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SUBTRACTIVE PARTIAL Γ-SUBSEMIMODULES OF PARTIAL Γ-SEMIMODULES

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ABSTRACT. In this paper, we study the concepts of subtractive partial Γ-subsemimodules of partial Γ-semimodules. Also, we introduce the notion of left austere partial Γ-semimodule and study its characterization.

1. INTRODUCTIOIN

In 1995, M Murali Krishna Rao [3] developed the thory of Γ-semirings and showed that this class is the common extension of semirings and Γ-rings. In 2014, M. Siva Mala [4] defined the concept of partial Γ-semiring by replacing the binary addition in Γ-semirings to infinitary partial addition and showed that this class is a common extension of partial semirings introduced by Arbib, manes [1] and Benson [2] and M. Murali Krishna Rao [3] Γ-semirings. Also M. Siva Mala [5], [6] and [7] studied theory of ideals for the Γ-so-rings. In [8], we introduced the concepts of left (right) partial Γ-semimodules over partial Γ-semirings.

In this paper, we study the concepts of subtractive partial Γ-subsemimodules of partial Γ-semimodules. Also, we introduce the notion of left austere partial Γ-semimodule and study its characterization.

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2. PRELIMINARIES

In the preliminaries, we recollect the necessary concepts from the literature. Throughout this paper, we use the following notations.

(1) **PM** stands for partial monoid.
(2) **PΓSR** stands for partial Γ-semiring.
(3) **L(R)PΓI** stands for left (right) partial Γ-ideal.
(4) **SL(R)PΓI** stands for subtractive left (right) partial Γ-ideal.
(5) **L(R)PΓSM** stands for left (right) partial Γ-semimodule.
(6) **PΓSSM** stands for partial Γ-subsemimodule.
(7) **SPΓSSM** stands for subtractive partial Γ-subsemimodule.
(8) **LA** stands for left austere.

A mapping $a : \Delta \to G$ from a set Δ to a nonempty set G is called a Δ-family in G. It is denoted by $(a_l : l \in \Delta)$, where $a_l = la ,\forall\ l \in \Delta$. A $sub\ family$ of $(a_l : l \in \Delta)$ is a family $(a_k : k \in K)$ where $K \subseteq \Delta$. The family $(a_l : l \in \emptyset)$ is called an $empty\ family$. Since $\Sigma(a_l : l \in \Delta) \notin G$ for all $(a_l : l \in \Delta)$ in G, Σ is called an *infinitary partial addition*. If $\Sigma_{l\in\Delta}a_l \in G$ then $(a_l : l \in \Delta)$ is called a *summable* family.

Definition 2.1. *[2] Let G be a nonempty set and Σ be an infinitary partial addition on G. Then the structure (G, Σ) is called a* **PM** *if it satisfies the following conditions:*

(M1) *If $(g_l : l \in \Delta)$ is in G and $\Delta = \{k\}$, then $\Sigma_{l\in\Delta}g_l = g_k \in G$.*
(M2) *If $(g_l : l \in \Delta)$ is in G and $(\Delta_k : k \in K)$ is a partition of Δ, then $\Sigma_{l\in\Delta}g_l \in G \iff \Sigma_{l\in\Delta_k}g_l \in G\ \forall\ k \in K$ and $\Sigma_{k\in K}(\Sigma_{l\in\Delta_k}g_l) \in G$, and $\Sigma_{l\in\Delta}g_l = \Sigma_{k\in K}(\Sigma_{l\in\Delta_k}g_l)$.*

Example 1. *[2] Let $Pfn(A, B)$ be the set of all partial functions from a set A to a set B. Define Σ on $Pfn(A, B)$ as follows: Let $(f_l : l \in \Delta)$ be a family in $Pfn(A, B)$. Then $\Sigma_{l\in\Delta}f_l \in Pfn(A, B) \iff$ for $l,\ k$ in Δ such that $l \neq k$, $dom(f_l) \bigcap dom(f_k) = \emptyset$ and for any $a \in A$,*

$$a(\Sigma_l f_l) = \begin{cases} af_l,\ if\ a \in dom(f_l)\ for\ some\ l \in \Delta; \\ undefined,\ otherwise. \end{cases}$$

Then $(Pfn(A, B), \Sigma)$ is a **PM**.

