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ON THE ACHROMATIC NUMBER OF MESH LIKE TREES

SHARMILA MARY ARUL

ABSTRACT. The achromatic number for a graph $G = (V, E)$ is the largest integer m such that there is a partition of V into disjoint independent sets (V_1,... ,V_m) satisfying the condition that for each pair of distinct sets $V_i, V_j,\ V_i \cup V_j$ is not an independent set in G. In this paper, we present $O\left(2^n\right)$ approximation algorithms to determine the achromatic number of mesh like trees.

1. INTRODUCTION

A *proper* coloring of a graph G is an assignment of colors to the vertices of G such that adjacent vertices are assigned different colors. A proper coloring of a graph G is said to be *complete* if for every pair of colors i and j there are adjacent vertices u and v colored i and j respectively. The *achromatic number* of the graph G is the largest number m such that G has a complete coloring with m colors. Thus the achromatic number for a graph $G = (V, E)$ is the largest integer m such that there is a partition of V into disjoint independent sets $(V_1,\ V_2, \ldots,\ V_m)$ such that for each pair of distinct sets $V_i, V_j,\ V_i \cup V_j$ is not an independent set in G .

Graph coloring problem is expected to have wide variety of applications such as scheduling, frequency assignment in cellular networks, timetabling, crew assignment etc. Scheduling problems with inter-processor communication delays whose number of processors is increasing are recent problems arising with the development of new message-passing architectures more and more. Small

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communication time task systems show that the achromatic number of the co-comparability graph is an upper bound on the minimum number of processors [1].

2. OVERVIEW OF THE PAPER

Harary, Hedetniemi and Prins [10, 11] introduced the achromatic number. The survey articles by Hughes and MacGillivray [12] and Edwards [5] contain huge collection of references of research papers related to achromatic problem. Computing achromatic number of a general graph was proved to be *NP-complete* by Yannakakis and Gavril [20]. A simple proof of this fact appears in [8]. M. Farber, G. Hahn, P. Hell and D. J. Miller [7] show that the problem is NP- hard on bipartite graphs. It was further proved that the achromatic number problem remains NP- complete even for connected graphs, which are both interval graphs and cographs simultaneously [2]. Cairnie and Edwards [3], Edwards and McDiarmid [6] show that the problem is *NP- hard* even on trees. Further it is polynomially solvable for paths, cycles [5], union of paths [14] and [13] gives the approximation algorithm for the achromatic number problem on bipartite graphs.

Since achromatic optimization problem is *NP*-hard, most of the research studies related to achromatic problem focus on approximation algorithms. An *approximation algorithm* for a problem, loosely speaking, is an algorithm that runs in polynomial time and produces an "approximate solution" to the problem. We say that an algorithm is α -approximation algorithm for a maximization problem if it always delivers a solution whose value is at least a factor $\frac{1}{\alpha}$ of the optimum. The parameter α is called the *approximation ratio* [4].

It is stated in [3] that "for achromatic numbers, there appear to be only a few results on special graphs apart from those for paths and cycles". Geller and Kronk [9] proved that there is almost optimal coloring for families of paths and cycles. This result was extended to bounded degree trees [3]. Roichman gives a lower bound on the achromatic number of Hypercubes [15]. In this paper, we determine an approximation algorithm for the achromatic number of mesh like trees.

3. Main Results

Lemma 3.1. *Any partial complete coloring can be extended to a complete coloring of the entire graph.*

By this lemma, the achromatic number of a subgraph H remains a lower bound for the achromatic number of the given graph. This key observation has motivated us to design efficient approximation algorithms for certain interconnection networks.

Thus throughout the paper our strategy is as follows.

Strategy: We identify an induced subgraph of the given graph whose achromatic number is easily computable. The achromatic number of this subgraph is then used as a lower bound for the achromatic number of the given graph.

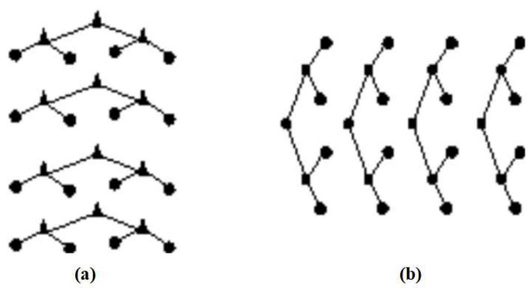


FIGURE 1. Vertices and edges added to form (a) Row trees (b) Column trees

3.1. Mesh of Trees.

Here we define the two-dimensional mesh of trees, which has a very natural and regular structure. There are many algorithms, which utilize simple and identical operations in the row and column trees to perform complex global tasks with surprising speed.

Definition 3.1. *The $N \times N$ mesh of trees see Figure 2a denoted by $MT(n)$ where $N = 2n$ is constructed from an $N \times N$ grid of vertices by adding vertices and edges to form a complete binary tree in each row see Figure 1(a) and each column see Figure 1(b).*

The leaves of the tree are precisely the original vertices of the grid, and the added vertices are precisely the internal vertices of the trees. The mesh of trees is a recurrent network. By removing the root vertices and their incident edges in all trees, the mesh of trees is partitioned into 4 disconnected $2^{n-1} \times 2^{n-1}$ mesh of trees. Overall, the network has $3N^2 - N$ vertices. The leaf and root vertices have degree 2 and all other vertices have degree 3.

