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A NOTE ON COVERING $L-$LOCALLY UNIFORM SPACES

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ABSTRACT. The purpose of this manuscript is to identify the topological spaces that generates a unique covering $L-$locally uniform spaces [1]. For this we have introduced the notion of strong completeness, compactness and totally boundedness in covering $L-$locally uniform spaces. Completeness and compactness are found equivalent in totally bounded covering $L-$locally uniform spaces. Further, we have shown that strong completeness satisfies closed hereditary property. Towards the end of this paper, we have shown that both the notion strong completeness and compactness satisfies the local uniform property and lastly compact regular $L-$topology generates unique covering $L-$locally uniform spaces.

1. INTRODUCTION

In topological spaces one cannot study properties such as uniform continuous, completeness and unform convergence, to study these, uniform structures were developed. Two equivalent structures were developed namely entourage Uniformity [2] and covering uniform spaces [3]. Efforts were made on to developed weaker spaces, wherein results in uniformity could possibility developed. As a result different generalisation of uniformity has been developed. One of the generalisation was developed by William [4] via localisation of the triangle

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axiom through entourage approach and it was characterised in term of covering by Vasudevan and Goel in [5] and many spectacular result were obtain.

The theory of uniform spaces on different categories of fuzzy topological spaces have been carried out by different authors viz, Hutton, Katsarsas, Lowen, Hu Cheng-Ming et.al. [6–9] in terms of entourage approached. However, the aspect of covering uniform space has consider by Soetens et.al [10], Chandrika et. al. [11, 12] and García et. al. [13]. Further, different generalisation of uniform spaces have been developed leads to theory and its applications in associated fields. The generalisation of uniform spaces, namely, $L-$locally uniform spaces through the entourage approached is carried out by Mitra and Hazarika [14,15]. In [14], an $L-$ topological space was shown to have compatible $L-$locally uniform spaces if and only if it is regular $L-$topological and many spectacular results were also obtain on Compactness, completeness and pseudo-metrisability.

In, [1] we generalised definition of García et al. [13] to introduced covering $L-$locally uniform spaces in the category $C-$**TOP**. Subsequently, many interesting results were obtain such as weakly uniformly continuous function, pseudo-metrisable in the context of covering $L-$locally uniform spaces. In this paper we, consider the problems of completeness and compactness in the context of developed notion covering $L-$locally uniform spaces [1]. In future paper, we will consider the problems of paracompactness and proximity relation in covering $L-$locally uniform spaces.

Throughout this paper $(L, \leq, \bigwedge, \bigvee)$ denotes a fuzzy lattice with order reversing involution $'$; 0_L and 1_L are respectively $\inf$ and $\sup$ in L. X is an arbitrary (ordinary) set and L^X denotes the collection of all mappings $A : X \to L$. Any member of L^X is an L-fuzzy set. The L-fuzzy sets $x_\alpha : X \to L$ defined by $x_\alpha(y) = 0_L$ if $x \neq y$ and $x_\alpha(y) = \alpha$ if $x = y$ are the L-fuzzy points. The mappings $A : X \to L$ and $B : X \to L$ defined by $A(x) = 1_L,\ \forall x \in X$ and $B(x) = 0_L,\ \forall x \in X$ are denoted by $\underline{1}$ and $\underline{0}$ respectively. For any $A,\ B \in L^X$, the union and intersection of A and B are defined as $A \cup B(x) = A(x) \vee B(x)$ and $A \bigcap B(x) = A(x) \wedge B(x)$ respectively. Further, we say that $A \subseteq B$ if and only if $A(x) \leq B(x)$ and $x_\alpha \in A$ if and only if $\alpha < A(x)$, where x_α is an L-fuzzy point; complement A' of A is defined as $A'(x) = A(x)'$. An L-topology $\mathbb{F}$ on L^X is a subset of L^X closed under finite intersection and arbitrary union. In this case, the pair $(L^X, \mathbb{F})$ is known as L-topological space. The elements of $\mathbb{F}$ are called open sets and its complements are called closed sets. For any $A \in L^X$, the interior and closure of A in L-topological space $(L^X, \mathbb{F})$, are respectively denoted by A^o and $\overline{A}$.

2. Preliminaries

This section includes basic definitions and results used in the main sections.

