

A NEW APPROACH FOR THE OPTIMIZATION OF PORTFOLIO SELECTION PROBLEM IN FUZZY ENVIRONMENT

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ABSTRACT. This paper presents a new approach for the optimization of portfolio selection (PS) problem in fuzzy environment. The PS problem is considered with data represented by piecewise quadratic fuzzy numbers. One of the good approximations of intervals, namely close interval approximation is used for piece-wise quadratic fuzzy numbers. To optimize the approximation of closed intervals for the objective function, the order relations empowered by the preferences of policy maker are implemented on the left limit, right limit, center and on the interval-width. The minimization problem is converted into multi-objective optimization problem which can be solved using the Weighting Tchebycheff program for obtaining the optimal compromise solution. The applicability and effectiveness of the suggested solution approach is illustrated through two examples.

1. INTRODUCTION

The process of portfolio selection (PS) is undertaken to create satisfaction of the securities or derivatives included in the portfolio. In a similar way, the PS problem is to choose a portfolio for securities (or assets) which gives the investor with a pre-decided expected value of return and minimizes the risk.

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One of severe shortcomings that occurs with the applicability of mathematical program is that the coefficients in the derivation do not have the fixed value. However, they keep on fluctuating and show the uncertainty. The existence of vague, ambiguity, imprecision, and uncertainty on the securities returns, leads to the difficulty to decide which should be selected. As a result of uncertainty of the parameters and variables of the problem, the accurate value of the return of every security is not possible to pre-specify in well advance.

The theory of fuzzy sets, proposed by Zahdeh [2], in 1965, has been applied in different disciplines like business, engineering, natural sciences, and management of financial risk. It follows the fuzziness concept, allowing the description and treatment of any elements of uncertainty in a problem. The concept is so crucial that it could be used to introduce uncertainty and imperfection of financial markets' behavior into the matrix as fuzzy quantities in the portfolio. The numerical value of fuzzy data is expressed by means of fuzzy subsets on real numbers, referred as fuzzy numbers.

1.1. Related works. In 1952, Markowitz [1] described the first work of portfolio selection problem with applications in financial risk management, engineering, and business. Fuzzy set theory developed by Zadeh [2] was further studied by several researchers. Dubois and Prade [4] proposed the extension of the applicability of algebraic operations on real numbers to fuzzy numbers. The processing time of a job vary in many ways and may be lead to different work places. To avoid these factors, the processing time of a job are represented in the form of piecewise quadratic fuzzy numbers. Chankong and Hamies [5] presented the theory and methodology for the multi objective decision making problems. Tanaka and asai [6] proposed the fuzzy LPP with fuzzy numbers. Kaufmann and Gupta [7] presented the fuzzy mathematical formulation of various problems in engineering with several applications in finance and management science. Ishibuchi and Tanaka [8] investigated the multiple objective programming in optimization assuming the objective function as of the interval number. Tanaka et al. [9] gave the proposition of fuzzy possibilities and possibilities distribution as the two best models for portfolio selection.

Several researchers presented their research work on interval numbers and their applications in various domains of science and engineering (Moore [3],

Alefeld and Mayer [10]). In later years, relevant studies Para et al. [11], Sngupta et al. [12], Lai et al. [13], and Ida [14] improved upon the portfolio selection in terms of large number of applications, using linear programming, and interval numbers. Li and Tian [25] presented a numerical solution technique for general interval quadratic programming. Deng et al. [30] introduced an investment PS problem related to the satisfaction index of interval inequality. The PS model by Bhattacharyya et al. [32] was fuzzy mean-variance-skewness, while Liu's proposal was based on mean-absolute deviation as the PS problem with interval-valued returns. Chen et al. [34] investigated the robust portfolio optimization with interval random uncertainty set. Alefeld and Herzberger [36] presented a study on the introduction to interval computation. Alolyan [38] developed an algorithm for interval linear programming involving interval constraints. Wu et al. [40] presented a study on interval portfolio selection problem. Li and Qin [41] derived some interval PS problems under the framework of theory of uncertainty. Xu et al. [46] presented the solution methodology for non-linear optimization problems with interval analysis. Liu et al. [48] studied the multi-period cardinality constrained portfolio selection models with interval coefficients.