Definition 2.2. *[4] Let* (S, Σ) *and* (Γ, Σ^*) *be two* **PM***s. Then* S *is called a* **P**Γ**SR** *if there is an operation* $S \times \Gamma \times S \longrightarrow S : (a, \mu, b) \mapsto a\mu b \, \forall \, a, \ b \in S$ *and* $\mu \in \Gamma$ *subject to the following conditions* $\forall \, a, \ b, \ c, \ (a_l : l \in \Delta) \in S$ *and* μ, γ, $(\mu_l : l \in \Delta) \in \Gamma$

(S1) $a\mu(b\gamma c) = (a\mu b)\gamma c$,

(S2) $\Sigma_{l\in\Delta} a_l \in S$ *implies that* $\Sigma_{l\in\Delta}(a\mu a_l) \in S$ *and* $a\mu[\Sigma_{l\in\Delta} a_l] = \Sigma_{l\in\Delta}(a\mu a_l)$, $[\Sigma_{l\in\Delta} a_l]\mu a = \Sigma_{l\in\Delta}(a_l \mu a)$,

(S3) $\Sigma^*_{l\in\Delta}\mu_l \in \Gamma$ *implies that* $\Sigma_{l\in\Delta}(a\mu_l b) \in S$ *and* $a(\Sigma^*_{l\in\Delta}\mu_l)b = \Sigma_{l\in\Delta}(a\mu_l b)$.

Example 2. *[4] Consider the* **PM***s* $(Pfn(A, B), \Sigma)$ *and* $(Pfn(B, A), \Sigma^*)$ *as defined in the Example 1. Now define an operation* $Pfn(A, B) \times Pfn(B, A) \times Pfn(A, B) \longrightarrow Pfn(A, B) : (g, \mu, h) \mapsto g\mu h$ *where* $a(g\mu h) = (((ag)\mu)h)$*, for any* $a \in A$*. Then* $Pfn(A, B)$ *is a* **P**Γ**SR** *where* $\Gamma = Pfn(B, A)$.

In general $Pfn(A, B)$ need not be a Γ-semiring, because an arbitrary family in the **P**Γ**SR** $Pfn(A, B)$ need not be summable. Here $\Gamma = Pfn(B, A)$.

Definition 2.3. *[5] Let* S *be a* **P**Γ**SR***,* $K \subseteq S$ *(*$K \neq \emptyset$*) and* $\Omega \subseteq \Gamma$ *(*$\Omega \neq \emptyset$*). Then* (K, Ω) *of* (S, Γ) *is called a* **L(R)P**Γ**I** *of* S *if the following conditions hold in K:*

(I1) $\Sigma_l a_l \in S$ *and* $a_l \in K \ \forall l \in \Delta$ *implies* $\Sigma_l a_l \in K$,

(I2) $\Sigma^*_l \mu_l \in \Gamma$ *and* $\mu_l \in \Omega \ \forall l \in \Delta$ *implies* $\Sigma^*_l \mu_i \in \Omega$*, and*

(I3) $s\mu a \in K$ *(*$a\mu s \in K$*)* $\forall \ s \in S, \ a \in K$ *and* $\mu \in \Omega$.

Definition 2.4. *[8] Let* S *be a* **P**Γ**SR** *and* (N, Σ') *be a* **PM***. Then* N *is called a* **L(R)P**Γ**SM** *over* S *if* $\exists$ *an operation* $S \times \Gamma \times N \to N : (s, \mu, n) \mapsto s\mu n$ *(*$N \times \Gamma \times S \to N : (n, \mu, s) \mapsto n\mu s$*) which satisfies the following axioms:*

(SM1) *if* $\Sigma'_l n_l \in N$ *then* $\Sigma'_l(s\mu n_l) \in N$ *and* $s\mu(\Sigma'_l n_l) = \Sigma'_l(s\mu n_l)$*,b*

(SM2) *if* $\Sigma^*_l \mu_l \in \Gamma$ *then* $\Sigma'_l(s\mu_l n) \in N$ *and* $s(\Sigma^*_l \mu_l)n = \Sigma'_l(s\mu_l n)$
(where Σ^* *is the partial addition in* Γ*),b*

(SM3) *if* $\Sigma_l s_l \in S$ *then* $\Sigma'_l(s_l\mu n) \in N$ *and* $(\Sigma_l s_l)\mu n = \Sigma'_l(s_l \mu n)$
(where Σ *is the partial addition in* S*),b*

(SM4) $(s\mu t)\alpha n = s\mu(t\alpha n)$,

(SM5) $0_S \mu n = s 0_\Gamma n = s\mu 0_N = 0_N$ *for every* $n, \ n_i \in N, \ \mu, \ \mu_i, \ \alpha \in \Gamma, \ s, \ s_i, \ t \in S$.