Definition 2.1. *[16] For any ordinary mapping $f : X \to Y$, the induced L-fuzzy mapping $f^{\to} : L^X \to L^Y$ and its L-fuzzy reverse mapping $f^{\leftarrow} : L^Y \to L^X$ respectively are defined as:*

$f^{\to}(A)(y) = \bigvee\{A(x) \mid x \in X,\ f(x) = y\},\ \ \forall\, A \in L^X,\ \forall\, y \in Y.$

$f^{\leftarrow}(B)(x) = B(f(x)),\ \ \forall\, B \in L^Y,\ \forall\, x \in X.$

Symbol $f^{\to}$ and $f^{\leftarrow}$ always denote $f^{\to}$ to be the L-fuzzy mapping induced from an ordinary mapping f and $f^{\leftarrow}$ is the L-fuzzy reverse mapping of $f^{\to}$. Both the L-fuzzy mappings $f^{\to}$ and $f^{\leftarrow}$ are order preserving. Also $f^{\to}$ is bijective iff f is bijective.

Theorem 2.1. *[16] Let L^X and L^Y be $L-$fuzzy spaces, $f : X \to Y$ an ordinary mapping. then $f^{\leftarrow}$ is bijective iff $f^{\leftarrow} \circ f^{\to} = id_{L^X}$, $f^{\to} \circ f^{\leftarrow} = id_{L^Y}$.*

Definition 2.2. *[16] For any $x_\alpha, A, B \in L^X$, x_α is said to be quasi-coincident with A, denoted as $x_\alpha \ll A$ if $x_\alpha \nsubseteq A'$, i.e.,$\alpha \nleq A'(x)$.*
A is called quasi-coincident with B at y if $A(y) \nleq B'(y)$. A is called quasi-coincident with B, denoted as $A\widehat{q}B$, if A quasi-coincident with B at some $y \in X$.

Definition 2.3. *[16] Let $x_\alpha \in Pt(L^X)$. Then an $L-$fuzzy set U is said to be a quasi-coincident neighbourhood (Q-nbd) at x_α in an $L-$topological space $(L^X, \mathbb{F})$, if there is $G \in \mathbb{F}$ such that $x_\alpha \ll G \subseteq U$.*
The family of all Q-nbd at x_α in an $L-$topological space $(L^X, \mathbb{F})$ is denoted by $\mathscr{Q}(x_\alpha)$

Definition 2.4. *[16] A subfamily $\mathscr{A} \subseteq \mathscr{Q}(x_\alpha)$ is called a Q-nbd base of x_α, if for every $U \in \mathscr{Q}(x_\alpha)$, there exits $V \in \mathscr{A}$ such that $V \subseteq U$.*

Theorem 2.2. *[16] Let $(L^X, \mathbb{F})$ be an $L-$topological space. Then for any $x_\alpha \in M(L^X)$, $\mathscr{Q}(x_\alpha)$ is a down-directed set in L^X and $\underline{0} \notin \mathscr{Q}(x_\alpha)$.*

Theorem 2.3. *[16] Let $(L^X, \mathbb{F})$ be an L-topological space and $A \in L^X$. Then an L-fuzzy point $x_\alpha \in \bar{A}$ iff each $Q-$nbd at x_α is quasi-coincident with A.*

Definition 2.5. *[16] Let $(L^X, \mathbb{F})$ be an $L-$fts, $A \in L^X$, $\mathscr{A}, \mathscr{B} \subset L^X$. $\mathscr{A}$ is called a cover of A, if $\bigcup \mathscr{A} \supseteq A$; particularly, if $\bigcup \mathscr{A} = \underline{1}$, then $\mathscr{A}$ is called a cover of $L-$topological space $(L^X, \mathbb{F})$. $\mathscr{B}$ is called subcover of $\mathscr{A}$, if $\mathscr{B} \subset \mathscr{A}$ and $\mathscr{B}$ is still a cover of A.*

Definition 2.6. *[9, 17] A non-empty sub collection $\mathscr{F}$ of L^X is said to be a filter in an L-topological space, if*

(F1) $\underline{0} \notin \mathscr{F}$.
(F2) $U_1, U_2 \in \mathscr{F} \Rightarrow U_1 \bigcap U_2 \in \mathscr{F}$.
(F3) $U \in \mathscr{F}$ *and* $V \in L^X$ *such that* $U \subseteq V$ *then* $V \in \mathscr{F}$.

$\mathscr{F}$ is said to be proper in $(L^X, \mathbb{F})$, if $\mathscr{F} \neq L^X$.

Definition 2.7. *[9,17] A subfamily $\mathscr{B}$ of L^X is called a filter base in an $L-$topological spaces, if*

(B1) $\underline{0} \notin \mathscr{B}$
(B2) *for any $U, V \in \mathscr{B}$, there exits $W \in \mathscr{B}$ such that $W \subseteq U \cap V$.*

Definition 2.8. *[9, 17] A non-empty sub collection $\mathscr{F}^*$ of L^X is said to be an $L-$fuzzy ultrafilter on L^X, if*

(U1) *For every $A \in L^X$, either $A \in \mathscr{F}^*$ or $A' \in \mathscr{F}^*$.*
(U2) *$A \bigcup B \in \mathscr{F}^*$ implies that either $A \in \mathscr{F}^*$ or $B \in \mathscr{F}^*$.*

An $L-$fuzzy filter $\mathscr{F}$ is $L-$fuzzy ultrafilter iff every $A \in L^X$, either $A \in \mathscr{F}$ or $A' \in \mathscr{F}$.

