STRONGLY CONNECTED INTERVAL-VALUED FUZZY GRAPHS

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ABSTRACT. In interval-valued fuzzy graphs (IVFGs) strong paths need not exist between every two nodes in contrast with fuzzy graphs. Based on this, we define a particular class of interval - valued fuzzy graphs called strongly connected interval valued fuzzy graphs (SCIVFGs). A connected IVFG in which a strong path always exists between every two nodes is called a SCIVFG. We prove several sufficient conditions for an IVFG to be strongly connected. Finally we show that strong connectedness is preserved under isomorphism and co-weak isomorphism.

1. INTRODUCTION

In 1975, Rosenfeld [11] developed the concept of fuzzy graphs whose basic idea was introduced by Kaufmann [4] in 1973. Since then fuzzy graph theory is witnessing an exponential growth and nowadays research is being done actively in this area. Various generalizations of fuzzy graph theory are defined and studied by researchers in this field. Among them IVFGs introduced by Hongmei and Lianhua [3] in 2009 is a simple and very important generalization of fuzzy graph theory.

In the crisp graph theory, every arc is of same nature. But in fuzzy graph theory and interval - valued fuzzy graph theory, since different arcs have different

membership degrees, they differ in their nature. So study of different types of arcs is a significant topic of research in fuzzy graph theory as well as interval - valued fuzzy graph theory. The study of arcs in fuzzy graphs was initiated by Bhutani and Rosenfeld [2]. They defined strong arcs in fuzzy graphs. Later, Mathew and Sunitha [5] classified strong arcs into α strong arcs and β strong arcs. They also defined δ arcs. Different types of arcs in IVFGs were studied by the authors in [10]. In this paper we define a particular class of IVFGs, viz., strongly connected IVFGs depending upon the nature of arcs in a path joining any two nodes of an IVFG.

2. Preliminaries

Graph theoretic terms used in this work are either standard or are explained as and when they first appear. We consider only simple connected undirected graphs. Throughout the paper, we take $G = (A, B)$ as an IVFG on the crisp graph $G^* = (V, E)$. For all the IVFGs given in examples, nodes can be given any membership degree without violating the conditions of an IVFG. The concepts such as path, strongest path, unique strongest path, IVF bridge, IVF cutnode, weakest arc, weak arc etc. are as defined in [7] , [8] and [10].

Following are some definitions which are necessary for the understanding of the subsequent results.

Definition 2.1. *[1] Let $G^\star = (V, E)$ be a crisp graph. Then an interval - valued fuzzy graph (IVFG) G on $G^\star$ is defined as a pair $G = (A, B)$, where $A = [\mu_A^-(x), \mu_A^+(x)]$ is an interval - valued fuzzy set on V and $B = [\mu_B^-(xy), \mu_B^+(xy)]$ is an interval - valued fuzzy set on E such that $\mu_B^-(xy) \leq \min\{\mu_A^-(x), \mu_A^-(y)\}$ and $\mu_B^+(xy) \leq \min\{\mu_A^+(x), \mu_A^+(y)\}$ for all $xy \in E$.*
If $\mu_B^-(xy) = min(\mu_A^-(x), \mu_A^-(y))$ and $\mu_{B^+}(xy) = \min(\mu_{A^+}(x), \mu_{A^+}(y))$ for all $x, y \in V$, then G is called a complete IVFG(CIVFG).

Definition 2.2. *[6] The degree of a node u in an IVFG $G = (A, B)$ is defined as $d(u) = [d^-(u), d^+(u)]$ where $d^-(u) = \displaystyle\sum_{(u,v) \in E} \mu_B^-(u, v)$ is called the negative degree of $u \in V$ and $d^+(u) = \displaystyle\sum_{(u,v) \in E} \mu_B^+(u, v)$ is called the positive degree of $u \in V$.*

Definition 2.3. *[6] The total degree of a node u in an IVFG $G = (A, B)$ is defined as $td(u) = [td^-(u), td^+(u)]$ where $td^-(u) = \displaystyle\sum_{(u,v) \in E} \mu_B^-(u,v) + \mu_A^-(u)$ is called the negative total degree of $u \in V$ and $td^+(u) = \displaystyle\sum_{(u,v) \in E} \mu_B^+(u,v) + \mu_A^+(u)$ is called the positive total degree of $u \in V$.*

Definition 2.4. *[6] An IVFG $G = (A, B)$ is said to be a regular IVFG(RIVFG) if every node has the same degree.*

