

APPLICATION OF HOMOTOPY PERTURBATION METHOD IN TERMS OF 4TH ORDER BVP PROBLEMS

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ABSTRACT. This paper comprises the solution of integral equations with the fourth-order BVPs through the HPM. The outcome of the study is that it can be obtained in a rapidly convergent series. For this, the implementation technique is quite easy and more reliable than the other methods.

1. INTRODUCTION

If a nonlinear problem encompasses singularities or multiple solutions, then the whole process can have numerical difficulties. The noticeable thing is that numerical and analytical methods used to solve the nonlinear problems have both advantages and disadvantages. Hence we cannot give importance to one by neglecting the other. To solve nonlinear problems, we have some analytical techniques like perturbation techniques [1], [2], [3], [4], which are used widely. When this perturbation technique is employed, many significant characteristics and existing facts of nonlinear problems have come out.

Various analytical and computational approaches have been used in recent years to solve integral equations, including perturbation and decomposition approaches. The most powerful tools for the non-linear study of engineering problems are perturbation approaches. The main disadvantage of the traditional perturbation approach is the fact that small parameters are over-dependent.

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2010 *Mathematics Subject Classification.* 45K05, 45Q05.

Key words and phrases. BVP, Homotopy Perturbation Method, Fourth order.

This condition is excessively stringent and affects the perturbation techniques applications significantly. Some non-linear problems do not even involve the so-called low setting; however, it is complicated to evaluate the specific parameter and require different techniques. Such results helped to establish alternative approaches such as an interpretation method for homotopies, decomposition, and the iteration process for variation. To cope up these advantages, the usual homotopy method is merged with perturbation, and through this, we get a new approach, which is specially regarded as homotopy perturbation approach. Noor and Mohyud-Din [5] carried out their research by using this method to solve BVPs in the fifth order. We use the Noor and Mohyud-Din principle to construct a homotopy perturbation method to solve integrated equations in a system linked to problems with the 4th order BVP problems. This method seems to be effective for giving solution of rapid convergence. Also, this method is reliable to use and found effective than the Adomian method [6], [7]. It is noticeable that to use this Adomian method, the derivatives of Adomian polynomial should be determined, which is very complicated. To cope with this, we can use the homotopy perturbation technique.

Here, general problems with the 4th order BVP are considered.

Here, we select the general BVPs of fourth-order which looks like as follows

$$y^{(4)}(x) = f(t, y, y', y'', y'''),$$

having the boundary condition

$$y(m) = A_1, y'(m) = A_2, y(n) = B_1, y'(n) = B_2,$$

where f is a function which is continuous on $[m, n]$, A_i and B_i are constant parameters which are real in nature and $i = 1, 2, 3, \dots$

In general these kinds of boundary value problems are seen in visco-elastic flow statistical models, the deformity of the plates, the theory of plate deflection, and in the field of scientific, engineering and physical science. For solving problems of the 4th boundary value, many computational methods were developed, including the finite differences and spline. Noor and mohydu-Din solve the 4th order BVPs through the use of variational iteration. Here to solve this, we try to use the HPM merged with the integral equations.

2. MATHEMATICAL FORMULATION AND DISCUSSION

Here, we choose some examples of 4th order BVPs and try to solve these by HPM. We discuss the details formulation of HPM and the obtained results are described below:

Example 1. *The following nonlinear initial BVP are taken into consideration:*

$$y^{(4)} = y^2 - t^{10} + 4(t^9 - t^8 - t^7 + 2t^6 - t^4 + 30t - 12)$$

and

$$y(0) = 0, y(1) = A, y'(0) = 0, y'(1) = B$$

are the boundary conditions.

The exact solution of the above problem is

$$y(t) = t^5 - 2t^4 + 2t^2.$$

We use the transformation:

$$y_t = q(t), q_t = f(t), f_t = z(t),$$

$$z_t = y^2 - t^{10} + 4(t^9 - t^8 - t^7 + 2t^6 - t^4 + 30t - 12)$$

with

$$z(0) = B, q(0) = 0, y(0) = 0, f(0) = A.$$

Using above transformation, we are able to express the equation in the form of a system of equations:

$$(2.1) \quad y(t) = 0 + \int_0^t q(g)dg, q(t) = 0 + \int_0^t f(g)dg, f(t) = A + \int_0^t z(g)dg,$$

$$z(t) = B + \int_0^t (y^2(g - g^10 + 4g^9 - 4g^8 - 4g^7 + 8g^6 - 4g^4 + 120g - 48))dg.$$