Definition 2.9. *[16] Let $x_\alpha \in L^X$ and $\mathscr{F}$ be a filter in $L-$topology, then $\mathscr{F}$ is said to be convergent to x_α, denoted by $\mathscr{F} \to x_\alpha$ if for any $U \in \mathscr{Q}(x_\alpha)$ there exits $F \in \mathscr{F}$ such that $F \subseteq U$, i.e., $\mathscr{Q}(x_\alpha) \subseteq \mathscr{F}$.*
An $L-$fuzzy point x_α is called a cluster point of $\mathscr{F}$, dentated by $\mathscr{F} \propto x_\alpha$ if every $U \in \mathscr{Q}(x_\alpha)$ and $F \in \mathscr{F}$, $U \bigcap F \neq \underline{0}$.

Definition 2.10. *[1] A non-empty family $\mathcal{U}$ of $L-$covers of L^X is said to be a covering $L-$ locally uniformity on L^X, if it satisfies the following axioms:*

(lc1) $\mathscr{A} \preccurlyeq \mathscr{B}, \mathscr{A} \in \mathcal{U} \Rightarrow \mathscr{B} \in \mathcal{U}$.
(lc2) *For every* $\mathscr{A}, \mathscr{B} \in \mathcal{U}, \mathscr{A} \bigcap \mathscr{B} \in \mathcal{U}$.

(lc3) *For each* $\mathscr{A} \in \mathcal{U}$ *and for all* $x_\alpha \in L^X$*, there exits* $\mathscr{B} \in \mathcal{U}$ *such that* $st(x_\alpha, st(\mathscr{B})) \subseteq st(x_\alpha, \mathscr{A})$.

Definition 2.11. *[1] Let* $(L^X, \mathcal{U})$ *and* $(L^X, \mathcal{V})$ *be two covering* $L-$*locally uniform spaces. Then a function* $f^{\rightarrow} : (L^X, \mathcal{U}) \rightarrow (L^X, \mathcal{V})$ *is called weakly uniformly continuous iff* $f^{\leftarrow}(\mathscr{C}) \in \mathcal{U}$*, whenever* $\mathscr{C} \in \mathcal{V}$*, where* $f^{\leftarrow}(\mathscr{C}) = \{f^{\leftarrow}(C) : C \in \mathscr{C}\}$.

Definition 2.12. *[15] Let* $f^{\rightarrow} : (L^X, \mathcal{U}) \rightarrow (L^Y, \mathcal{V})$ *be a function, then* $f^{\rightarrow}$ *is said to be weakly uniform isomorphism iff* $f^{\rightarrow}$ *is bijective and both* $f^{\rightarrow}$ *and* $f^{\leftarrow}$ *are weakly uniform continuous.*

3. Completeness In Covering $L-$locally Uniform Spaces

In this section, we introduce Cauchy filters and Strongly completeness and study about hereditary property and isomorphic in the context of covering $L-$locally uniform spaces.

Definition 3.1. *Let* $(L^X, \mathcal{U})$ *be a covering* $L-$*locally uniform spaces, then a filter* $\mathscr{F}$ *is called cauchy filter for each* $\mathscr{A} \in \mathcal{U}$*, there exits* $F \in \mathscr{F}$ *and* $A \in \mathscr{A}$ *such that* $F \subseteq A$.

Definition 3.2. *A filter* $\mathscr{F}$ *to be weakly Cauchy if each* $\mathscr{A} \in \mathcal{U}$*, there is a filter* $\mathscr{G}$ *containing* $\mathscr{F}$ *and* $G \in \mathscr{G}$ *such that* $G \subset A$*, for some* $A \in \mathscr{A}$.

Clearly, Cauchy filters are weakly Cauchy filters.

Definition 3.3. *A covering* $L-$*locally uniform space* $(L^X, \mathcal{U})$ *is said to be (strongly) complete if every (weakly) Cauchy filter in* $(L^X, \mathcal{U})$ *converges.*

Proposition 3.1. *Let* $\mathscr{A}$ *and* $\mathscr{B}$ *be* $L-$*covers and let* $A \in L^X$ *be* $L-$*fuzzy subset, then we have* $st(A, \mathscr{A} \bigcap \mathscr{B}) \subseteq st(A, \mathscr{A}) \bigcap st(A, \mathscr{B})$.

