

**BIJECTION BETWEEN PERMUTATIONS IN HYPEROCTAHEDRAL GROUP
OF TYPE B_N AND THE SYMMETRIC ALTERNATIVE TABLEAUX**A. TAMILSELVI¹ AND R. ANITHA

ABSTRACT. In this paper, we analyse the alternative tableaux corresponding to the permutations in hyperoctahedral group of type B_n (denoted by B_n) using exchange fusion and exchange delete algorithms and obtain symmetric alternative tableaux. Also we discuss the reverse algorithms and establish a bijection between permutations in B_n and the symmetric alternative tableaux of size $2n$.

1. INTRODUCTION

Xavier Viennot introduced the concept of alternative tableaux in [3] and established a bijection between permutations in S_n and alternative tableaux of size $n - 1$ through different algorithms in [4]. These algorithms were based on advance and going backward elements of a permutation in S_n . We extend these algorithms from S_n to B_n , by defining advance and going backward elements of negative numbers in B_n and adding steps to do these algorithms. The type B_n version of alternative tableaux are called symmetric alternative tableaux, defined by Nadeau in [2]. We define symmetric alternative tableaux from [1], by replacing left and down arrows by blue and red colored dots. Thus establishing a bijection between permutations in B_n and the symmetric alternative tableaux of size $2n$.

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2. PRELIMINARIES

Definition 2.1. A Ferrers diagram is a finite collection of boxes or cells, arranged in left-justified rows, with length of the rows $\lambda_1 \geq \dots \geq \lambda_n$ from bottom to top.

Definition 2.2. In a Ferrers diagram if there exists horizontal or vertical extended lines then it is called an extended Ferrers diagram. That is, extended Ferrers diagram is a Ferrers diagram possibly with empty rows or columns.

The following figures are examples for the extended Ferrers diagram.

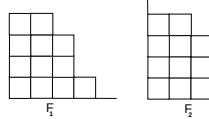


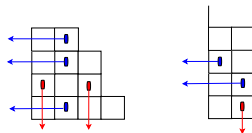
FIGURE 1

Size of the extended Ferrers diagram $n = \# \text{ rows} + \# \text{ columns}$ (empty rows columns should be considered). In Figure 1, size of F_1 is 9 and size of F_2 is 8.

Definition 2.3. Alternative tableaux is an extended Ferrers diagram such that some cells are colored with blue or red dots and it satisfies the following conditions.

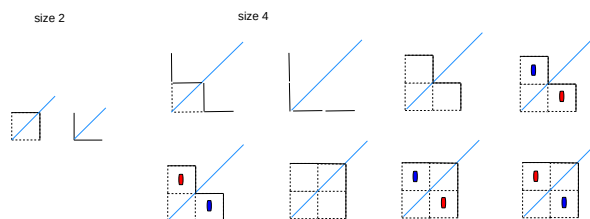
- (1) No colored cell to the left of a blue cell;
- (2) No colored cell below a red cell.

The number of alternative tableaux of size n is $(n + 1)!$. The diagrams given below are examples for alternative tableaux .



Definition 2.4. A symmetric alternative tableaux is an alternative tableaux obtained by interchanging colored dots in the reflection of the given alternative tableaux with respect to its main diagonal.

Note 2.5. Let $n = 1$ and $n = 2$, then the symmetric alternative tableaux of size 2 and size 4 are given below:



Hence the number of symmetric alternative tableaux of size $2n$ is $2^n n!$.

Definition 2.6. The hyperoctahedral group of type B_n is a subgroup of S_{2n} , symmetric group on $2n$ symbols, such that $B_n = \{\mu \in S_{2n} : \mu(i) + \mu(-i) = 0\}$ where $2n = \{-n, -n+1, \dots, -1, 1, \dots, n\}$. The number of elements in B_n is $2^n n!$.

Definition 2.7. Let $\mu \in B_n$ be the given permutation and let $x = \mu(i)$. If $x > 0$ then x is said to be an advance element of μ if $x+1 = \mu(j)$ with $i < j$, that is in the permutation $x+1$ is on the right side of x . If $x < 0$ then x is said to be an advance element of μ if $x-1 = \mu(j)$ with $i < j$. That is in the permutation $x-1$ is on the right side of x , where $i, j \in \{-n, -n+1, \dots, -1, 1, \dots, n\}$. By convention $x = -n$ is an advance element of μ .

