result follows similar way. □

Example 2. *Condition 1.*

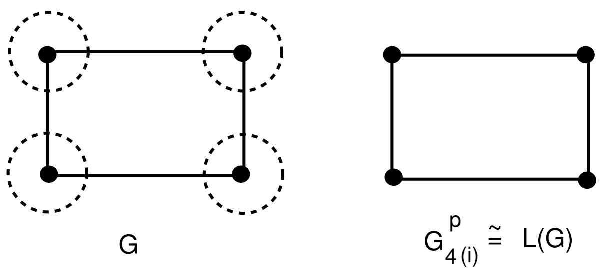

FIGURE 2

Condition 2.

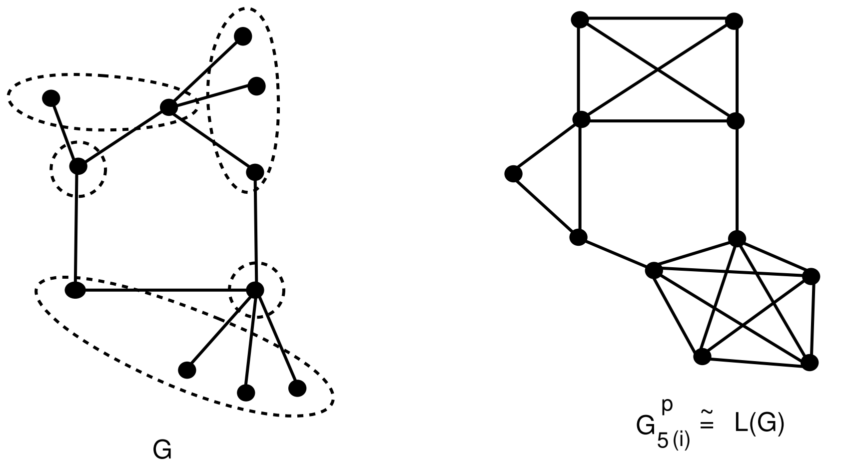

FIGURE 3

2. CHARECTERIZATION OF GENERALIZED COMPLEMENTS OF EULER GRAPHS

Definition 2.1. *A connected graph G is called Eulerian graph if it contains a closed trail containing all the edges of G.*

Theorem 2.1. *If $G(V, E)$ is Eulerian and $P = \{V_1, V_2, \dots, V_k\}$ partition for V of order $k \geq 2$, such that for at least one i, $i = 1, 2, \dots, k$, $|V_i| = 2$ then $G^p_{k(i)}$ is non Eulerian.*

Proof. Let $G(V, E)$ be any Eulerian graph. Then every vertex of G is of even degree. Suppose $P = \{V_1, V_2, \dots, V_k\}$ partition for V of order $k \geq 2$. Let V_i be a partition having only two vertices say u and v. Then in $G^p_{k(i)}$, degree of u and v will be increased by 1 if u and v are not adjacent in $< V_i >$ and degree of u and v will be decreased by 1 if u and v are adjacent in $< V_i >$. Thus in either of the cases $G^p_{k(i)}$ has a vertex of odd degree. Hence $G^p_{k(i)}$ is not Eulerian. □

Theorem 2.2. *If G is any Eulerian graph and $P = \{V_1, V_2, \dots, V_k\}$ partition for V of order $k \geq 2$, then $G^p_{k(i)}$ is Eulerian if all the following conditions hold good.*

(1) *$|V_i|$ is odd for each $i = 1, 2, \dots, k$.*
(2) *For every vertex v of odd degree in $< V_i >$ there must be odd number of vertices in V_i not adjacent to v for $i = 1, 2, \dots, k$.*

(3) *For every vertex v of even degree in $< V_i >$ there must be even number of vertices in V_i not adjacent to v for $i = 1, 2, \dots, k$.*

Proof. Let G be any Eulerian graph. Then degree of every vertex of G is even. Let $P = \{V_1, V_2 \dots, V_k\}$ be a partition for V of order $k \geq 2$ for which above three conditions hold. Let vertex $v \in V_i$ for some $i = 1, 2, \dots, k$. Suppose degree of v in G is d and degree of v in $< V_i >$ is d_i. If d_i is odd then there are odd number of vertices say n_i in $< V_i >$ which are not adjacent to v. Then degree of v in G_k^p is $d - d_i + n_i$ which is even. On the other hand, if d_i is even then there are even number of vertices say l_i in $< V_i >$ which are not adjacent to v. Then degree of v in $G_{k(i)}^p$ is $d - d_i + l_i$ which is even. This is true for every v and each $i = 1, 2, \dots, k$. Thus every vertex of $G_{k(i)}^p$ is of even degree. Hence $G_{k(i)}^p$ is Eulerian. □

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