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UNITARY DIVISOR ADDITION CAYLEY GRAPHS

C. THILAGA[1] AND P. B. SARASIJA

ABSTRACT. Let $n \geq 1$ be an integer and S be the set of Unitary Divisor of n. Then the set $S^* = \{s, n-s/s \in S \ and \ n \neq s\}$ is a symmetric subset of the group $(Z_n, \oplus)$,the additive Abelian group of integers modulo n.The Cayley graph of $(Z_n, \oplus)$ associated with the above symmetric subset S^* is called the Unitary Divisor Addition Cayley graph and it is denoted by $G_n(D)$. That is the graph $G_n(D)$ is the graph whose vertex set is $V = \{0, 1, \ldots, (n-1)\}$ and the edge set E is the set of all ordered pairs of vertices x, y such that $x + y \in S^*$. In this paper, we discuss the degree of the vertices and the total number of edges and some properties of Unitary Divisor Addition Cayley graph $G_n(D)$.

1. INTRODUCTION

A graph G is a pair (V, E), where $V = \{0, 1, \ldots, (n-1)\}$, be the vertex set and E is a set of unordered pairs of elements of V are the edges of G. The degree of a vertex v, $d(v)$ in G is the number of edges incident at v.If the degree of each vertex is equal, say r in G, then G is called r-regular graph. A graph is called (r_1, r_2)-semi regular if its vertex set can be partitioned into two subsets V_1 and V_2 such that all the vertices in V_i are of degree for $i = 1, 2$.Vertices u and v of a graph G are adjacent if $uv \in E(G)$. Throughout the text, we consider non-trivial,finite,undirected graph with no loops or multiple edges. For standard terminology and notation in graph theory we follow [1],[4].

[1]*corresponding author*

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For a positive integer $n > 1$, the Unitary Addition Cayley graph G_n is the graph whose vertex set is Z_n, the integers modulo n and if U_n denotes set of all units of the ring Z_n, then two vertices a, b are adjacent if and only if $a + b \in U_n$ [2] and also refer [5], [6].

Let $n \geq 1$ be an integer and S be the set of unitary divisor of n. Then the set $S^* = \{s, n - s/s \in S, n \neq s\}$ is a symmetric subset of the group $(Z_n, \oplus)$, the additive abelian group of integers modulo n. The Cayley graph of (Z_n, U_n), associated with the above symmetric subset S^* is called the Unitary Divisor Cayley graph $G\,(Z_n, U_n)$. That is $G\,(Z_n, U_n)$ is the graph whose vertex set $V = \{0, 1, \ldots, (n-1)\}$ and the edge set E is the set of all ordered pairs of vertices x, y, such that $x - y \in S^*$ or $y - x \in S^*$, see [3].

Now motivated by the Unitary Addition Cayley graph and Unitary Divisor Cayley graph we introduce the Unitary Divisor Addition Cayley graph as follows: Let $n \geq 1$ be an integer and S be the set of Unitary Divisor of n. Then the set $S^* = \{s, n - s/s \in S \; and \; n \neq s\}$ is a symmetric subset of the group $(Z_n, \oplus)$, the additive abelian group of integers modulo n the Cayley graph of $(Z_n, \oplus)$ associated with the above symmetric subset S^* is called the Unitary Divisor Addition Cayley graph and it is denoted by $G_n(D)$. That is the graph $G_n(D)$ is the graph whose vertex set is $V(G_n(D)) = \{0, 1, \ldots, (n-1)\}$ and the edge set E is the set of all ordered pairs of vertices x, y, such that $x + y \in S^*$.

The diameter of graph is the maximum distance between the pair of vertices. The number of edges in the maximum matching of G is called the matching number. Let $G(V, E)$ be a connected with vertex set V and edge set E. A subset of V is called independence if its vertices are mutually non-adjacent. A graph is Eulerian if the graph both connected and has a closed trial(a walk with no repeated edges) containing all edges of the graph. A connected graph has an Euler cycle if and only if every vertex has even degree. A Hamiltonian path is an undirected or directed graph that visits each vertex exactly once.

2. Some examples of Unitary Divisor Addition Cayley graph

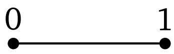

Figure: $G_2(D)$

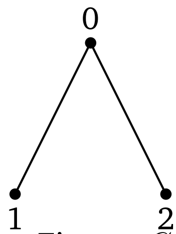

Figure: $G_3(D)$

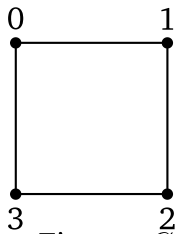

Figure: $G_4(D)$

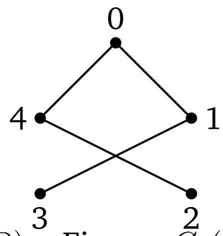

Figure: $G_5(D)$

Theorem 2.1. *Let m be any vertex of Unitary Divisor Addition Cayley graph $G_n(D)$, then the degree of m is* $d(m) = \begin{cases} |S^*| \textit{ if } 2m(modn) \notin S^* \\ |S^*| - 1 \textit{ if } 2m(modn) \in S^* \end{cases}$.