Lemma 3.1. *Let* $(L^X, \mathcal{U})$ *be a covering* $L-$*locally uniform spaces, then the collection* $\{st(x_\alpha, \mathscr{A}) : \mathscr{A} \in \mathcal{U}\}$ *is the family of all Q-nbd at* x_α *in* $(L^X, \mathcal{U}(\mathbb{F}))$.

Proof. By regularity for $x_\alpha \in L^X$ and $\mathscr{A} \in \mathcal{U}$, there exits an $L-$fuzzy open set G such that $x_\alpha \subseteq G \subseteq \overline{G} \subseteq st(x_\alpha, \mathscr{A})$. Which implies $x_\alpha \in \overline{G}$ and then by Theorem 2.3, $x_\alpha \ll G \subseteq st(x_\alpha, \mathscr{A})$. By Definition 2.3, we have $st(x_\alpha, \mathscr{A})$ is $Q-$nbd at x_α in $(L^X, \mathbb{F}(\mathcal{U}))$. $\square$

Theorem 3.1. *Convergent filter in covering $L-$locally uniform spaces is weakly Cauchy filter.*

Proof. Let $(L^X,\mathcal{U})$ be a covering $L-$locally uniform space and $\mathscr{F}$ be a filter such that for some $x_\alpha \in L^X, \mathscr{F} \to x_\alpha$ in $(L^X, \mathbb{F}(\mathcal{U}))$. Let $\mathscr{Q}(x_\alpha) = \{st(x_\alpha, \mathscr{A}) : \mathscr{A} \in \mathcal{U}\}$, then by Lemma 3.1, $\mathscr{Q}(x_\alpha)$ is $Q-$nbd in $\mathbb{F}(\mathcal{U})$. Since $\mathscr{F}$is convergent then by Definition 2.9, for any $U = st(x_\alpha, \mathscr{B}) \in \mathscr{Q}(x_\alpha)$, there exits $F \in \mathscr{F}$ such that $F \subseteq U$. Now let $\mathscr{G} = \{st(U, \mathscr{A}) : \mathscr{A} \in \mathcal{U}\}$, then $\mathscr{G} \neq \underline{0}$ and as by Proposition 3.1, $st(U, \mathscr{A} \bigcap \mathscr{B}) \subseteq st(U, \mathscr{A}) \bigcap st(U, \mathscr{B})$. Again by Definition 2.10, $(\mathscr{A} \bigcap \mathscr{B}) \in \mathcal{U}$, so $st(U, \mathscr{A} \bigcap \mathscr{B}) \in \mathscr{G}$, implies $\mathscr{G}$ is base for a filter. Also $F \subseteq U = st(x_\alpha, \mathscr{A}) \subseteq st(U, \mathscr{A})$ implies $\mathscr{F}$ is weakly Cauchy filter.

□

Definition 3.4. *Let $(L^X,\mathcal{U})$ be a covering $L-$locally uniform spaces and $A \in L^X$. Let for each $\mathscr{B} \in \mathcal{U}$ define $\mathcal{U}_A = \{A \bigcap B : B \in \mathscr{B} \in \mathcal{U}\}$. Then $\mathcal{U}_A$ is a covering $L-$locally uniform spaces on A which we call a sub covering $L-$locally uniform spaces on A and $(A,\mathcal{U}_A)$ said to be the subspace. $\mathcal{U}_A$ is open or closed sub covering $L-$uniform spaces according to $A \in \mathbb{F}(\mathcal{U})$ or $A' \in \mathbb{F}(\mathcal{U})$.*

Proposition 3.2. *Let $\mathscr{F}$ be a filter on a subspace of $(A,\mathcal{U}_A)$, then $\mathscr{F}$ is also filter on $(L^X,\mathcal{U})$.*

Proof. Let $(L^X,\mathcal{U})$ be a covering $L-$locally uniform space and let $A \in L^X$ be a $L-$fuzzy subset, then by Definition 3.4, $(A,\mathcal{U}_A)$ is a subspace. Suppose $\mathscr{F}$ be filter on $(A,\mathcal{U}_A)$, then for any $F \in \mathscr{F} \subset L^A$ implies $F \in \mathscr{F} \subset L^X$ as $A \in L^X$, so $\mathscr{F}$ is also filter on L^X. □

Lemma 3.2. *Let $\mathscr{F}$ be a weakly Cauchy filter in a covering $L-$locally uniform space $(L^X,\mathcal{U})$ and let $A \in L^X$. Then $\mathscr{F}_A = \{A \bigcap F : F \in \mathscr{F}\}$ is weakly Cauchy filter in $(A,\mathcal{U}_A)$.*