Ida [15] presented the solutions of PS problem with the coefficients represented by fuzzy and interval type uncertainty. Lu et al. [16] studied the LLP model based on interval numbers for the PS problem. Papahristodoulou and Dotzauer [17] investigated the PS problem, and used the LPP models. Giove et al. [18] developed a PS problem with intervals by the use of regret function. Huang [19] studied PS model by using the fuzzy returns. Lobo et al. [20] investigated the optimization of PS problem with linear costs of transportation. Fernandez and Gomez [21] studied the PS problem with the applicability of artificial neural network. Mansini et al. [22] presented the conditional value at risk and related LPP for portfolio optimization. Ammar [23] presented a study on the determination of optimal solution of quadratic programming with multiple objective functions in fuzzy-random environment. They further studied the applications in portfolio problem. Li and Tsai [24] proposed a distributed computation algorithm for solving portfolio problems with integer variables. Tiriyaki and Ahlatcioglu [26] presented a fuzzy portfolio selection problem using fuzzy analytic hierarchic process. Li and Xu [27] derived a new PS model with hybrid uncertainty. Bonami and Lejeune [28] development of portfolio selection was

established under stochastic and integer constraints by introducing an exact solution approach. Jain [29] presented a study on close interval approximation for the piecewise quadratic fuzzy number in case of fuzzy fractional programming problem. Anagnostopoulos and Mamanis [31] developed a five multiple objective evolutionary algorithms based on experimental work. They also presented the mean-variance cardinality constrained portfolio optimization problem. Castro et al. [33] derived an algebraic approach to integer portfolio problems.

Bermudez et al. [37] formulated the PS problem as a tri-objective problem to determine the tradeoffs between return, risk, and the number of securities in PS problem. Cesarone et al. [39] developed a novel methodology for mean-variance PS problem with cardinality constraints. Khalifa and ZeinEldin [42] studied PS problem with fuzzy objective function coefficients, and applied fuzzy programming approach to obtain the α -optimal compromise. Dutta and Kumar's [43] application was based on a fuzzy goal programming that was tailored to the model of multi-objective linear fractional inventory. Qin [44] studied the mean-variance model for portfolio optimization problem in the simultaneous presence of random and uncertain returns. Nazemi et al. [45] proposed the solution approach for portfolio selection models with uncertain returns using an artificial neural network scheme. Huang and Di [47] studied the uncertain portfolio selection with background risk. Zhai and Bai [49] investigated the uncertain portfolio selection with background risk and liquidity constraint. Yan et al. [50] made a proposal for an interval PS model that could be used in banking institutions. Zhou and Xu [51] presented the portfolio selection and risk investment under the hesitant fuzzy environment. Vaezi et al. [52] made a suggestion for a knapsack problem PS model established under uncertainty. Their main findings are the development of portfolio model with uncertainty.

1.2. Our contributions. We first attempt to present a PS problem in fuzzy environment. Thereafter, a new approach is proposed for the optimization purpose. To the best of our expertise, this approach is the first time used for the solution of PS problem that undertakes the piecewise quadratic fuzzy numbers. Our contributions are as follows:

- (i) We proposed a new approach for solving a PS problem by using the piecewise quadratic fuzzy numbers.

- (ii) The close interval approximation of piecewise quadratic fuzzy number is used.
- (iii) The order relations are defined by the right limit, the left limit, the center and the width of an interval.
- (iv) The multi objective optimization problem is solved using the Weighting Tchebycheff Program (Chankong and Haimes [5]) for obtaining the optimal compromise solution.

1.3. Paper organization. The remainder of the manuscript is organized as follows. In the next section, basic concepts and arithmetic operations related to piecewise quadratic fuzzy numbers and the operations related to computations are introduced. Section 3 introduces notations and assumptions needed in the problem formulation. Section 4 defines portfolio selection problem. In section 5, two numerical examples to illustrate the approach are introduced. In the last, some concluding remarks are mentioned in section 6.

2. PRELIMINARIES AND DEFINITIONS

The section presents some of basic definitions and results regarding fuzzy number, piecewise quadratic fuzzy number as well as their mathematical operations.