For the convenience of study the symbol Σ is used hereafter instead of the partial additions Σ in S, Σ^* in Γ and Σ' in N irrespective of the context.

Definition 2.5. *[8] Let* S *be a* **P**Γ**SR***,* N *be a* **L(R)P**Γ**SM** *over* S *and* $K \subseteq N$ *(*$K \neq \emptyset$*). Then* K *is called a* **P**Γ**SSM** *of* N *if the following holds in K:*

(SSM1) *if* $\Sigma_l a_l \in N$ *and* $a_l \in K\ \forall\ l \in \Delta$ *then* $\Sigma_l a_l \in K$*, and*
(SSM2) *if* $s \in S,\ \mu \in \Gamma,\ a \in K$ *then* $s\mu a \in K$ *(*$a\mu s \in K$*).*

Definition 2.6. *[8] Let* N *be a* **LPΓSM** *over a* **PΓSR** S*,* K *be a* **PΓSSM** *of* N *and* $n^* \in N$*. Then* $(K : n^*) = \{a \in S \mid a\mu n^* \in K\ \forall \mu \in \Gamma\}$*.*

Theorem 2.1. *[8] Let* N *be a* **LPΓSM** *over a* **PΓSR** S*,* K *be a* **PΓSSM** *of* N *and* $n^* \in N$*. Then* $(K : n^*)$ *is a* **LPΓI** *of* S*.*

Definition 2.7. *[8] If* K *is a* **PΓSSM** *of* N *and* $C \subseteq K$ *(*$C \neq \emptyset$*) then* $(K : C) = \bigcap\{(K : c) \mid c \in C\}$*.*

3. Subtractive Partial Γ-Subsemimodules

In this section we define **SPΓSSM** and **LAPΓSM**.

Definition 3.1. *Let* N *be a* **LPΓSM** *over a PΓSR* S *and* $C \subseteq N$ *(*$C \neq \emptyset$*). Then* C *is called subtractive subset of* N *if for any* $p,\ q \in N$*,* $p \in C$ *and* $p + q \in C$ *implies* $q \in C$*.*

Example 3. *Let* $S = \{0, c_1, c_2, c_3, c_4, c_5\}$*. Define* Σ *on* S *as*

$$\Sigma_l x_l = \begin{cases} x_k,\ if\ x_l = 0\ \forall l \neq k,\ for\ some\ k, \\ c_4,\ if\ (x_j = c_1,\ x_k = c_2 \text{ or } x_j = c_2,\ x_k = c_3\ for\ some\ j,\ k) \text{ and } x_l = 0\ \forall l \neq j, k, \\ undefined,\ otherwise. \end{cases}$$

Then S *is a* **PM***.*
Let $\Gamma = \{0^*, 1^*\}$*. Define* Σ^* *on* Γ *as*

$$\Sigma^*_l \mu_l = \begin{cases} 0^*,\ if\ \mu_l = 0^*\ \forall l \in \Delta \\ 1^*,\ if\ \mu_l = 0^*\ \forall l \neq k\ for\ some\ k \\ undefined,\ otherwise. \end{cases}$$

Then Γ *is a* **PM***. Define an operation* $S \times \Gamma \times S \to S$ *as:* $x0^*y = 0\ \forall\ x,\ y \in S$*,* $x1^*y = 0\ \forall\ x,\ y \in S \setminus \{c_5\}$ *and* $c_5 1^* y = y1^*c_5 = y\ \forall\ y \in S$*. Then* S *is a* **PΓSR***. Take* $N := S$*. Then* N *is a* **LPΓSM** *over* S*. Now* $C = \{0,\ c_1,\ c_2\}$ *is a subtractive subset of* N *whereas* $D = \{0,\ c_1,\ c_2,\ c_4\}$ *is not a subtractive subset of* N *(since* $c_2 \in D$ *and* $c_2 + c_3 = c_4 \in D$ *but* $c_3 \notin D$*.*

Theorem 3.1. *Let N be a* **LPΓSM** *over a* **PΓSR** *S, K be a* **SPΓSSM** *of N and $C \subseteq N$ ($C \neq \emptyset$). Then $(K : C)$ is a* **SLPΓI** *of S.*