Definition 2.5. *[7] An IVFG G is said to be connected if any two nodes are joined by a path.*

Definition 2.6. *[7] The maximum of the μ^- strength (minimum of the μ_B^- values of the arcs in the path) of various paths connecting u and v is called the μ^- strength of connectedness between u and v and is denoted by $(\mu_{B-})^\infty(u,v)$ or $NCONN_G(u,v)$. The maximum of the μ^+ strength(minimum of the μ_B^+ values of the arcs in the path) of various paths connecting u and v is called the μ^+ strength of connectedness between u and v and is denoted by $(\mu_{B+})^\infty(u,v)$ or $PCONN_G(u,v)$.*

Definition 2.7. *[10] Two arcs e_1 and e_2 are said to be comparable if their membership degrees are such that either $\mu_{B-}(e_1) > \mu_{B-}(e_2)$ and $\mu_{B+}(e_1) > \mu_{B+}(e_2)$ or $\mu_{B-}(e_1) < \mu_{B-}(e_2)$ and $\mu_{B+}(e_1) < \mu_{B+}(e_2)$. They are said to be equal if their lower and upper membership degrees are equal.*

There are nine different types of arcs in an IVFG [10]. See the table 1 for various types of arcs (u,v) in an IVFG, G and their requirements to be of that type.

Definition 2.8. *[10] A path comprising only strong arcs is called a strong path.*

Definition 2.9. *[9] Let $G = (A, B)$ be a connected IVFG. Then G is called an interval - valued fuzzy tree(IVFT) if it has a spanning subgraph $F = (A, C)$ which is a tree and $\mu_B^-(u,v) < NCONN_F(u,v), \mu_B^+(u,v) < PCONN_F(u,v)$ for all arcs $(u,v) \notin F$.*

Definition 2.10. *[10] A path comprising only strong arcs is called a strong path.*

Definition 2.11. *[9] Let $G = (A, B)$ be a connected IVFG. Then G is called an interval - valued fuzzy tree(IVFT) if it has a spanning subgraph $F = (A, C)$ which*

is a tree and $\mu_B^-(u,v) < NCONN_F(u,v), \mu_B^+(u,v) < PCONN_F(u,v)$ *for all arcs* $(u,v) \notin F$.

Name	Requirement
α^- strong	$\mu_B^-(u,v) > NCONN_{G-(u,v)}(u,v)$
α^+ strong	$\mu_B^+(u,v) > PCONN_{G-(u,v)}(u,v)$
α strong	α^- strong and α^+ strong
β^- strong	$\mu_B^-(u,v) = NCONN_{G-(u,v)}(u,v)$
β^+ strong	$\mu_B^+(u,v) = PCONN_{G-(u,v)}(u,v)$
β strong	β^- strong and β^+ strong
$\alpha\beta$ strong	α^- strong and β^+ strong
$\beta\alpha$ strong	β^- strong and α^+ strong
Type I strong	α strong or β strong
Type II strong	$\alpha\beta$ strong or $\beta\alpha$ strong
Strong arc	Type I strong or Type II strong
δ^-arc	$\mu_B^-(u,v) < NCONN_{G-(u,v)}(u,v)$
δ^+arc	$\mu_B^+(u,v) < PCONN_{G-(u,v)}(u,v)$
δ arc(weak arc)	δ^-arc and δ^+arc
$\alpha\delta$	$\alpha^-strong$ and δ^+
$\beta\delta$	$\beta^-strong$ and δ^+
right feeble	$\alpha\delta$ or $\beta\delta$
$\delta\alpha$	δ^- and $\alpha^+strong$
$\delta\beta$	δ^- and $\beta^+strong$
left feeble	$\delta\alpha$ or $\delta\beta$
feeble arc	*left feeble or right feeble*

TABLE 1. Types of Arcs

Theorem 2.1. *[10] Let* $G = (A, B)$ *be an IVFG and* (u, v) *be an arc in* G. *Then* (u, v) *is a* weak *arc if and only if* G *contains atleast one cycle* C *whose unique weakest arc is* (u, v) .