Definition 2.8. Let $\mu \in B_n$ be the given permutation and let $x = \mu(i)$. If $x > 0$ then x is said to be a going backwards element of μ if $x+1 = \mu(j)$ with $j < i$, that is in the permutation $x+1$ is on the left side of x . By convention $x = n$ is a going backwards element of μ . If $x < 0$ then x is said to be a going backwards element of μ if $x-1 = \mu(j)$ with $j < i$. That is in the permutation $x-1$ is on the left side of x , where $i, j \in \{-n, -n+1, \dots, -1, 1, \dots, n\}$.

Example 1. Let $\mu = \begin{pmatrix} -3 & -2 & -1 & 1 & 2 & 3 \\ 1 & -2 & -3 & 3 & 2 & -1 \end{pmatrix}$. Here 1, -2, -3 are advance elements of μ and 3, 2, -1 are going backward elements of μ .

3. EXCHANGE FUSION ALGORITHM FOR B_n AND ITS REVERSE

3.1. Exchange fusion algorithm. Let $\mu \in B_n$ be the given permutation. Let $a_1, \dots, a_k$ be the advance elements and $b_1, \dots, b_l$ be the going backward elements of μ such that $k+l = 2n$. Draw a diagram of $2n$ vertices in such a way that:

- Place $2n$ vertices in first row.
- Mark the vertices of the first row labelled by advance elements of μ as blue dot with label a_i and going backward elements of μ by red dot with label b_j .
- First l vertices of second row are red dots and the remaining k vertices are blue dots.
- Join the corresponding labels by curves in such a way that:
 - join the label blue dots (red dots) in first row and the label blue dots (red dots) in the second row by a blue colored curve (red curve);

– the curve with same color do not intersect each other.

- Note 3.1.** (1) *The number of points we obtained from the intersection of blue colored and red colored curve gives the number of cells in the alternative tableau of μ .*
- (2) *The number of red colored curves give the number of rows and the total number of blue colored curves give the number of columns of the Ferrers diagram.*
- (3) *The number of cells in the i^{th} column is equal to the number of intersection of i^{th} blue colored curve with all red colored curves and the number of cells in the j^{th} row is equal to the number of intersection of j^{th} red colored curve with all blue colored curves.*

Now draw the Ferrer's diagram with l -rows and k -columns. Fill the entries of the alternative tableaux by the following steps.

- Step 1:** Select any pair of vertices such that blue dot is followed by red dot and bring the numbers down through the curves.
- Step 2:** If both the numbers are positive then follow the same steps as in exchange fusion algorithm for S_n [4].
- Step 3:** If the numbers are different in sign then the numbers will meet at the intersection of blue and red colored curves and follow the respective colored curves without assigning colored dots.
- Step 4:** If both the numbers are negative and are not consecutive, then the numbers will meet at the intersection of blue and red colored curves and follow the respective colored curves without assigning colored dots.
- Step 5:** If both the numbers are negative and consecutive, then at the intersection of blue and red colored curves assign a blue or red dot according to the color of the smallest number. Label the new dot as a block with entries as consecutive numbers and the new block will follow the respective colored curve only.
- Step 6:** When a pair of blocks having negative numbers meet at the intersection of a blue and red colored curves then assign a colored dot, if the union of the two blocks is formed through consecutive numbers. Label the dot with a new block by concatenating the blocks, assign the color of the block having smallest letters and the new block will follow the

same colored curve. If not, the blocks meet at the intersection and will follow the respective colored curves.

Continue the steps until all the $a_1, \dots, a_k$'s and $b_1, \dots, b_l$'s reach the bottom. The resulting alternative tableau is a symmetric alternative tableau of size $2n$.