Proof. Note that $(K : C) = \bigcap_{c \in C}(K : C) = \bigcap_{c \in C}\{s \in S \mid s\mu c \in K \ \forall \ \mu \in \Gamma\}$. By the Theorem 2.1, $(K : C)$ is a **LPΓI** of S. So it is enough to prove that $(K : C)$ is subtractive. Let $p, \ q \in N \ni p \in (K : C)$ and $p + q \in (K : C)$. Then $p \in (K : c)$ and $p + q \in (K : c) \ \forall \ c \in C$. $\Rightarrow p\mu c \in K$ and $(p + q)\mu c \in K \ \forall \ \mu \in \Gamma, \ c \in C$. $\Rightarrow p\mu c \in K$ and $p\mu c + q\mu c \in K \ \forall \ \mu \in \Gamma, \ c \in C$. Since K is a **SPΓSSM** of N, $q\mu c \in K \ \forall \ \mu \in \Gamma, \ c \in C$. $\Rightarrow q \in (K : c) \ \forall \ c \in C$. $\Rightarrow q \in (K : C)$. Hence $(K : C)$ is a **SLPΓI** of S. □

Theorem 3.2. *Let N be a* **LPΓSM** *over a* **PΓSR** *S. If $L, \ L^*, \ L^{**}$ are* **PΓSSM***s of N $\ni L$ is subtractive and $L^* \subseteq L$. Then $L \bigcap (L^* + L^{**}) = L^* + (L \bigcap L^{**})$.*

Proof. Since $L^* \subseteq L$, it is trivial to prove that $L \bigcap (L^* + L^{**}) \supseteq L^* + (L \bigcap L^{**})$. Now take $p \in L \bigcap (L^* + L^{**})$. Then $p \in L$ and $p \in (L^* + L^{**})$. $\Rightarrow p \in L$ and $p = q + r$ where $q \in L^*, \ r \in L^{**}$. $\Rightarrow p = q + r \in L$ where $q \in L^*, \ r \in L^{**}$. Since $L^* \subseteq L$, $q \in L$ and $q + r \in L$. Since L is a **SPΓSSM** of N, $r \in L$. $\Rightarrow p = q + r$ where $q \in L^*$ and $r \in L \bigcap L^{**}$. $\Rightarrow p \in L^* + (L \bigcap L^{**})$. Hence $L \bigcap (L^* + L^{**}) \subseteq L^* + (L \bigcap L^{**})$. Hence $L \bigcap (L^* + L^{**}) = L^* + (L \bigcap L^{**})$. □

Theorem 3.3. *Let N be a* **LPΓSM** *over a* **PΓSR** *S and Q be a* **L(R)PΓI** *of S. Then the set $L = \{n \in N \mid Q\Gamma n = 0\}$ is a* **SPΓSSM** *of N.*

Proof. Let $\Sigma_l n_l \in N \ni n_l \in L, \ l \in \Delta$. Then $Q\Gamma n_l = 0, \ l \in \Delta$. $\Rightarrow Q\Gamma(\Sigma_l n_l) = \Sigma_l(Q\Gamma n_l) = 0$ and so $\Sigma_l n_l \in L$. Let $s \in S$, $\mu \in \Gamma$ and $n \in L$. Then $Q\Gamma n = 0$. Let $m \in Q\Gamma(s\mu n)$. Then $m = \Sigma_l m_l \mu_l (s\mu n)$, $m_l \in Q$ and $\mu_l \in \Gamma$. $\Rightarrow m = \Sigma_l(m_l\mu_l s)\mu n$. Since $m_l \in Q$, $m_l\mu_l s \in Q$, $l \in \Delta$ (Since **Q** is **L(R)PΓI**). $\Rightarrow \Sigma_l(m_l\mu_l s)\mu n \in Q\Gamma n = 0$, $l \in \Delta$. $\Rightarrow m = \Sigma_l m_l\mu_l(s\mu n) = 0$. $\Rightarrow Q\Gamma(s\mu n) = 0$ and so $s\mu n \in L$. Hence L is a **PΓSSM** of N.

To prove L is subtractive, let $p, \ p^* \in N$ such that $p \in L$ and $p + p^* \in L$. Then $Q\Gamma p = 0$ and $Q\Gamma(p + p^*) = 0$. $\Rightarrow Q\Gamma p^* = 0$ and so $n \in L$. Hence L is a **SPΓSSM** of N. □

Definition 3.2. *A* **LPΓSM** *U over a* **PΓSR** *S is called* **LA** *if $\{0\}$ and U are the only* **SPΓSSM***s of N.*

Theorem 3.4. *Let S be a commutative* **PΓSR***. If U is a* **LAPΓSM** *over S then $(0 : U) = (0 : u) \ \forall \ 0 \neq u \in U$.*