Definition 2.1. (Jain [29]). A piecewise quadratic fuzzy number (PQFN) is denoted by $\tilde{a}_{PQ} = (a_1, a_2, a_3, a_4, a_5)$, where $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$ are real numbers, and is defined by if its membership function $\mu_{\tilde{a}_{PQ}}$ is given by

$$\mu_{\tilde{a}_{PQ}} = \begin{cases} 0, & x < a_1; \\ \frac{1}{2} \frac{1}{(a_2 - a_1)^2} (x - a_1)^2, & a_1 \leq x \leq a_2; \\ \frac{1}{2} \frac{1}{(a_3 - a_2)^2} (x - a_3)^2 + 1, & a_2 \leq x \leq a_3; \\ \frac{1}{2} \frac{1}{(a_4 - a_3)^2} (x - a_3)^2 + 1, & a_3 \leq x \leq a_4; \\ \frac{1}{2} \frac{1}{(a_5 - a_4)^2} (x - a_5)^2, & a_4 \leq x \leq a_5; \\ 0, & x > a_5. \end{cases}$$

Definition 2.2. (Jain [29]). An interval approximation $[A] = [a_\alpha^L, a_\alpha^U]$ of a PQFN $\tilde{A}$ is called closed interval approximation if $a_\alpha^L = \inf\{x \in \mathbb{R} : \mu_{\tilde{A}} \geq 0.5\}$, and $a_\alpha^U = \sup\{x \in \mathbb{R} : \mu_{\tilde{A}} \geq 0.5\}$.

Definition 2.3. Associated ordinary number (Jain [29]). If $[A] = [a_\alpha^L, a_\alpha^U]$ is close interval approximation of PQFN, the associated ordinary number of $[A]$ is defined as $\tilde{A} = \frac{a_\alpha^L + a_\alpha^U}{2}$.

Definition 2.4. (Jain [29]). Let $[A] = [a_\alpha^L, a_\alpha^U]$, and $[B] = [b_\alpha^L, b_\alpha^U]$ be two interval approximations of PQFN. Then the arithmetic operations are:

- 1- Addition: $[A] \oplus [B] = [a_\alpha^L + b_\alpha^L, a_\alpha^U + b_\alpha^U]$,
- 2- Subtraction: $[A] \ominus [B] = [a_\alpha^L - b_\alpha^L, a_\alpha^U - b_\alpha^U]$,
- 3- Scalar multiplication: $\alpha[A] = \begin{cases} [\alpha a_\alpha^L, \alpha a_\alpha^U], & \alpha > 0; \\ [\alpha a_\alpha^U, \alpha a_\alpha^L], & \alpha < 0. \end{cases}$,
- 4- Multiplication: $[A] \otimes [B], \left[\frac{a_\alpha^U b_\alpha^L + a_\alpha^L b_\alpha^U}{2}, \frac{a_\alpha^L b_\alpha^L + a_\alpha^U b_\alpha^U}{2} \right]$,
- 5- Division: $[A] \oslash [B], \begin{cases} \left[2 \left(\frac{a_\alpha^L}{b_\alpha^L + b_\alpha^U} \right), 2 \left(\frac{a_\alpha^U}{b_\alpha^L + b_\alpha^U} \right) \right], & [B] > 0, b_\alpha^L + b_\alpha^U \neq 0, \\ \left[2 \left(\frac{a_\alpha^U}{b_\alpha^L + b_\alpha^U} \right), 2 \left(\frac{a_\alpha^L}{b_\alpha^L + b_\alpha^U} \right) \right], & [B] < 0, b_\alpha^L + b_\alpha^U \neq 0, \end{cases}$

It is noted that $P(\mathbb{R}) \subset F(\mathbb{R})$, where $F(\mathbb{R})$, and $P(\mathbb{R})$ are the sets of all piece-wise quadratic fuzzy numbers and close in interval approximation of PQFN, respectively.