Using HPM in the equation(2.1) we get

$$y_0 + py_1 + p^2y_2 + \dots = 0 + p \int_0^t (v_0 + pv_1 + p^2v_2 + \dots)dt,$$

$$v_0 + pv_1 + p^2v_2 + \dots = 0 + p \int_0^t (s_0 + ps_1 + p^2s_2 + \dots)dt,$$

$$s_0 + ps_1 + p^2s_2 + \dots = A + p \int_0^t (t_0 + pt_1 + p^2t_2 + \dots)dt,$$

$$g_0 + pg_1 + p^2g_2 + \dots = B + p \int_0^t ((y_0 + py_1 + p^2y_2 + \dots)^2 - t^{10} + 4(t^9 - t^8 - t^7 + 2t^6 - t^4 + 30t - 12))dt.$$

Simply, comparing the coefficients of same powers of p , we have

$$\begin{aligned}
p^0 : y_0 &= 0, v_0 = 0, s_0 = A, y_0 = B, \\
p^1 : y_1 &= v_0 t = 0, v_1 = s_0 t = At, s_1 = g_0 t = Bt, \\
g_1 &= y_0^2 - \frac{1}{11}y^1 1 + \frac{4}{10}y^1 0 - \frac{4}{9}y^9 - \frac{4}{8}y^8 + \frac{8}{7}y^7 - \frac{4}{5}y^5 + 60y^2 - 48y \\
&= -\frac{1}{11}y^1 1 + \frac{4}{10}y^1 0 - \frac{4}{9}y^9 - \frac{4}{8}y^8 + \frac{8}{7}y^7 - \frac{4}{5}y^5 + 60y^2 - 48y, \\
p^2 : y_2 &= \int_0^t v_1 t dt = \int_0^t At dt = \frac{A}{2}t^2, v_2 = \int_0^t s_1 dt = \int_0^t Bt dt = \frac{B}{2}t^2, \\
s_2 &= \int_0^t g_1 dt = \int_0^t \left(-\frac{1}{11}t^1 1 + \frac{4}{10}t^1 0 - \frac{4}{9}t^9 - \frac{4}{8}t^8 + \frac{8}{7}t^7 - \frac{4}{5}t^5 + 60t^2 - 48t\right) dt, \\
&= -\frac{1}{132}t^1 2 + \frac{2}{55}t^1 1 - \frac{2}{45}t^1 0 - \frac{1}{18}t^9 + \frac{1}{7}t^8 - \frac{2}{15}t^6 + 20t^3 - 24t^2, \\
g_2 &= \int_0^t 2y_0 y_1 dt = 0 \\
p^3 : y_3 &= \int_0^t v_2 dt = \int_0^t \left(\frac{B}{2}t^2\right) dt = \frac{B}{6}t^3, \\
v_3 &= \int_0^t s_2 dt = \int_0^t \left(-\frac{1}{132}t^1 2 + \frac{2}{55}t^1 1 - \frac{2}{45}t^1 0 - \frac{1}{18}t^9 + \frac{1}{7}t^8 - \frac{2}{15}t^6 + 20t^3 - 24t^2\right) dt, \\
&= -\frac{1}{1716}t^1 3 + \frac{2}{330}t^1 2 - \frac{2}{15}t^1 1 - \frac{1}{180}t^1 0 + \frac{1}{63}t^9 - \frac{2}{105}t^7 + 5t^4 - 8t^3, \\
s_2 &= \int_0^t g_2 dt = 0, g_3 = \int_0^t (y_1^2 + 2y_0 y_2) dt = 0, \\
p^4 : y_4 &= \int_0^t v_3 dt = \int_0^t \left(-\frac{1}{1716}t^1 3 + \frac{2}{330}t^1 2 - \frac{2}{495}t^1 1 - \frac{1}{180}t^1 0 + \frac{1}{63}t^9 - \frac{2}{105}t^7 + 5t^4 - 8t^3\right) dt, \\
&= -\frac{1}{24024}t^1 4 + \frac{1}{2145}t^1 3 - \frac{1}{2970}t^1 2 - \frac{1}{1980}t^1 1 + \frac{1}{630}t^1 0 - \frac{1}{420}t^8 + t^5 - 2t^4, \\
v_4 &= \int_0^t s_3 dt = 0, s_4 = \int_0^t g_3 dt = 0, g_4 = \int_0^t (2y_1 y_2 + 2y_0 y_3) dt = 0.
\end{aligned}$$