Theorem 2.2. *[8] A CIVFG does not contain any feeble arcs or weak arcs.*

Theorem 2.3. *[9] Let $G = (A, B)$ be a connected IVFG on $G^* = (V, E)$, where G^* is not a tree. Then G is an IVFT if and only if G contains only α strong arcs and weak arcs.*

Theorem 2.4. *[12] Let $G = (A, B)$ be an IVFG on a graph $G^* = (V, E)$ such that $G^* = (V, E)$ is an odd cycle. Then G is a RIVFG if and only if $B = [\mu_B^-, \mu_B^+]$ is a constant function.*

Theorem 2.5. *[12] If an IVFG G is both edge regular and totally edge regular, then $B = [\mu_B^-, \mu_B^+]$ is a constant function.*

3. STRONGLY CONNECTED IVFGS

Strong arcs in fuzzy graphs were defined and discussed by Bhutani and Rosenfeld [2]. Analogous to this, strong arcs in IVFGs were defined by the authors in [10].

In [2], it is proved that every two nodes x and y of a connected fuzzy graph G, is joined by a strong path. But this is not in general true in the case of IVFGs which is clear from the following example.

Example 1. In the IVFG G given in Figure 1, we cannot find a strong path joining the nodes a and c.
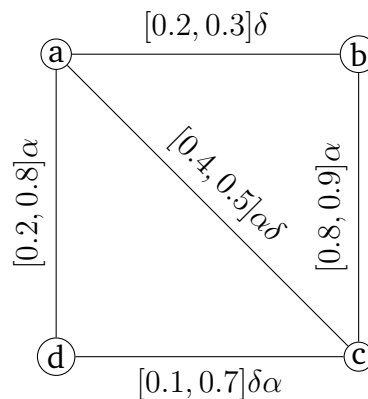

FIGURE 1. Example to show that strong paths need not exist between every two nodes of an IVFG

In this context, we define strongly connected IVFGs.

Definition 3.1. *A connected IVFG G is said to be strongly connected if there exists a strong path between every two nodes of G.*

Definition 3.2. *A path P in a connected IVFG G that contains combination of strong arcs and feeble arcs is called a partially strong path.*

Definition 3.3. *A path P in a connected IVFG G that contains feeble arcs alone is called a feeble path.*

Definition 3.4. *Suppose that $G = (A, B)$ is a strongly connected IVFG on $G^* = (V, E)$. The strong degree of a node $v \in V$ denoted by $d_s(v)$ is defined as $d_s(v) = [d_s^-(v), d_s^+(v)]$ where $d_s^-(v)$ and $d_s^+(v)$ are respectively the sum of the lower and upper membership degrees of all strong arcs incident at v.*

Definition 3.5. *Suppose that $G = (A, B)$ is a strongly connected IVFG on $G^* = (V, E)$. The partially strong degree of a node $v \in V$ denoted by $d_p(v)$ is defined as $d_p(v) = [d_p^-(v), d_p^+(v)]$ where $d_p^-(v)$ and $d_p^+(v)$ are respectively the sum of the lower and upper membership degrees of all feeble arcs incident at v.*

Based on the discussions above, we have the following obvious proposition.

Proposition 3.1. *Suppose that $G = (A, B)$ is a connected IVFG. Then there exists a strong path or partially strong path between any two nodes of G.*

By proposition 3.1, we can find either a strong arc or a feeble arc incident at each node of a non trivial connected IVFG. But if G is strongly connected, there exists atleast one strong arc incident at each node of G. Hence we have the following obvious propositions.

Proposition 3.2. *Suppose that $G = (A, B)$ is a non trivial connected IVFG on $G^* = (V, E)$. Then $0 < d_s(v) + d_p(v) \leq d(v)$ for all nodes $v \in V$.*

Proposition 3.3. *Suppose that $G = (A, B)$ is a non trivial SCIVFG on $G^* = (V, E)$. Then $0 < d_s(v) \leq d(v)$ for all nodes $v \in V$.*

Now we have another obvious proposition.

Proposition 3.4. *Suppose that $G = (A, B)$ is a SCIVFG defined on $G^* = (V, E)$. Then the sum of strong degrees of all nodes in V is equal to twice the sum of the membership degrees of all strong arcs in G.*

Theorem 3.1. *Every CIVFG is strongly connected.*

Proof. Suppose that $G = (A, B)$ is a CIVFG. Then by Theorem 2.2, G does not contain any feeble arcs or weak arcs. Hence there exist a strong path joining every two nodes of G. Hence G is strongly connected. ☐

Theorem 3.2. *Suppose that $G = (A, B)$ is a connected IVFG such that its arcs are either comparable or equal. Then G is strongly connected.*

Proof. Suppose that $G = (A, B)$ is an IVFG such that every two arcs are either comparable or equal. Then by Theorem 2.3, G contains only combinations of Type I strong arcs and weak arcs. Let u and v be any two nodes of G. Now there arises three cases.