3. NOTATION AND ASSUMPTIONS

3.1. Notations: The notions needed in the problem formulation

S_j : securities ($j = 1, \dots, n$),

r_j : Return for S_j ,

x_j : Proportion of total investment funds,

r^0 : Average vector of returns over m periods,

Q : Covariance matrix ($Q = q_{ij}^2$), where $q_{ij}^2 = \sum_{k=1}^m (r_{ki} - r_i^0)(r_{kj} - r_j^0)$, $i, j = \overline{1, n}$,

r_{av} : Average return,

$\mathbb{R}$: set of real numbers.

3.2. Assumptions:

- The portfolio selection problem in fuzzy environment is considered.

- The future returns of assets (or securities) are represented as piecewise quadratic fuzzy numbers.
- The piecewise quadratic fuzzy number are introduced with their close interval approximation.

4. PROBLEM STATEMENT

Consider the following close interval approximation portfolio selection model: (Problem-I)

$$Z = \min_x (x^t [Q] x)$$

subject to

$$x \in M = \left\{ x^t [r^0] = [r_a v], \quad \sum_{j=1}^n x_j = 1, \quad x_j \geq 0 \right\}.$$

Here,

$$[Q] = [q_{ij}^2] = ((q_{ij}^2)_\alpha^L, (q_{ij}^2)_\alpha^U),$$

$$[r^0] = ((r_j^0)_\alpha^L, (r_j^0)_\alpha^U), \quad \text{and}$$

$$[r_{av}] = ((r_{av})_\alpha^L, (r_{av})_\alpha^U) \in P(\mathbb{R}).$$

Here, M denotes the set of all feasible solutions of Problem-I, and $P(\mathbb{R})$ denotes the set of all piecewise quadratic fuzzy numbers.

Definition 4.1. (Ishibuchi and Tanaka [8]). the order relation ($\leq_{UC}$) between $[A] = [a_\alpha^L, a_\alpha^U]$, and $[B] = [b_\alpha^L, b_\alpha^U]$ is defined as:

$$[A] \leq_{UC} [B] \Leftrightarrow a_\alpha^U \leq b_\alpha^U, \quad \text{and} \quad a_\alpha^C \leq b_\alpha^C,$$

$$[A] <_{UC} [B] \Leftrightarrow [A] \leq_{UC} [B], \quad \text{and} \quad [A] \neq [B],$$

where

$$a_\alpha^C = \frac{1}{2}(a_\alpha^L + a_\alpha^U).$$

Definition 4.2. $x_j \in M$ is a solution of Problem-I if and only if there is no $\hat{x}_j \in M$ such that

$$Z(\hat{x}_j) <_{UC} Z(x_j).$$

The solution of Problem-I is defined according to Definition 4.1 and can be obtained by solving the following bi-objective programming problem (Problem-II)

$$\begin{aligned} &\text{Main}\{Z_U, Z_C\} \\ &\text{subject to} \\ &x \in M. \end{aligned}$$

In Problem-II, Z_U and Z_C are respectively the upper bound and the center of Z , where Z is given by Problem-I.

Definition 4.3. A point $\hat{x} \in M$ is said to be an efficient solution of Problem (II) if and only if there does not exist another $x \in M$ such that $Z_U(x) \leq Z_U(\hat{x})$ or $Z_C(x) \leq Z_C(\hat{x}) \neq Z_U(\hat{x})$ or $Z_C(x) \neq Z_C(\hat{x})$.

Therefore, the Problem-II can be treated using the weighting Tchebycheff Program (Chankong and Haimes [5]), in the form of a single objective quadratic programming problem, as in Problem-III:

$$\text{Problem-III)} \quad \left\{ \begin{array}{l} \text{Min } \gamma, \\ \text{subject to} \\ w_1(Z_U(x) - Z_U^*) \leq \gamma, \\ w_2(Z_C(x) - Z_C^*) \leq \gamma, \\ x \in M, \\ w_1 \geq 0, \\ w_2 \geq 0, \\ Z_U^* \text{ and } Z_C^* \text{ are the ideal targets.} \end{array} \right.$$

5. PROPOSED APPROACH

The steps of the approach to determine the best compromise solution of Problem-III illustrated as:

Input: Dividend payments and Investment stock with fuzzy return for all mutual funds.

Step 1: Consider Problem-*I*.