Combining all the terms we have

$$(2.2) \quad y(t) = \frac{A}{2}t^2 + \frac{B}{6}t^3 - 2t^4 - t^5 - \frac{1}{420}t^8 + \frac{A^2}{6720}t^8 + O(t^{10}),$$

$$(2.3) \quad y'(t) = At + \frac{B}{2}t^2 - 8t^3 - 5t^4 - \frac{8}{420}t^7 + \frac{8A^2}{6720}t^7 + O(t^{10}).$$

Using the boundary conditions at $t = 1$, solving equation (2.2) and equation (2.3) we get

$$A = 3.000003938, B = 3.19339701 \times 10^{13}.$$

The obtained series solution is

$$y(t) = 1.9999t^2 + (5.3223 \times 10^{-14})t^3 - 2t^4 + t^5 - (1.2141 \times 10^{-16})t^8 + o(t^9).$$

t	Exact solution	Series Solution
0.0	0.00000	0.00000
0.1	0.01998	0.01997
0.2	0.07421	0.07420
0.3	0.18641	0.18639
0.4	0.28849	0.28848
0.5	0.42162	0.42160
0.6	0.54296	0.54295
0.7	0.68530	0.68531
0.8	0.79426	0.79425
0.9	0.88542	0.88541
1.0	1.00000	0.99999

TABLE 1. Error Estimation

The value of error is obtained by taking the difference between analytical and numerical solution. The comparison between the exact solution and series solution via HPM is provided in Table 1. When defining more $y(t)$ terms then more precision can be obtained.

Example 2. Considering the following nonlinear 4th order BVP

$$(2.4) \quad y^{(4)}(t) = e^t(t - 3) + y(t) + y''t.$$

The related boundary conditions

$$(2.5) \quad y(0) = 1, y(1) = 0, y'(0) = 0, y'(1) = -e.$$

The exact solution of the above mentioned equation is

$$y(t) = (1 - t)e^t.$$

We are going to use the following transformation

$$y_t = q(t), q_t = f(t), f_t = z(t),$$

we can rewrite the 4th order BVPs (2.4) and (2.5) as the system of differential equations which are given below

$$y_t = q(t), q_t = f(t), f_t = z(t), z_t = u(t) + f(t) + e^t(t - 3),$$

with

$$f(0) = A, z(0) = B, y(0) = 1, q(0) = 0.$$

Using the above transformation, we are able to express the equation in the form of a integral equations:

$$(2.6) \quad \begin{aligned} y(t) &= 1 + \int_0^t q(g)dg, q(t) = 0 + \int_0^t f(g)dg, f(t) = A + \int_0^t z(g)dg, \\ z(t) &= B + \int_0^t (y(g) + f(g) + e^g(g - 3))dg. \end{aligned}$$

Using HPM for equation (2.6) we have

$$\begin{aligned} a_0 + pa_1 + p^2a_2 + \dots &= 1 + p \int_0^t (b_0 + pb_1 + p^2b_2 + \dots)dt, \\ b_0 + pb_1 + p^2b_2 + \dots &= 0 + p \int_0^t c(b_0 + pc_1 + p^2c_2 + \dots)dt, \\ C_0 + pc_1 + p^2c_2 + \dots &= A + p \int_0^t (d_0 + pd_1 + p^2d_2 + \dots)dt, \end{aligned}$$

$$d_0 + pd_1 + p^2d_2 + \dots = B + p \int_0^t [(a_0 + pa_1 + p^2a_2) + (C_0 + pc_1 + p^2c_2 + \dots) + e^t(t - 3)]dt.$$

Simply comparing the coefficients of power of p we have

$$\begin{aligned} p^0 : a_0 &= 1, b_0 = 0, c_0 = A, d_0 = B, \\ p^1 : a_1 &= \int_0^t b_0dt = 0, b_1 = \int_0^t c_0dt = \int_0^t A dt = At, c_1 = \int_0^t d_0dt = \int_0^t B dt = Bt, \\ d_1 &= \int_0^t (a_0 + c_0 + e^t(t - 3))dt = \int_0^t (1 + A + e^t(t - 3))dt = 4 + t + At - 4e^t + te^t \\ p^2 : a_2 &= \int_0^t b_1dt = \int_0^t At dt = \frac{A}{2}t^2, b_2 = \int_0^t c_1dt = \int_0^t Bt dt = \frac{B}{2}t^2, \\ c_2 &= \int_0^t d_1dt = \int_0^t (4 + t + At - 4e^t + te^t)dt = 4t + \frac{1}{2}t^2 + \frac{A}{2}t^2 - 5e^t + 5 + te^t, \end{aligned}$$