Case 1: Arc (u, v) exists and is a Type I strong arc.

In this case, $P_1 = (u, v)$ is a strong path between u and v.

Case 2: Arc (u, v) exists and is a weak arc.

Here since (u, v) is a weak arc, by Theorem 2.1, (u, v) is the unique weakest arc of atleast one cycle in G and let C be one such cycle. Then $P_2 = C - (u, v)$ is a strong path between u and v.

Case 3: Arc (u, v) does not exists Then there exists three possibilities.

  (i) Both u and v belongs to a cycle.

 (ii) Any one of them belongs to a cycle.

(iii) None of them belongs to a cycle.

We consider the three possibilities one by one and in each case prove the needed.

  (i) Suppose both u and v belongs to a cycle say, C'. If C' contains a weak arc, consider that $u - v$ path which does not contain weak arcs. Obviously, that path contains only α strong arcs and hence it is a strong path between u and v.

 (ii) Suppose that u belongs to a cycle say, C^0 and v does not belong to any cycle. Now consider a $u-v$ path $P : u = u_1, u_2, u_3, \ldots, u_{n-1}, u_n = v$. Since G is connected, such a path exists. Let the path P be such that $w = u_i$ be the last node of P that belongs to C^0 and $u_{i+1}, u_{i+2} \ldots, u_{n-1}$ does not belong to any cycle. Then using the above arguments there exist a unique a $u - w$ path containing α strong arcs alone. Clearly, $(u_{i+1}, u_{i+2}), (u_{i+2}, u_{i+3}), \ldots, (u_{n-1}, u_n)$ are

α strong arcs as they are bridges and hence IVF bridges. This $u - w$ path together with the arcs $(u_{i+1}, u_{i+2}), (u_{i+2}, u_{i+3}) \ldots (u_{n-1}, u_n)$ form a strong $u - v$ path. Similarly, if v belongs to a cycle and u does not belong to any cycle, replacing u by v and v by u and arguing as above we obtain the required result.

(iii) Suppose both u and v does not belong to a cycle. Now consider a $u - v$ path $P : u = u_1, u_2, u_3, \ldots, u_{n-1}, u_n = v$. Since G is connected such a path exists. If the internal nodes of the path also does not belong to any cycle, all the arcs in the paths are bridges and hence IVF bridges or in otherwords α strong arcs. If some of the internal nodes belong to a cycle, using the above arguments, we can conclude that a strong $u - v$ path exists.

Thus in all the above three cases, there exists a strong path joining u and v. Hence G is strongly connected.

<div align="right">□</div>

Theorem 3.3. *IVFTs are strongly connected.*

Proof. Suppose that $G = (A, B)$ is an IVFT. Then there arises two cases.

<u>Case 1.</u> The underlying graph G^* is a tree.

In this case every arc will be an α strong arc and hence the theorem.

<u>Case 2.</u> The underlying graph G^* is not a tree.

In this case, by Theorem 2.3, G contains only α strong arcs and *weak* arcs. Let u and v be any two arbitrary nodes. If arc (u, v) is α strong, then obviously it is a strong path joining u and v. If arc (u, v) is a weak arc, then by Theorem 2.1, G contains atleast one cycle C whose unique weakest arc is (u, v). Then obviously $C - (u, v)$ will be a strong path joining u and v.

Thus in either cases there is a strong path joining every two nodes. Hence IVFTs are strongly connected.

<div align="right">□</div>

Theorem 3.4. *Suppose that $G = (A, B)$ is a RIVFG defined on an odd cycle. Then G is strongly connected.*

Proof. Suppose that $G = (A, B)$ is a RIVFG defined on an odd cycle. Then by theorem 2.4, $B = [\mu_{B^-}, \mu_{B^+}]$ is a constant function. Hence again from theorem, all the arcs are β strong. Thus G is strongly connected.