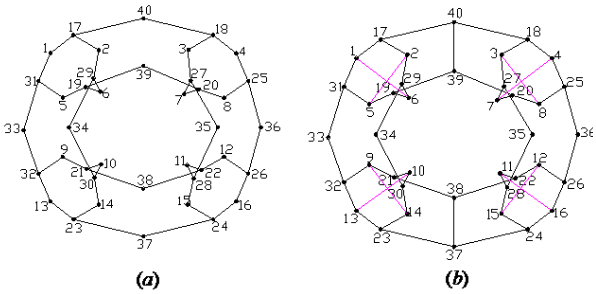


FIGURE 2. (a) Mesh of Trees (b) 3-Regular Mesh of Trees

Let G be an $N \times N$ mesh of trees where $N = 2n$. It has $2^n\left(2^{n+1} + 2^n - 2\right)$ vertices, for $n \geq 2$.
Equivalent form of Figure 2 is given in Figure 3.

Theorem 3.1. *Let $MT\left(2^n\right)$ denote the $2^n \times 2^n$ mesh of trees. Then $\psi\left(MT\left(2^n\right)\right) \geq 5n - 3$.*

Proof. It is easy to see that the coloring in Figure 3(a) represents proper coloring. □

Remark 3.1. *Since $MT\left(2^n\right)$ follows exponential growth, therefore in this paper we have restricted ourselves till $n = 3$ only.*

The following result is due to the number of edges of $MT\left(2^n\right)$ is $2^n\left(2^{n+1} + 2^n - 2 + n2^{n-2}\right)$.

Theorem 3.2. *Let $MT\left(2^n\right)$ denote the $2^n \times 2^n$ mesh of trees. Then*

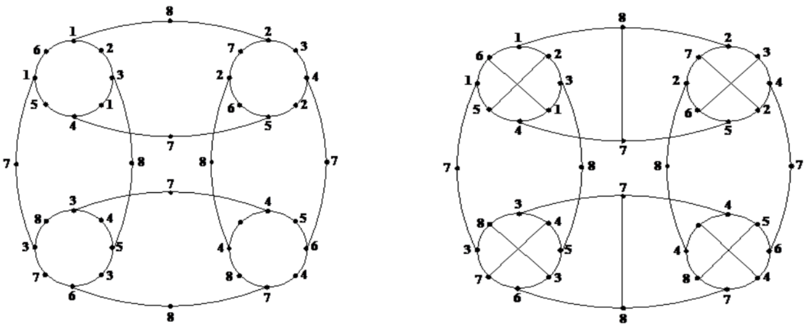
$$\psi\left(MT\left(2^n\right)\right) \leq \frac{1 + \sqrt{1 + 2^{n+2}\left(2^{n+1} + 2^n - 2 + n2^{n-2}\right).}}{2}.$$

Theorem 3.3. *There is an $O\left(2^n\right)$ -approximation algorithm to determine the achromatic number of a two dimensional mesh of trees $MT\left(2^n\right)$.*

Proof. The expected achromatic number for $MT\left(2^n\right)$ denoting the $2^n \times 2^n$ mesh of trees is

$$\frac{1 + \sqrt{1 + 2^{n+2}\left(2^{n+1} + 2^n - 2 + n2^{n-2}\right)}}{2},$$

and the achromatic number realized is $5n - 3$. This proves the theorem. □



(a) (b)

FIGURE 3. Achromatic labeling of (a) Mesh of Trees (b) 3-Regular Mesh of Trees

3.2. **3-Regular Mesh of Trees.**

Definition 3.2. *One can modify G by adding new edges to G so that the modified graph is 3-regular. The modified graph is denoted by $RMT(n)$ See Figure 2b. The edges added do not reduce the diameter of the new graph. Thus it follows that the diameter of a $RMT(n)$ is also $4n$.*

Theorem 3.4. *Let $RMT(n)$ denote a two-dimensional 3-regular mesh of trees. Then*

$$5n-3 \leq \psi(MT(n)) \leq \frac{1+\sqrt{1+2^{n+2}\left(2^{n+1}+2^{n}-2+n2^{n-2}\right)}}{2}.$$

Proof. Since the graph is 3-regular, the number of edges is

$$3.2^{n}\left(2^{n+1}+2^{n}-2+n2^{n-2}\right).$$

See Figure 3b. □

Theorem 3.5. *There is an $O(2^{n})$ -approximation algorithm to determine the achromatic number of a two dimensional 3-regular mesh of trees $RMT(n)$.*

Remark 3.2. *Here also since $RMT(n)$ follows exponential growth, therefore in this paper we have restricted ourselves till $n=3$ only.*

4. Conclusion

Here we present an $O(2^n)$ - approximation algorithms to determine the achromatic number of mesh like trees. Already the same is obtained in [16–19] for Butterfly and Benes Networks, extended toroid graphs, *H*-graphs, Silicate networks, honeycomb networks and Circulant graphs. Finding efficient approximation algorithms to determine achromatic number for other interconnection networks is quite challenging.

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DIVISION OF MATHEMATICS
SAVEETHA SCHOOL OF ENGINEERING
SIMATS, CHENNAI, TAMIL NADU, INDIA
Email address: sharmilamaryarul@gmail.com