Proof. Consider Unitary Divisor Addition Cayley graph $G_n(D)$ with vertex set $V(G_n(D)) = \{0, 1, \ldots, (n-1)\}$. Let m be any vertex in $V(G_n(D)$. Let S be the set of unitary divisor of n and $S^* = \{s, n - s/s \in S \textit{ and } n \neq s\}$.The graph $G_n(D)$ is $|S^*|$ (equal to 2)-regular if n is 2^γ, $\gamma > 1$, and $(|S^*|, |S^*| - 1)$-semi regular otherwise.

The vertex $0 \in V(G_n(D))$ and $0 \notin S^*$, 0 is adjacent to all the vertices in S^*.Hence degree of 0 is $|S^*|$. That is $d(0) = |S^*|$. If the vertex $m \in V(G_n(D))$ then either $m \in S^*$ or $m \notin S^*$.

Suppose $m \in S^*$ and $m = n - m$. If $m = n - m$ implies $2m = n$. Take $(modn)$ on both sides we get $2m(modn) = n(modn)$. That is $2m(modn) \equiv 0 \notin S^*$. The m vertex is adjacent to all the vertices in S^*.Hence degree of m is $|S^*|$.That is $d(m) = |S^*|$.

Suppose $m \in S^*$ and $m \neq n - m$.Also $m \notin S^*$. Then $2m(modn) \neq n(modn)$.Let $2m(modn) \equiv k(modn)$. If $k \notin S^*$ then the vertex m is adjacent to all the vertices in S^*. Hence degree of m is $|S^*|$. That is $d(m) = |S^*|$. If $k \in S^*$ then $d(m) = |S^*| - 1$.

Hence $d(m) = \begin{cases} |S^*| \text{ if } 2m(modn) = k \notin S^* \\ |S^*| - 1 \text{ if } 2m(modn) = k \in S^* \end{cases}$. □

Remark 2.1. *If $n = 2^\gamma$, $\gamma > 1$, then the Unitary Divisor is 1. Hence $S^* = \{1, n-1\}$. The Unitary Divisor Addition Cayley graph is a cycle. Thus $d(n) = 2$.*

Remark 2.2. *If $n = p^{\alpha}$, $\alpha \geq 1$, then $S^* = \{1, n-1\}$. The Unitary Divisor Addition Cayley graph is a path. The vertices $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$ have degree 1 and all other vertices have degree 2.*

Theorem 2.2. *The total number of edges in the Unitary Divisor Addition Cayley graph $G_n(D)$ is* $\begin{cases} n \text{ if } n = 2^{\gamma}, \gamma > 1 \\ \frac{|S^*|(n-1)}{2} \text{ if } n \text{ is odd} \\ \frac{|S^*|(n-1)+1}{2} \text{ if } n = 2p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r} \\ \frac{|S^*|(n-1)}{2} + 1 \text{ if } n = 2^{\alpha}p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r} \end{cases}$

Proof. Suppose that $n = 2^{\gamma}$, $\gamma > 1$.
Then 1 is the only unitary divisor of n, so that $S^* = \{1, n-1\}$.The Unitary Divisor Addition Cayley graph is a cycle. So each vertex of degree is 2. Therefore number of edges is n.
Suppose n is odd, then $n - |S^*|$ vertices having degree $|S^*|$ and $|S^*|$ vertices having degree $|S^*|-1$. Therefore number of edges in $G_n(D)$ is $\frac{(n-|S^*|)|S^*|+|S^*|(|S^*|-1)}{2} = \frac{|S^*|(n-1)}{2}$.
Suppose $n = 2p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r}$. Here $(n - |S^*| + 1)$ vertices having degree $|S^*|$ and $|S^*| - 1$ vertices having degree $|S^*| - 1$. Therefore number of edges is $\frac{(n-|S^*|+1)|S^*|+(|S^*|-1)(|S^*|-1)}{2} = \frac{|S^*|(n-1)+1}{2}$.
Suppose $n = 2^{\alpha}p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r}$
Here $(n-|S^*|+2)$ vertices having degree $|S^*|$ and $|S^*|-2$ vertices having degree $|S^*| - 1$. The number of edges is $\frac{(n-|S^*|+2)|S^*|+(|S^*|-2)(|S^*|-1)}{2} = \frac{|S^*|(n-1)}{2} + 1$. □