Proof. Let $\mathscr{F}$ be a weakly Cauchy in a covering $L-$locally uniform spaces $(L^X,\mathcal{U})$. Also let $A \in L^X$, then $\mathscr{F}_A = \{A \bigcap F : F \in \mathscr{F}\}$. Since $\mathscr{F}$ is weakly Cauchy filter there exits another filter $\mathscr{G}$(say) containing $\mathscr{F}$. So, $\mathscr{G}_A = \{A \bigcap G : G \in \mathscr{G}\}$ and $A \bigcap F \subseteq A \bigcap G$, as $F \subseteq G$, implies $\mathscr{G}_A$ is filter containing $\mathscr{F}_A$, therefore $\mathscr{F}_A$ is weakly Cauchy filter in $(A,\mathcal{U}_A)$. □

Theorem 3.2. *Every closed subspace in strongly complete covering $L-$locally uniform spaces is strongly complete.*

Proof. Let $(L^X,\mathcal{U})$ be a strongly complete covering $L-$locally uniform space. Let A is closed subset of L^X, and then by Definition 3.4, $(A,\mathcal{U}_A)$ be a closed subspace of $(L^X,\mathcal{U})$. Let $\mathscr{F}$ be a weakly Cauchy filter on $(A,\mathcal{U}_A)$, and then by proposition 3.2, $\mathscr{F}$ is weakly Cauchy filter on $(L^X,\mathcal{U})$. Since $(L^X,\mathcal{U})$ is strongly complete, so $\mathscr{F} \to x_\alpha \in L^X$. Since A is closed subset of L^X so we must have $x_\alpha \in A$. So, $\mathscr{F}$ is converges in $(A,\mathcal{U}_A)$. Hence $(A,\mathcal{U}_A)$ is strongly complete. □

Theorem 3.3. *Let $(L^X,\mathcal{U})$ and $(L^Y,\mathcal{V})$ be covering $L-$locally uniform spaces and $f^{\to} : (L^X,\mathcal{U}) \to (L^Y,\mathcal{V})$ be weakly uniform continuous. If $\mathscr{F}$ is weakly filter in $(L^X,\mathcal{U})$, then $f^{\to}(\mathscr{F})$ is weakly cauchy filter in $(L^Y,\mathcal{V})$.*

Proof. Let $\mathscr{F}$ be a weakly Cauchy filter in $(L^X,\mathcal{U})$ and let $\mathscr{C} \in \mathcal{V}$. Since $f^{\to} : (L^X,\mathcal{U}) \to (L^Y,\mathcal{V})$ be weakly uniform continuous, therefore $f^{-1}(\mathscr{C}) \in \mathcal{U}$, where $f^{-1}(\mathscr{C}) = \{f^{\leftarrow}(C) : C \in \mathscr{C}\}$. As $\mathscr{F}$ is a weakly Cauchy filter on $(L^X,\mathcal{U})$, then by Definition 3.2, there exits a filter $\mathscr{G}$ containing $\mathscr{F}$ such that $G \subseteq f^{\leftarrow}(A)$ for some $f^{\leftarrow}(A) \in f^{-1}(\mathscr{C}) \Rightarrow f^{\to}(A) \in \mathscr{C}$.Since $f^{\to}$ is order preserving and hence $f^{\to}(G) \subseteq f^{\to}(A)$. Hence $f^{\to}(\mathscr{G})$ is a filter containing $f^{\to}(\mathscr{F})$, with $f^{\to}(G) \subseteq f^{\to}(A)$. Which implies $f^{\to}(\mathscr{F})$ is weakly Cauchy filter on $(L^Y,\mathcal{V})$.

□

Theorem 3.4. *Let $(L^X,\mathcal{U})$ and $(L^Y,\mathcal{V})$ be two covering $L-$locally uniform spaces and $f^{\to} : (L^X,\mathcal{U}) \to (L^Y,\mathcal{V})$ be weakly uniform isomorphism, then $(L^X,\mathcal{U})$ is strongly complete iff $(L^Y,\mathcal{V})$ is strongly complete.*