Step 2: Convert Problem-*I* into Problem-*II* as in Definition 4.1.

Step 3: Solve the following two problems as mentioned in Equation (5.1) and (5.2):

$$(5.1) \quad \begin{cases} Z_U^* = \text{Min } Z_U \\ \text{subject to} \\ x \in M, \end{cases}$$

$$(5.2) \quad \begin{cases} Z_C^* = \text{Min } Z_C \\ \text{subject to} \\ x \in M. \end{cases}$$

Step 4: Calculate the individual maximum and minimum for Z_U , and Z_C under the given constraints, respectively.

Step 5: Calculate the weights from the following Equation (5.3):

$$(5.3) \quad \begin{cases} w_1 = \frac{\bar{Z}_U - \underline{Z}_U}{(\bar{Z}_U - \underline{Z}_U) + (\bar{Z}_C - \underline{Z}_C)}, \\ w_2 = \frac{\bar{Z}_C - \underline{Z}_C}{(\bar{Z}_C - \underline{Z}_C) + (\bar{Z}_U - \underline{Z}_U)}. \end{cases}$$

In equation (5.2), $\bar{Z}_U$, and $\bar{Z}_C$ are the individual maximum; $\underline{Z}_U$, and $\underline{Z}_C$ are individual minimum, respectively.

Step 6: Apply Problem-*III* for Problem-*II* with the help of step 3, and step 4.

Step 7: Using the MATLAB software, we obtain the solution of Problem-*I*, which is the optimal compromise solution.

Output: Optimal compromise solution.

The flowchart of the proposed approach is presented as in Figure 1.

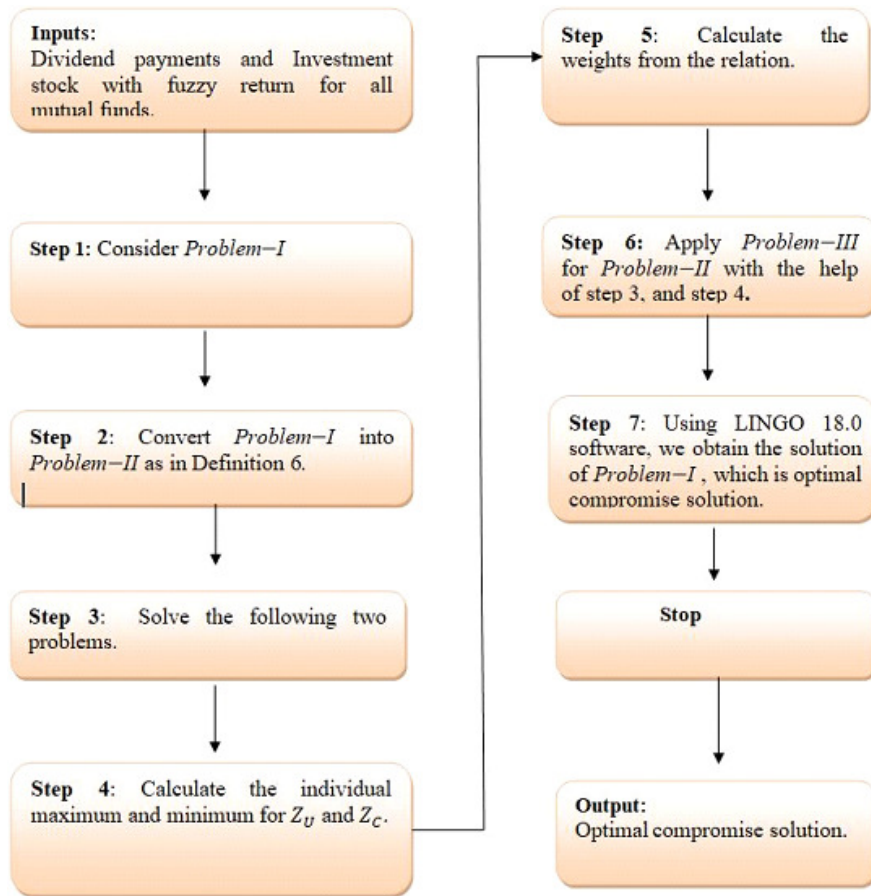


FIGURE 1. The flowchart of the proposed approach.