Theorem 2.3. *The Unitary Divisor Addition Cayley graph is Eulerian iff $n = 2^{\gamma}$, $\gamma > 1$.*

Proof. Suppose $G_n(D)$ is Eulerian. If possible n is other than 2^{γ}, $\gamma > 1$.
A graph is Eulerian iff G is connected and its vertices all have even degree Therefore $G_n(D)$ is not Eulerian , a contradiction to our assumption.
Next suppose $n = 2^{\gamma}$ and $\gamma > 1$, then the degree of each vertex is $|S^*|$ and $|S*|$=2 is even. Hence the result. □

Theorem 2.4. *The Unitary Divisor Addition Cayley graph is Hamiltonian for n is even and also connected for all n.*

Proof. Now we construct a cycle $C = (0, n-1, 2, n-3, 4, \dots, n-4, 3, n-2, 1, 0)$.

Since the cycle C contains all the vertices of $G_n(D)$ exactly once, C is a Hamilonian cycle of $G_n(D)$. Thus $G_n(D)$ is Hamiltonian.

Suppose n is even the Unitary Divisor Addition Cayley graph is Hamiltonian , then find a Hamiltonian path $P = (n-1, 2, n-3, 4, \ldots, n-4, 3, n-2, 0)$. Then $G_n(D)$ is connected.

Suppose n is odd, there exists a path$(\frac{n+1}{2}, \ldots, 3, n-2, 1, 0, n-1, 2, n-3, \ldots, \frac{n-1}{2})$. Thus $G_n(D)$is connected. □

Theorem 2.5. *The Unitary Divisor Addition Cayley graph is bipartite iff $n = 2^\alpha$, $\alpha > 1$ and $n = p^m$, $m \geq 1$; p is prime.*

Proof. Suppose $n = 2^\alpha$, $\alpha > 1$ and $n = p^m$, $m \geq 1$. We can split the vertex set into two parts $X = \{0, 2, 4, \ldots, n-2\}$ and $Y = \{1, 3, \ldots, n-1\}$and $V = X \bigcup Y$. Hence $G_n(D)$ is bipartite. □

Theorem 2.6. *The diameter of Unitary Divisor Addition Cayley graph is*

$$\begin{cases} n-1 \textit{ if } n = p^m \\ \frac{n}{2} \textit{ if } n = 2^\alpha \\ 2 \textit{ if } n = 6, 12 \\ 3 \; otherwise. \end{cases}.$$

Proof. Suppose $n = p^m$, $m \geq 1$; it is a bipartite graph with $X = \{1, 3, \ldots, n-1\}$and $Y = \{0, 2, \ldots, n-2\}$. It is also a path graph. Hence diameter=$n-1$.

Suppose $n = 2^\alpha$, $\alpha > 1$; the Unitary Divisor Addition Cayley graph is bipartite, the vertex set can split into two parts $X = \{0, 2, 4, \ldots., n-2\}$ and $Y = \{1, 3, 5, \ldots, n-1\}$,it is a cycle. Hence diameter is $\frac{n}{2}$.

Suppose $n = 6$ and $n = 12$, there exists a path $(1, 0, n-1)$ of length 2. Hence diameter=2.

Suppose n is odd. Then there exists two non adjacent vertices x and y in vertex set in $G_n(D)$ such that x is even and y is odd and have no common neighbour. Let z be a neighbour of x in S^* and odd , there have a common neighbour u. Hence we take a path $x - z - u - y$ of length 3. Hence diameter =3. □

Theorem 2.7. *The matching number of Unitary Divisor Addition Cayley graph is*

$$\begin{cases} \frac{n}{2} \textit{ if } n \textit{is even} \\ \frac{n-1}{2} \textit{ if } n \textit{ is odd} \end{cases}.$$

Proof. In $G_n(D)$, the generating set S^* must contain 1.

Suppose n is even. The edge set $E_1 = \{(0,1),(n-1,2),(n-2,3),\dots,(\frac{n+2}{2},\frac{n}{2})\}$ is an independent set in $G_n(D)$ and $|E_1| = \frac{n}{2}$. So the matching number is $\frac{n}{2}$.

Suppose n is odd. The edge set $E_2 = \{(0,1),(n-1,2),\dots,(\frac{n+3}{2},\frac{n-1}{2})\}$is an independent set in $G_n(D)$, and $|E_2| = \frac{n-1}{2}$. Therefore matching number is $\frac{n-1}{2}$. □

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DEPARTMENT OF MATHEMATICS
NOORUL ISLAM CENTRE FOR HIGHER EDUCATION
KUMARACOIL-629175
Email address: thilagakmoorthy7@gmail.com

DEPARTMENT OF MATHEMATICS
NOORUL ISLAM CENTRE FOR HIGHER EDUCATION
KUMARACOIL-629175
Email address: sijavk@gmail.com