Proof. $(\Rightarrow)$ Let $(L^Y,\mathcal{V})$ be a strongly complete and let $\mathscr{F}$ be a weakly Cauchy filter on $(L^X,\mathcal{U})$. Then by Theorem 3.3, $f^{\to}(\mathscr{F})$ is weakly Cauchy filter on $(L^Y,\mathcal{V})$. Therefore $f^{\to}(\mathscr{F})$ is converges to $x_\alpha \in L^Y$ being $(L^Y,\mathcal{V})$ be a strongly complete i.e., $f^{\to}(\mathscr{F}) \to x_\alpha \in L^Y$. Also $f^{\to} : (L^X,\mathcal{U}) \to (L^Y,\mathcal{V})$ being weakly uniform isomorphism, then by Definition 2.12, $f^{\leftarrow}$ is weakly uniform continuous, so $f^{\leftarrow}(f^{\to}(\mathscr{F}))$ is weakly Cauchy filter on $(L^X,\mathcal{U})$. Since $f^{\to}$ is an $L-$fuzzy homeomorphism being weakly uniform isomorphism and so, $f^{\leftarrow}(f^{\to}(\mathscr{F})) \to f^{\leftarrow}(x_\alpha)$ and being $f^{\to}$ bijective as weakly uniform isomorphism, therefore by Theorem 2.1, $f^{\leftarrow}(f^{\to}(\mathscr{F})) = \mathscr{F} \to f^{\leftarrow}(x_\alpha) \in L^X$ and hence $(L^X,\mathcal{U})$ is strongly complete.
$(\Leftarrow)$ It follows the other way implication.
Hence the theorem. □

4. Compactness and totally boundedness

In this section, we establish the equivalence of the compactness, completeness and uniqueness in the context of covering $L-$locally uniform spaces.

Definition 4.1. *[18] A family $\mathscr{A}$ of L^X has the finite intersection property if the intersection of the members of each finite subfamily of $\mathscr{A}$ is nonempty.*

Definition 4.2. *[16] An $L-$topology $(L^X, \mathbb{F})$ is called compact, if every open cover of $(L^X, \mathbb{F})$ has a finite subcover.*

Theorem 4.1. *[17] Let$(L^X, \mathbb{F})$ be an $L-$topology, then $(L^X, \mathbb{F})$ is compact iff*

(1) *Every open cover $\mathscr{C}$ of closed subset A of L^X has a finite subcover.*
(2) *Every closed collection with finite intersection property has non-empty intersection.*

Definition 4.3. *Let $(L^X,\mathcal{U})$be a covering $L-$locally uniform space then it is said to be totally bounded if for all $\mathscr{A} \in \mathcal{U}$ there is a finite $L-$fuzzy set F such that $st(F, \mathscr{A}) = \underline{1}$.*

Lemma 4.1. *If $f^{\rightarrow}$ is weakly uniform continuous then $\mathscr{A} \in f^{\rightarrow}(\mathcal{U})$ iff $f^{\leftarrow}(\mathscr{A}) \in \mathcal{U}$.*

Proof. Straight forward. □

Theorem 4.2. *Let $f^{\rightarrow} : (L^X,\mathcal{U}_1) \rightarrow (L^Y,\mathcal{U}_2)$, weakly uniformly continuous. If $(L^X,\mathcal{U}_1)$ compact then $(L^Y, f^{\rightarrow}(\mathcal{U}_1))$ is compact.*

Proof. Let $\mathscr{C}$ be an open covering of L^Y. Then $f^{\leftarrow}(\mathscr{C}) = \{f^{\leftarrow}(C) : C \in \mathscr{C}\}$ is open covering of L^X as $f^{\rightarrow}$ is weakly uniformly continuous. By compactness there exits a finite subcover of $f^{\leftarrow}(\mathscr{C})$, i.e., $\bigcup_{i=1}^{n} f^{\leftarrow}(C_i) = \underline{1}$, $n \in \mathbb{N}$ where $f^{\leftarrow}(C_i) \in f^{\leftarrow}(\mathscr{C})$. Now, by Lemma 4.1, $\bigcup_{i=i}^{n} C_i = \underline{1}$. Hence $(L^X, f^{\rightarrow}(\mathcal{U}_1))$ is compact.

□

Theorem 4.3. *Every compact covering $L-$locally uniform spaces is totally bounded.*

Proof. Let $(L^X,\mathcal{U})$ be a compact space. Then for any $\mathscr{A} \in \mathcal{U}$, the collection

$$\{st(x_\alpha, \mathscr{A}) : x_\alpha \in Pt(L^X)\}$$

is an open cover of $\underline{1}$.
Now, since $\underline{1}$ is closed. Therefore, by compactness, there exits finite subcover of

$$\{st(x_\alpha, \mathscr{A}) : x_\alpha \in Pt(L^X)\}$$

For some finite $L-$fuzzy points x_{α_i}, $1 \leq i \leq n$, $n \in \mathbb{N}$ such that $\bigcup_{i=1}^{n} st(x_{\alpha_i}, \mathscr{A}) = \underline{1} \Rightarrow st(\bigcup_{i=1}^{n} x_{\alpha_i}, \mathscr{A}) = \underline{1} \Rightarrow st(F, \mathscr{A}) = \underline{1}$, where $\bigcup_{i=1}^{n} x_{\alpha_i} = F$. Since F is finite and hence $(L^X, \mathcal{U})$ is totally bounded. □