3.2. Reverse exchange fusion algorithm for B_n . Consider a symmetric alternative tableau of size $2n$ with $2n = k + l$ where l be the number of rows and k be the number of columns. Construct a diagram with k blue and l red curves in such a way that i^{th} blue curve should intersect as much red curves as the cells in the i^{th} column of the alternative tableau and j^{th} red curve should intersect as much blue curves as the cells in the j^{th} row of the alternative tableau. If there is a colored dot in the $(i, j)^{th}$ position in the given alternative tableau, then assign the same coloured dot at the intersection of i^{th} blue line and j^{th} red line. Delete all the extended blue curves from the red dots and all the extended red curves from the blue dots. The resulting diagram will be a tree form with blue and red curves as roots. Roots are labelled as follows:

- Start labelling the roots which are red in color and label the left most root as $n \ n - 1 \ \dots \ n - (m - 1)$, where m is the number of branches of that root. Similarly label the next root starting with the number $n - m$ and continue this labelling till the end of roots which are in color red.
- Label the roots which are blue in color in such a way the left most root is labelled as $-1 \ -2 \ -3 \ \dots \ -p$ where p is the number of branches of the that root. Similarly label the next root starting with the number $-(p + 1)$ and continue this labelling till the end of roots which are in color blue.

Move the labels to the top of the curves by following the steps given below:

Step 1: If the labels are positive follow the same steps as in reverse exchange fusion algorithm for S_n [4].

Step 2: If the labels are negative, while moving if we meet a blue dot then move the labels of lesser denomination to blue curve and the label of higher denomination to the red curve. If we meet a red dot then move labels of lesser denomination to the red curve and the label of higher denomination to the blue curve.

Continue the steps until all labels reach top of the curves which will result in required permutation.

4. EXCHANGE DELETE ALGORITHM AND ITS REVERSE

4.1. Exchange delete algorithm. Let $\mu \in B_n$ be the permutation. Let $a_1, \dots, a_k$ be the advance elements and $b_1, \dots, b_l$ be the going backward elements of μ such that $k + l = 2n$. Draw a diagram of $2n$ vertices and the Ferrers diagram as described in exchange fusion algorithm. In Exchange delete algorithm, the steps to fill the entries in alternative tableau is similar to exchange fusion algorithm till Step 4. Note that in step 2, we follow exchange delete algorithm for S_n [4]. Remaining steps are as follows:

Step 5: If both the numbers are negative and consecutive, then at the intersection of the blue and red colored curves assign a blue or red dot according to the color of the smallest number. Label the dot with the smallest number and delete the highest number from the permutation.

Step 6: After deletion, two numbers are consecutive in the sense that there does not exist any numbers in between the two values of the permutation.

Continue the steps until all the $a_1, \dots, a_k$'s and $b_1, \dots, b_l$'s reach the bottom except the deleted numbers. The resulting alternative tableau is a symmetric alternative tableau of size $2n$.

4.2. Reverse exchange delete algorithm. Consider a symmetric alternative tableau of size $2n$ with $2n = k + l$ where l be the number of rows and k be the number of columns. Construct a diagram with k blue and l red curves by following the same method as in reverse exchange fusion algorithm for B_n . Note that there is no need to delete the extended curves from colored dots since tree form is not required. Label the curves in such a way that

- First take the left most red curve without blue dots at the bottom and label it as n .
- Next label the second red curve without blue dots with the number $n - 1$. Continue this labeling till the end of the red curves without blue dots.
- Label similarly the blue curves without red dots beginning from the left most curve starting with the next number.

Move the labels to the top of the curves by following the steps given below:

Step 1: While moving if the label meet a blue or red dot then the corresponding label splits into two consecutive numbers by subtracting 1 from the label.

Step 2: If the labels after splitting are positive then we follow the same method as in the same algorithm for S_n .

Step 3: If the labels after splitting are negative then smallest label move to the curve which has same colour of the dot and the highest label move to the other coloured curve.

Step 4: If the resultant label is already existing in the labels then subtract 1 from the previous same label and subtract 1 from the labels with lesser denomination and consider -1 and 1 are consecutive.

Continue the steps until all labels reach top of the curves which will result in required permutation.

From the above algorithms we conclude that

Theorem 4.1. *There exists a bijection between permutations in B_n and the symmetric alternative tableaux of size $2n$.*

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