6. NUMERICAL EXAMPLES

In this section, we solve two numerical examples.

Example 1. Consider a company who identified two mutual funds for investment as attractive opportunities. Over the last four years, the dividend payments (\$) are shown in Table 1:

Funds	Years			
	1 st	2 nd	3 rd	4
Inv. 1	(10, 11, 12, 13, 14)	(7, 8, 9, 10, 11)	(14, 15, 16, 17, 18)	(18, 19, 20, 21, 22)
Inv. 2	(5, 6, 7, 8, 9)	(12, 13, 14, 15, 16)	(16, 17, 18, 19, 20)	(20, 21, 22, 23, 24)

TABLE 1. Investment (Inv.) stock with fuzzy returns (Input data)

Based on Definition 2.2 Table 1 as in Table 2 below.

Funds	Years			
	1 st	2 nd	3 rd	4
Investment 1	[11, 13]	[8, 10]	[15, 17]	[19, 21]
Investment 2	[6, 8]	[13, 15]	[17, 19]	[20, 23]

TABLE 2. Investment stock with fuzzy closed interval approximation return

The expectation and covariance estimations for Z_U are as in Table 3.

	x_{i1}	x_{i2}	x_{i1}^2	x_{i2}^2	$x_{i1}x_{i2}$
1	13	8	169	64	104
2	10	15	100	225	150
3	17	19	289	361	323
4	21	23	441	625	483
Sum	61	65	999	1275	1060

TABLE 3. Expectation and Covariance estimation for Z_U

$$r_1^0 = 15.25, \quad r_2^0 = 16.25$$

$$Q_U = q_{ij}^2 = \begin{pmatrix} -60.64 & 53.4 \\ 53.4 & 86.0 \end{pmatrix}$$

Then,

$$Z_U = \text{Min } (-60.64x_1^2 + 86x_2^2 + 106.8x_1x_2)$$

subject to

$$15.25x_1 + 16.25x_2 \leq 10000,$$

$$x_1 + x_2 = 1, \quad x_2 \geq 0.$$

Applying MATLAB software,

$$\overline{Z}_U = +86.0000 \quad \text{for} \quad x_1 = 0.0000, \quad x_2 = 1.0000$$

and

$$\underline{Z}_U = -60.6400 \quad \text{for} \quad x_1 = 1.0000, \quad x_2 = 0.0000$$

	x_{i1}	x_{i2}	x_{i1}^2	x_{i2}^2	$x_{i1}x_{i2}$
1	12	7	144	49	84
2	9	14	81	196	126
3	16	18	256	324	288
4	20	22	400	484	440
Sum	57	61	881	1053	938

TABLE 4. Expectation and Covariance estimation for Z_C

$$r_1^0 = 14.25, \quad r_2^0 = 15.25$$

$$Q_C = q_{ij}^2 = \begin{pmatrix} 46.24 & 48.52 \\ 48.52 & 61.75 \end{pmatrix}$$

Step 3:

$$Z_C = \text{Min } (46.24x_1^2 + 61.75x_2^2 + 97.04x_1x_2)$$

subject to

$$14.25x_1 + 15.25x_2 \leq 10000,$$

$$x_1 + x_2 = 1, \quad x_2 \geq 0.$$

Applying MATLAB software,

$$\overline{Z}_C = +61.7500 \quad \text{for} \quad x_1 = 0.0000, \quad x_2 = 1.0000$$

and

$$\underline{Z}_C = +46.2400 \quad \text{for} \quad x_1 = 1.0000, \quad x_2 = 1312369E - 7 = 0.1312369 \times 10^{-7}.$$

Step 5:

$$w_1 = \frac{\overline{Z}_U - \underline{Z}_U}{(\overline{Z}_U - \underline{Z}_U) + (\overline{Z}_C - \underline{Z}_C)} = 0.90435,$$

$$w_2 = \frac{\overline{Z}_C - \underline{Z}_C}{(\overline{Z}_U - \underline{Z}_U) + (\overline{Z}_C - \underline{Z}_C)} = 0.09565.$$