Theorem 4.4. *In $(L^X, \mathcal{U})$ be a covering $L-$locally uniform spaces every ultra-filter is a weakly Cauchy filter.*

Proof. Let $\mathscr{F}^*$ be ultra-filter on $(L^X, \mathcal{U})$. Then the collection $\mathscr{B} = \{st(x_\alpha, \mathscr{A}) : x_\alpha \in Pt(L^X)\}$ is base for $\mathcal{U}$ by totally boundedness there is finite $L-$fuzzy points x_{α_i}, $1 \leq i \leq n$, $n \in \mathbb{N}$ such that $\bigcup_{i=1}^{n} st(x_{\alpha_i}, \mathscr{A}) = \underline{1}$. But $\underline{1} \in \mathscr{F}^*$, then by Definition 2.8, $F \in \mathscr{F}^*$ such that $F \subseteq st(x_\alpha, \mathscr{A})$ for some $x_\alpha \in Pt(L^X)$. Hence $\mathscr{B}$ is filter base containing $\mathscr{F}^*$, so $\mathscr{F}^*$ is a weakly Cauchy filter. □

Remark 4.1. *Every $L-$fuzzy filter has the finite intersection property.*

Lemma 4.2. *Let $(L^X, \mathcal{U})$ be a compact $L-$locally uniform spaces, then every filter has a cluster point.*

Proof. Let $(L^X, \mathcal{U})$ be compact and $\mathscr{F}$ be a filter. Then $\mathscr{G} = \{\overline{F} : F \in \mathscr{F}\}$ is closed filter. Then by Theorem 4.1 and Remark 4.1, $\mathscr{G}$ has a finite intersection property with non-empty intersection, i.e., $\bigcap \mathscr{G} \neq \underline{0}$. This implies there exits $x_\alpha \in L^X$ such that $x_\alpha \in \bigcap \overline{F} \Rightarrow x_\alpha \in \overline{F}$, with $F \in \mathscr{F}$. Hence x_α is a cluster point of $\mathscr{F}$. □

Theorem 4.5. *Every compact covering $L-$locally uniform spaces is a strongly complete.*

Proof. Let $(L^X, \mathcal{U})$ be a compact covering $L-$locally uniform spaces. Let $\mathscr{F}$ be a weakly Cauchy filter on $(L^X, \mathcal{U})$, then for each $\mathscr{A} \in \mathcal{U}$ there exits a filter $\mathscr{G}$ containing $\mathscr{F}$ with $G \in \mathscr{G}$ such that $G \subset A$ for some $A \in \mathscr{A}$. Also for $A \in \mathscr{A}$ there exits $F \in \mathscr{F}$ such that $F \subseteq A$ as $\mathscr{F}$ is Cauchy filter. By Theorem 4.2, $\mathscr{G}$ is has a cluster point, i.e., there exits $x_\alpha \in L^X$ such that $x_\alpha \in \bigcap \overline{G}, G \in \mathscr{G}$. This implies $x_\alpha \in \overline{G}$, then by Theorem 2.3, $st(x_\alpha, \mathscr{A})\widehat{q}G$ [Since the family of Q-nbd at x_α is $\mathscr{Q}(x_\alpha) = \{st(x_\alpha, \mathscr{A}) : \mathscr{A} \in \mathcal{U}\}$]. Also $st(x_\alpha, \mathscr{A}) \bigcap G \neq \underline{0}$, there exits $y_\beta \in L^X$ such that $y_\beta \in st(x_\alpha, \mathscr{A}) \bigcap G \Rightarrow y_\beta \in st(x_\alpha, \mathscr{A})$ and $y_\beta \in G$. Now also we have, $y_\beta \in st(y_\beta, st(\mathscr{A})) \Rightarrow G \subseteq st(y_\beta, st(\mathscr{A})) \subseteq (x_\alpha, \mathscr{A})$ [as $y_\beta \in st(x_\alpha, \mathscr{A})$]. Now since $\mathscr{F}$ is weakly Cauchy filter for each $F \in \mathscr{F}$ with $F \subseteq G$, $F \subseteq G \subseteq st(x_\alpha, \mathscr{A}) \in \mathscr{Q}(x_\alpha)$. Which implies $\mathscr{F}$ converges at x_α. Hence strongly complete.