Proof. Since $(0:U) = \bigcap_{u \in U} (0:u)$, we have $(0:U) \subseteq (0:u)\ \forall\ 0 \neq u \in U$. Suppose if $(0:u) \nsubseteq (0:U)$ for some $0 \neq u \in U$. Then $(0:u) \nsubseteq (0:u^*)$ for some $0 \neq u^* \in U$. Take $V = \{z \in U \mid (0:u) \subseteq (0:z)\}$. Clearly $0 \neq u \in V$ and $0 \neq u^* \notin V$. Therefore $\{0\} \subset V \subset U$. Now we claim that V is a **SP**Γ**SSM** of U: Let $\Sigma_l z_l \in U \ni z_l \in V$, $l \in \Delta$. Then $(0:u) \subseteq (0:z_l)$, $l \in \Delta$. $\Rightarrow (0:u) \subseteq (0:\Sigma_l z_l)$ and so $\Sigma_l z_l \in V$. Let $s \in S$, $\mu \in \Gamma$ and $z \in V$. Then $(0:u) \subseteq (0:z)$. To prove $s\mu z \in V$ it is enough to prove that $(0:u) \subseteq (0:s\mu z)$. For this, let $y \in (0:u)$. Then $y \in (0:z)$. $\Rightarrow y\beta z = 0\ \forall\ \beta \in \Gamma$. $\Rightarrow y\mu z = 0$ (for $\beta = \mu$). $\Rightarrow s\alpha(y\mu z) = 0\ \forall\ \alpha \in \Gamma$. $\Rightarrow (s\alpha y)\mu z = 0\ \forall\ \alpha \in \Gamma$. $\Rightarrow (y\alpha s)\mu z = 0\ \forall\ \alpha \in \Gamma$ (since S is commutative). $\Rightarrow y\alpha(s\mu z) = 0\ \forall\ \alpha \in \Gamma$. $\Rightarrow y \in (0:s\mu z)$. $\Rightarrow (0:u) \subseteq (0:s\mu z)$ and so $s\mu z \in V$. Hence V is a **P**Γ**SSM** of U.

Let $p,\ q \in U \ni p \in V$ and $p+q \in V$. Then $(0:u) \subseteq (0:p)$ and $(0:u) \subseteq (0:p+q)$. Let $x \in (0:u)$. Then $x \in (0:p)$ and $x \in (0:p+q)$. $\Rightarrow x\mu p = 0$ and $x\mu(p+q) = 0\ \forall\ \mu \in \Gamma$. $\Rightarrow x\alpha q = 0\ \forall\ \mu \in \Gamma$. $\Rightarrow x \in (0:q)$ and so $(0:u) \subseteq (0:q)$. $\Rightarrow q \in V$. Hence V is a nontrivial **SP**Γ**SSM** of U, a contradiction to the fact that U is **LAP**Γ**SM**. Hence $(0:U) = (0:u)\ \forall\ 0 \neq u \in U$. □

REFERENCES

[1] M. A. ARBIB, E. G. MANES: *Partially Additive Categories and Flow-diagram Semantics*, Journal of Algebra, **62** (1980), 203 – 227.

[2] E. G. MANES, D. B. BENSON: *The Inverse Semigroup of a Sum-Ordered Partial Semiring*, Semigroup Forum, **31** (1985), 129 – 152.

[3] M. MURALI KRISHNA RAO: *Γ-semirings-I*, Southeast Asian Bulletin of Mathematics, **19**(1) (1995), 49 – 54.

[4] M. SIVA MALA, K. SIVA PRASAD: *Partial Γ-Semirings*, Southeast Asian Bulletin of Mathematics, **38** (2014), 873 – 885.

[5] M. SIVA MALA, K. SIVA PRASAD: *Ideals of Sum-Ordered partial Γ-Semirings*, Southeast Asian Bulletin of Mathematics, **40** (2016), 413 – 426.

[6] M. SIVA MALA, K. SIVA PRASAD: *Prime Ideals of Γ-So-rings*, International Journal of Algebra and Statistics(IJAS), **3**(1) (2014), 1 – 8.

[7] M. SIVA MALA, K. SIVA PRASAD: *Semiprime Ideals of Γ-So-rings*, International Journal of Algebra and Statistics(IJAS), **3**(1) (2014), 26 – 33.

[8] M. SIVA MALA, P. V. SRINIVASA RAO, K. KIRAN KUMAR: *Partial Γ-Semimodules over Partial Γ-Semirings*, communicated to Punjab University Journal of Mathematics(PUJM).

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