Step 6: Solve the following problem

Min γ

subject to

$$(54.8398x_1^2 - 77.7741x_2^2 - 96.585116x_1x_2 + \gamma \geq 54.8398),,$$

$$(4.4229x_1^2 + 5.9064x_2^2 + 9.2819x_1x_2 - \gamma \leq 4.422856),$$

$$15.25x_1 + 16.25x_2 \leq 10000,$$

$$14.25x_1 + 15.25x_2 \leq 10000,$$

$$x_1 + x_2 = 1, \quad x_1, x_2 \geq 0.$$

The solution is: $\gamma = 0.4400000E - 04 = 0.4400000 \times 10^{-4} = 0.000044$, $x_1 = 1.000000$ and $x_2 = 0.000000$.

Example 2. Consider an investor who identified three mutual funds as attractive opportunities. Over the last five years, the divided payments (\$) are shown in Table 5:

Funds	Years				
	1	2	3	4	5
Inv. 1	(3, 5, 8, 10, 12)	(0, 1, 3, 4, 5)	(7, 9, 11, 13, 15)	(8, 10, 12, 13, 15)	(1, 3, 5, 6, 8)
Inv. 2	(1, 3, 5, 6, 8)	(6, 7, 8, 9, 10)	(1, 3, 5, 6, 7)	(1, 2, 4, 5, 6)	(5, 6, 8, 9, 10)
Inv. 3	(13, 15, 16, 17, 19)	(0, 1, 2, 3, 4)	(6, 8, 10, 11, 12)	(15, 17, 18, 19, 21)	(0, 1, 2, 3, 5)

TABLE 5. Investment (Inv.) stock with fuzzy returns (Input data)

According to the Definition 2.2, the data of Table 5 transform to the following form as shown in Table 6

Funds	Years				
	1	2	3	4	5
Inv. 1	[5, 10]	[1, 4]	[9, 13]	[10, 13]	[3, 6]
Inv. 2	[3, 6]	[7, 9]	[3, 6]	[2, 5]	[6, 9]
Inv. 3	[15, 17]	[1, 3]	[8, 11]	[17, 19]	[1, 3]

TABLE 6. Investment (Inv.) stock with closed interval approximation return

The expectation covariance estimations for Z_U are as in Table 7

	x_{i1}	x_{i2}	x_{i3}	x_{i1}^2	x_{i2}^2	x_{i3}^2	$x_{i1}x_{i2}$	$x_{i1}x_{i3}$	$x_{i2}x_{i3}$
1	10	6	17	100	36	289	60	170	102
2	4	9	3	16	81	9	36	12	27
3	13	6	11	169	36	121	78	143	66
4	13	5	19	169	25	25	361	65	95
4	6	9	3	36	81	9	54	18	27
Sum	46	35	53	490	259	789	293	437	317

TABLE 7. Expectation and Covariance estimation for Z_U

We have $r_1^0 = 9.2$, $r_2^0 = 7$, $r_3^0 = 10.6$, and the Coovariance matrix is given by

$$Q_U = q_{ij}^2 = \begin{pmatrix} 13.36 & -5.8 & -10.12 \\ -5.8 & 2.8 & -10.8 \\ -10.12 & -10.8 & 45.44 \end{pmatrix}$$

Step 3:

$$Z_U = \text{Min}(13.36x_1^2 + 2.8x_2^2 + 45.44x_3^2 - 11.6x_1x_2 - 20.24x_1x_3 - 21.6x_2x_3)$$

subject to

$$9.2x_1 + 7x_2 + 10.6x_3 \leq 12000,$$

$$x_1 + x_2 + x_3 = 1,$$

$$x_1, x_2, x_3 \geq 0.$$

Step 4: Applying MATLAB software,

$$\bar{Z}_U = 45.44, \quad \text{for } x_1 = 0, \quad x_2 = 0, \quad x_3 = 1.0$$

and

$$\underline{Z}_U = -1.5878, \quad \text{for} \quad x_1 = 0.2560, \quad x_2 = 5.832, \quad x_3 = 0.1607.$$

Similarly, the expectation and covariance estimations for Z_C are as shown in Table 8.