<div align="right">□</div>

Theorem 3.5. *Suppose that* $G = (A, B)$ *is a connected IVFG on a graph* $G^* = (V, E)$ *such that* G *is both edge regular and totally edge regular. Then* G *is strongly connected.*

Proof. Suppose that $G = (A, B)$ is a connected IVFG on a graph $G^* = (V, E)$ such that G is both edge regular and totally edge regular. Then by Theorem 2.5, $B = [\mu_B^-, \mu_B^+]$ is a constant function. Hence, all the arcs which belong to atleast one cycle are β strong. If there are any arcs which do not belong to any cycles, obviously they will be α strong. Thus there is a strong path joining every two nodes. Hence G is strongly connected. $\qquad\square$

Lemma 3.1. *Suppose that* $G_1 = (A_1, B_1)$ *and* $G_2 = (A_2, B_2)$ *are two IVFGs such that* $G_1 \cong G_2$. *Then,*

$$NCONN_{G_1}(u, v) = NCONN_{G_2}(f(u), f(v)) \quad and$$
$$PCONN_{G_1}(u, v) = PCONN_{G_2}(f(u), f(v))$$

where f is the isomorphism between G_1 and G_2.

Proof. Suppose that $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are two IVFGs such that $G_1 \cong G_2$ and f be the corresponding isomorphism between them. Let u and v be any two nodes of G and P_1 be a strongest $u - v$ path in G. Also, let P_2 be the corresponding image path in G_2. Then as f is an isomorphism, $\mu_{B_1^-}(u, v) = \mu_{B_2^-}(f(u), f(v))$, $\mu_{B_1^+}(u, v) = \mu_{B_2^+}(f(u), f(v))$ for all $(u, v) \in E_1$. Thus, μ^- and μ^+ strengths of P_1 will be same as those of P_2. Hence,

$$NCONN_{G_1}(u, v) = NCONN_{G_2}(f(u), f(v)) \quad and$$
$$PCONN_{G_1}(u, v) = PCONN_{G_2}(f(u), f(v)).$$

$$\square$$

Lemma 3.2. *Suppose that G_1 and G_2 are two IVFGs such that $G_1 \cong G_2$. Then the image of strong arc in G_1 is strong in G_2 and viceversa.*

Proof. Suppose that $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are two IVFGs such that $G_1 \cong G_2$ and f be the corresponding isomorphism between them. Also let (u,v) be a strong arc in G_1. Then by the definitions of strong arc, isomorphism and by Lemma 3.1, we have

$$
\begin{aligned}
\mu_{B_1^-}(u,v) \quad &\geq\quad NCONN_{G_1-(u,v)}(u,v)\\
&=\quad NCONN_{G_2-(f(u),f(v))}(f(u),f(v))\\
\mu_{B_2^-}(f(u),f(v)) \quad &\geq\quad NCONN_{G_2-(f(u),f(v))}(f(u),f(v))\\
\text{Similarly,}\quad&\\
\mu_{B_2^+}(f(u),f(v)) \quad &\geq\quad NCONN_{G_2-(f(u),f(v))}(f(u),f(v))\,.
\end{aligned}
$$

Now by the definition of a strong arc, $(f(u), f(v))$ is a strong arc in G_2. Hence the image of a strong arc in G_1 is a strong arc in G_2.

Conversely, let (x, y) be a strong arc in G_2. Then by the bijective and isomorphism property of f, its preimage is a strong arc in G_1.   □

Theorem 3.6. *Suppose G_1 and G_2 are two IVFGs such that $G_1 \cong G_2$. Then G_1 is strongly connected if and only if G_2 is strongly connected.*

Proof. Suppose that $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are two IVFGs on $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively. Then by the definition of a SCIVFG and from Lemma 3.2, G_1 is strongly connected $\Longleftrightarrow \exists$ a strong path $P_1 : u = u_1, u_2, \ldots, u_n = v$ between every two nodes u and v in $G_1 \Longleftrightarrow \exists$ a strong path $P_2 : f(u) = f(u_1), f(u_2), \ldots, f(u_n) = f(v)$ between every two nodes $f(u)$ and $f(v)$ in $G_2 \Longleftrightarrow G_2$ is strongly connected .   □

The results of Lemma 3.1 and Lemma 3.2 are true even if the IVFG G_1 is co - weak isomorphic to G_2. Thus we have the following theorem.

Theorem 3.7. *Suppose that $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are two IVFGs on $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ such that G_1 is co - weak isomorphic to G_2. Then G_1 is strongly connected if and only if G_2 is strongly connected.*

4. CONCLUSION

In this paper, we have shown that a strong path need not exist between every two nodes of an IVFG. Based on this we introduced strongly connected IVFGs. We obtained several sufficient conditions for an IVFG G to be strongly connected. Finally, we have proved that strong connectedness is preserved under isomorphism and co-weak isomorphism.

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