$$d_2 = \int_0^t (a_1 + c_1)dt = \int_0^t Btdt = \frac{B}{2}t^2.$$

$$p^3 : a_3 = \int_0^t b_2dt = \int_0^t \frac{Bt^2}{2}dt = \frac{B}{6}t^3,$$

$$b_3 = \int_0^t c_2dt = \int_0^t (4t + \frac{1}{2}t^2 + \frac{A}{2}t^2 - 5e^t + 5 + te^t)dt = 5t + 2t^2 + \frac{(A+1)}{6}t^3 + (t-6)e^t + 6,$$

$$c_3 = \int_0^t d_2dt = \int_0^t (\frac{B}{2}t^2)dt = \frac{B}{6}t^3,$$

$$d_3 = \int_0^t (a_2 + c_2)dt = \int_0^t [\frac{A}{2}t^2 + (4t + \frac{1}{2}t^2 + \frac{A}{2}t^2 - 5e^t + 5 + te^t)] = 6 + 5t + 2t^2 + \frac{(2A+1)}{6}t^3 + (t-6)e^t.$$

Adding up all the terms we have

$$(2.7) \quad y(t) = 512 + 480t + \frac{(499+B)}{2}t^2 + \frac{(418+B)}{6}t^3 + \frac{(385+A)}{24}t^4 + \frac{(354+B)}{120}t^5 + o(t^6) + \dots$$

$$(2.8) \quad y'(t) = 480 + (499+A)t + \frac{(418+B)}{2}t^2 + \frac{(365+A)}{6}t^3 + \frac{(354+B)}{24}t^4 + \dots$$

Using the boundary conditions at $t = 1$, solving equations (2.7) and (2.8) we get

$$A = -0.995781, B = -2.00547$$

The obtained series solution is given below:

$$y(t) = 512 + 480t + 223.001t^2 + 69.298t^3 + 16.0028t^4 + 02.9320t^5 + o(t^6).$$

t	Exact solution	Series Solution
0.0	1.00000	1.00000
0.1	0.98791	0.98792
0.2	0.96715	0.96716
0.3	0.95471	0.95472
0.4	0.8974	0.8975
0.5	0.8481	0.8482
0.6	0.7362	0.7363
0.7	0.6262	0.6263
0.8	0.4486	0.4487
0.9	0.2842	0.2843
1.0	0.00000	-9.6824E-14

TABLE 2. Estimation of Errors

The value of error is obtained by taking the difference between analytical and numerical solution. The comparison between the exact solution and series solution via HPM is provided in Table 2. When defining more $y(t)$ terms then more precision can be obtained.

Example 3. Let us consider the following non linear BVP

$$y^{(4)}(t) = \sin t + \sin^2 t - y''(t)^2,$$

$$y(0) = 0, y(1) = \sin(1), y'(0) = 1, y'(1) = \cos(1)$$

are the boundary conditions.

Using the transformations

$$f_t = z(t), y_t = q(t), q_t = f(t), z_t = \sin t + \sin^2 t - (f(t))^2,$$

with

$$y(0) = 0, q(0) = 1, f(0) = A, z(0) = B.$$

Using the above transformation, we are able to express the equation in the form of a integral equations:

$$y(t) = 0 + \int_0^t q(g)dg, q(t) = 1 + \int_0^t f(g)dg, f(t) = A + \int_0^t z(g)dg,$$

$$(2.9) \quad z(t) = B + \int_0^t (\sin t + \sin 2t - (f(t))^2)dg.$$

Using HPM for equation (2.9) we have

$$y_0 + py_1 + p^2y_2 + \dots = 0 + p \int_0^t (v_0 + pv_1 + p^2v_2 + \dots)dt,$$

$$v_0 + pv_1 + p^2v_2 + \dots = 0 + p \int_0^t (s_0 + ps_1 + p^2s_2 + \dots)dt,$$

$$s_0 + ps_1 + p^2s_2 + \dots = A + p \int_0^t (g_0 + pg_1 + p^2g_2 + \dots)dt,$$

$$g_0 + pg_1 + p^2g_2 + \dots = B + p \int_0^t [(\sin t + \frac{1 - \cos 2t}{2}) - (s_0 + ps_1 + p^2s_2 + \dots)^2]dt.$$