	x_{i1}	x_{i2}	x_{i3}	x_{i1}^2	x_{i2}^2	x_{i3}^2	$x_{i1}x_{i2}$	$x_{i1}x_{i3}$	$x_{i2}x_{i3}$
1	7.5	4.5	16	56.25	20.25	256	33.75	120	72
2	2.5	8	2	6.25	64	4	20	5	16
3	11	4.5	9.5	121	20.25	90.25	49.5	104.5	42.75
4	11.5	3.5	18	132.25	12.25	324	40.25	207	63
4	4.5	7.5	2	20.25	56.25	4	33.75	9	15
Sum	37	28	47.5	336	173	678.25	177.25	445.5	208.75

TABLE 8. Expectation and Covariance estimation for Z_C

We have $r_1^0 = 7.4$, $r_2^0 = 5.6$, $r_3^0 = 9.5$, and the covariance matrix is given by:

$$Q_U = q_{ij}^2 = \begin{pmatrix} 12.44 & -5.99 & 18.8 \\ -5.99 & 3.24 & -11.45 \\ -18.8 & -11.45 & 45.4 \end{pmatrix}.$$

Then,

$$Z_C = \text{Min}(12.44x_1^2 + 3.24x_2^2 + 45.44x_3^2 - 11.98x_1x_2 + 37.6x_1x_3 - 22.9x_2x_3)$$

subject to

$$7.4x_1 + 5.6x_2 + 9.5x_3 \leq 12000,$$

$$x_1 + x_2 + x_3 = 1,$$

$$x_1, x_2, x_3 \geq 0.$$

Similarly, solving above problem, we obtain

$$\bar{Z}_C = 45.44, \quad \text{for} \quad x_1 = 0, \quad x_2 = 0, \quad x_3 = 1.0$$

and

$$\underline{Z}_C = 0.00913705, \quad \text{for} \quad x_1 = 0.1916334, \quad x_2 = 0.7088373, \quad x_3 = 0.09952934.$$

Step 5: we calculate the weight as follows:

$$w_1 = \frac{\overline{Z}_U - \underline{Z}_U}{(\overline{Z}_U - \underline{Z}_U) + \overline{Z}_C - \underline{Z}_C} = 0.5086,$$

and

$$w_2 = \frac{\overline{Z}_C - \underline{Z}_C}{(\overline{Z}_U - \underline{Z}_U) + \overline{Z}_C - \underline{Z}_C} = 0.4913.6$$

Step 6: We formulate the following programming problem:

Min γ

subject to

$$(-13.36x_1^2 - 2.8x_2^2 - 45.44x_3^2 + 11.6x_1x_2 + 20.24x_1x_3 + 21.6x_2x_3 + 1.9662\gamma \geq 1.5878),$$

$$(-12.44x_1^2 - 3.24x_2^2 - 45.44x_3^2 + 11.98x_1x_2 - 37.6x_1x_3 + 22.9x_2x_3 + 2.0352\gamma \geq 0.00913705),$$

$$28x_1 + 21x_2 + 42x_3 \leq 60000,$$

$$46x_1 + 35x_2 + 53x_3 \leq 60000,$$

$$x_1 + x_2 + x_3 = 1,$$

$$x_1, x_2, x_3 \geq 0.$$

The solution is: $\gamma = 0.0725863$ for $x_1 = 0.2122611$, $x_2 = 0.6574734$, $x_3 = 0.1302655$. This is the optimal compromise solution for the individual, having three mutual funds as attractive opportunities.

7. CONCLUSION

In this research work, the portfolio selection problem involving the data represented by piecewise quadratic fuzzy numbers was introduced. The proposed problem is formulated using the close interval approximation of piecewise quadratic fuzzy numbers. As a result, the Weighting Tchebycheff Program was applied to solve the corresponding optimization problem for the optimal compromise solution. The proposed approach is significantly useful when the financial managers encountering a difficulty due to ambiguity in the data.

Finally, as future research, one may consider the extension of proposed model to chaotic numbers as well as neutrosophic numbers. To handle the uncertainty of the financial markets more effectively, the hybrid solution approaches which make the fusion of theory of fuzzy sets and theory of probability will be an interesting further research direction of the proposed study.

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