□

Theorem 4.6. *Let $(L^X, \mathcal{U})$ be a covering $L-$locally uniform space, then the space is compact iff*

1. $(L^X, \mathcal{U})$ *is totally bounded, and*
2. $(L^X, \mathcal{U})$ *is strongly complete.*

Proof. $(\Rightarrow)$ Let $(L^X, \mathcal{U})$ be a compact covering $L-$locally uniform spaces then by Theorem 4.3 (i) $(L^X, \mathcal{U})$ is totally bounded and
Theorem 4.5 (ii) $(L^X, \mathcal{U})$ is strongly complete.
$(\Leftarrow)$ Let $(L^X, \mathcal{U})$ be a totally bounded and strongly complete covering $L-$locally uniform space. Let $\mathscr{A} \in \mathcal{U}$ be an open cover, then by totally boundedness there exits finite $L-$ fuzzy set F such that $st(F, \mathscr{A}) = \underline{1}$. For each $x_{\alpha_i} \in F$ we consider one A_i for some $x_\alpha \in A_i \in \mathscr{A}$. Then it is clear that $\bigcup A_i = \underline{1}$ implies $\{A_i\}$ is a finite subcover of $\mathscr{A}$ as F is finite $L-$fuzzy set. Hence $(L^X, \mathcal{U})$ is compact covering $L-$locally uniform space. □

Thus in a totally bounded covering $L-$locally uniform spaces, compactness and strong completeness are equivalent.

Definition 4.4. *[16] Let $(X, \mathbb{F})$ be an $L-$ topological space.$(L^X, \mathbb{F})$ is regular, if every $U \in \mathbb{F}$, there exits $\mathscr{V} \subseteq \mathbb{F}$ such that $\bigcup \mathscr{V} = U$ and $\overline{U} \subseteq U$ for every $V \in \mathscr{V}$.*

Lemma 4.3. *Let $(X, \mathbb{F})$ be a regular $L-$ topological space. For each open cover $\mathscr{U}$ such that there exits an open cover $\mathscr{V}$ such that $\overline{\mathscr{V}} \preccurlyeq \mathscr{U}$.*

Proof. Straight forward. □

Theorem 4.7. *Let $(L^X, \mathbb{F})$ compact regular $L-$topolgy. Then the $L-$topology generates a unique covering $L-$uniform space.*

Proof. Let $\mathcal{U}$ and $\mathcal{U}^*$ two covering $L-$locally uniform spaces on $(L^X, \mathbb{F})$ for the compact regular $L-$topological spaces. Let $\mathscr{A} \in \mathcal{U}$, then there exits finite subcover say $\{A_i : 1 \leq i \leq n\}$ also by Lemma 4.3 there exits a covering $\mathscr{B} \in \mathcal{U}$ such that $cl(\mathscr{B}) \preccurlyeq \{A_i : 1 \leq i \leq n\}$ i.e, for each i there exits some $cl(B) \in cl(\mathscr{B})$ such that $cl(B) \subseteq A_i$. Let k be a positive integer such that $k \leq n$. For each $x_\alpha \in cl(B) \subseteq A_k$ there exits $\mathscr{A}^* \in \mathcal{U}^*$ with $st(x_\alpha, st(\mathscr{B}^*)) \subseteq st(x_\alpha, \mathscr{A}^*)$ for some $\mathscr{B}^* \in \mathcal{U}^*$. Put $\mathscr{A}_k^* = \{st(x_\alpha, \mathscr{A}^*) : x_\alpha \in cl(B) \subseteq A_k\}$, since $cl(B)$ is compact so, $\mathscr{A}_k^*$ has finite sub cover $\mathscr{C}_k^*$. For each $\mathscr{A}^* \in \mathcal{U}^*$ there is a $\mathscr{A}_k^*$ such that

$st(x_\alpha, \mathscr{A}) \in \mathscr{D}_k^*$ for each $x_\alpha \in cl(B) \subseteq A_k$. which implies $st(x_\alpha, \mathscr{C}_k^*) \subseteq A_k$ for each $x_\alpha \in cl(B) \subseteq A_k$.

Next we choose $\mathscr{A}^* \in \mathcal{U}^*$ such that $\mathscr{A}^* \preccurlyeq \mathscr{A}_k^*$ for each $k = 1, 2, 3 \ldots n$. Let $x_\alpha \in L^X$, then $x_\alpha \in cl(B) \subseteq A_j$ for some $j \leq n$. Therefore $x_\alpha \in st(x_\alpha, \mathscr{A}_k^*) \subseteq A_k$. Consequently $\mathscr{A}^* \preccurlyeq \{A_i : 1 \leq i \leq n\}$. But then $\mathscr{A} \in \mathcal{U}$, we conclude that $\mathcal{U} \subset \mathcal{U}^*$. Similarly $\mathcal{U}^* \subset \mathcal{U}$. Hence the theorem. □

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