Simply comparing the coefficients of power of p we have

$$p^0 : y_0 = 0, v_0 = 0, s_0 = A, g_0 = B,$$

$$p^1 : y_1 = \int_0^t v_0 dt = 0, v_1 = \int_0^t s_0 dt = \int_0^t A dt = At, s_1 = \int_0^t g_0 dt = \int_0^t B dt = Bt,$$

$$\begin{aligned}
g_1 &= \int_0^t [(\sin t + \frac{1 - \cos 2t}{2}) - s_0^2] dt = -(\cos 2t - 1) + \frac{1}{2}t - \frac{1}{4} \sin 2t - A^2 t = 1 - A^2 t - \cos t + \frac{1}{2}t - \frac{1}{4} \sin 2t. \\
p^2 : y_2 &= \int_0^t v_1 dt = \int_0^t A t dt = \frac{A}{2} t^2, v_2 = \int_0^t s_1 dt = \int_0^t B t dt = \frac{B}{2} t^2, \\
s_2 &= \int_0^t g_1 dt = \int_0^t (1 - A^2 t - \cos t + \frac{1}{2}t - \frac{1}{4} \sin 2t) dt = t + \frac{1}{4} t^2 - \frac{A^2}{2} t^2 - \sin t + \frac{1}{8} \cos 2t - \frac{1}{8}, \\
g_2 &= \int_0^t (-2s_0 s_1) dt = \int_0^t (-2ABt) dt = \frac{-2AB}{2} t^2 = -ABt^2. \\
p^3 : y_3 &= \int_0^t v_2 dt = \int_0^t (\frac{B}{2} t^2) dt = \frac{B}{6} t^3, \\
v_3 &= \int_0^t s_2 dt = \int_0^t (t + \frac{1}{4} t^2 - \frac{A^2}{2} t^2 - \sin t + \frac{1}{8} \cos 2t - \frac{1}{8}) dt = \frac{1}{2} t^2 + \frac{1}{12} t^3 - \frac{A^2}{6} t^3 - \cos t - 1 + \frac{1}{16} \sin 2t - \frac{1}{8} t, \\
s_3 &= \int_0^t g_2 dt = \int_0^t (-ABt^2) dt = -\frac{AB}{3} t^3, \\
g_3 &= - \int_0^t (s_1^2 + 2s_0 s_1) dt = - \int_0^t [B^2 t^2 + 2(At + \frac{A}{4} t^2 - \frac{A^3}{2} t^2 - A \sin t + \frac{A}{8} \cos 2t - \frac{A}{8})] dt \\
&= -\frac{B^2}{3} t^3 - At^2 - \frac{A}{6} t^3 - A \cos t + A - \frac{A}{16} \sin 2t + \frac{A}{8} t.
\end{aligned}$$

Adding up all the terms we have

$$(2.10) \quad y(t) = 0.0313 + t + \frac{A}{2} t^2 + \frac{B}{6} t^3 + \frac{t}{48} (-48 - 3t + 8t^2 + t^3) + \dots$$

$$(2.11) \quad y'(t) = 1 + At + \frac{B}{2} t^2 - 1 - \frac{6}{48} t + \frac{24}{48} t^2 + \frac{4}{48} t^3 + \dots$$

Using the boundary conditions at $t = 1$, solving equations (2.10) and (2.11) we get

$$A = -0.0001529, B = -1.0005468.$$

The obtained series solution is given below:

$$y(t) = 11.347 - 262.827t - 3.407t^3 + 0.1421t^4 + \dots$$

The value of error is obtained by taking the difference between analytical and numerical solution. The comparison between the exact solution and series solution via HPM is provided in Table 3. When defining more $y(t)$ terms then more precision can be obtained.

t	<i>Exact solution</i>	<i>Series Solution</i>
0.0	1.00000	9.5834E-14
0.1	0.09841	0.098413
0.2	0.1989	0.19896
0.3	0.2965	0.29654
0.4	0.3798	0.37982
0.5	0.4898	0.48986
0.6	0.5786	0.57864
0.7	0.6794	0.67948
0.8	0.7468	0.74682
0.9	0.7964	0.79642
1.0	0.8664	86646

TABLE 3. Estimation of Errors

3. CONCLUSION

We found that this approach is effective in determining the analytical solutions to a large field of BVPs. This method offers a more realistic series of solutions, which quickly converges on physical challenges. Therefore, we conclude that the perturbation strategy can be regarded as an important way to treat the linear as well as non-linear problems. We think that this approach will be successful and very much effective in solving non-linear problems and paving the way for study.

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